BLIND IDENTIFICATION OF SECOND ORDER HAMMERSTEIN SERIES

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ABSTRACT

Blind identification of second order Hammerstein series is considered [1]. The output cumulants up to order 5 are used to determine the Volterra kernels, when the input is a stationary zero mean Gaussian white stochastic process. Both infinite and finite extent kernels are considered.

1 INTRODUCTION

Second order Hammerstein series constitute an interesting class of second order Volterra systems [1], where the second order homogeneous Volterra kernel is diagonal. They have the form

$$y(n) = \sum_{k=0}^{\infty} h_1(k)u(n-k) + \sum_{k=0}^{\infty} h_2(k)u^2(n-k) \qquad (1)$$

Blind identification is concerned with the determination of the Volterra kernels $h_1(k)$ and $h_2(k)$ on the basis of output information only. The complexity of the problem arises from the fact that the output statistics depend nonlinearly on the kernels even if the system is linear [2], [6]- [7]. This is in sharp contrast with conventional system identification whereby the use of crosscumulants between the output and the input copies depend linearly on the Volterra kernels [2]- [4]. Little is known about the blind identification of general Volterra systems [6]- [7]. Solution of the blind identification problem for the special case of a second order Hammerstein series presented in this paper will provide useful insight for the general case.

2 OUTPUT CUMULANTS AND SPECTRA

Let us consider the system (1) where the input $u(n)$ is a stationary zero mean Gaussian white stochastic process with power spectral density $C_{2u}(w) = \gamma_2$.

To determine the kernels we shall generate the output cumulants of all orders less than 5. This is a tedious exercise that makes use of the standard properties of cumulants and the Leonov-Shiryaev formula for manipulating products of random variables [5]. Detailed derivations are omitted. The resulting expressions are listed below.

Output mean

$$c_{1y} = E[y(n)] = \gamma_2 \sum_l h_2(l) = \gamma_2 H_2(0) \qquad (2)$$

Cumulants of order 2

$$\begin{aligned} c_{2y}(i_1) &= \text{cum}[y(n+i_1), y(n)] \\ &= \gamma_2 \sum_l h_1(l+i_1)h_1(l) + 2\gamma_2^2 \sum_l h_2(l+i_1)h_2(l) \end{aligned} \qquad (3)$$

Cumulants of order 3

$$\begin{aligned} c_{3y}(i_1, i_2) &= \text{cum}[y(n+i_1), y(n+i_2), y(n)] = \\ &= 2\gamma_2^2 \sum_l h_1(l+i_1)h_1(l+i_2)h_2(l) \\ &+ 2\gamma_2^2 \sum_l h_1(l+i_1)h_2(l+i_2)h_1(l) \\ &+ 2\gamma_2^2 \sum_l h_2(l+i_1)h_1(l+i_2)h_1(l) \\ &+ 8\gamma_2^3 \sum_l h_2(l+i_1)h_2(l+i_2)h_2(l) \end{aligned} \qquad (4)$$

Cumulants of order 4

$$\begin{aligned} c_{4y}(i_1, i_2, i_3) &= \text{cum}[y(n+i_1), y(n+i_2), y(n+i_3), y(n)] = \\ &= 8\gamma_2^3 \left(\phi_0(i_1, i_2, i_3, 0) + \phi_0(i_1, i_3, i_2, 0) + \phi_0(i_2, i_3, i_1, 0)\right. \\ &\quad \left. + \phi_0(i_1, 0, i_2, i_3) + \phi_0(i_2, 0, i_1, i_3) + \phi_0(i_3, 0, i_1, i_2)\right) \\ &+ 48\gamma_2^4 \sum_l h_2(l+i_1)h_2(l+i_2)h_2(l+i_3)h_2(l) \end{aligned} \qquad (5)$$

where

$$\phi_0(i_1, i_2, i_3, i_4) = \sum_l h_1(l+i_1)h_1(l+i_2)h_2(l+i_3)h_2(l+i_4)$$

Cumulants of order 5

$$c_{5y}(i_1,i_2,i_3,i_4) =$$

$$= \mathrm{cum}[y(n+i_1), y(n+i_2), y(n+i_3), y(n+i_4), y(n)]$$

$$= 48\gamma_2^4\,(\phi_1(i_1,i_2,i_3,i_4,0) + \phi_1(i_1,i_3,i_2,i_4,0)$$
$$+\phi_1(i_2,i_3,i_1,i_4,0)+\phi_1(i_1,i_4,i_2,i_3,0)+\phi_1(i_2,i_4,i_1,i_3,0)$$
$$+\phi_1(i_3,i_4,i_1,i_2,0)+\phi_1(i_1,0,i_2,i_3,i_4)+\phi_1(i_2,0,i_1,i_3,i_4)$$
$$+\phi_1(i_3,0,i_1,i_2,i_4) + \phi_1(i_4,0,i_1,i_2,i_3))$$
$$+384\gamma_2^5 \sum_l h_2(l+i_1)h_2(l+i_2)h_2(l+i_3)h_2(l+i_4)h_2(l) \quad (6)$$

where, $\phi_1(i_1,i_2,i_3,i_4,i_5) =$

$$= \sum_l h_1(l+i_1)h_1(l+i_2)h_2(l+i_3)h_2(l+i_4)h_2(l+i_5)$$

Using the multidimensional Fourier transform we convert cumulants to polyspectra.

Cumulant spectrum of order 2

$$C_{2y}(w_1) = \gamma_2|H_1(w_1)|^2 + 2\gamma_2^2|H_2(w_1)|^2 \quad (7)$$

Cumulant spectrum of order 3

$$C_{3y}(w_1,w_2) = 2\gamma_2^2 H_1(w_1)H_1(w_2)H_2^*(w_1+w_2)$$
$$+2\gamma_2^2 H_1(w_1)H_2(w_2)H_1^*(w_1+w_2)$$
$$+2\gamma_2^2 H_2(w_1)H_1(w_2)H_1^*(w_1+w_2)$$
$$+8\gamma_2^3 H_2(w_1)H_2(w_2)H_2^*(w_1+w_2) \quad (8)$$

Cumulant spectrum of order 4

$$C_{4y}(w_1,w_2,w_3) =$$
$$= 8\gamma_2^3\Big(\tilde\phi_0(w_1,w_2,w_3)+\tilde\phi_0(w_1,w_3,w_2)+\tilde\phi_0(w_2,w_3,w_1)$$
$$+\tilde\phi_1(w_1,w_2,w_3)+\tilde\phi_1(w_1,w_3,w_2)+\tilde\phi_1(w_2,w_3,w_1)\Big)$$
$$+48\gamma_2^4 H_2(w_1)H_2(w_2)H_2(w_3)H_2^*(w_1+w_2+w_3) \quad (9)$$

where,

$$\tilde\phi_0(w_1,w_2,w_3) = H_1(w_1)H_1(w_2)H_2(w_3)H_2^*(w_1+w_2+w_3)$$

and

$$\tilde\phi_1(w_1,w_2,w_3) = H_2(w_1)H_2(w_2)H_1(w_3)H_1^*(w_1+w_2+w_3)$$

Cumulant spectrum of order 5

$$C_{5y}(w_1,w_2,w_3,w_4) =$$
$$= 48\gamma_2^4\Big(\tilde\phi_2(w_1,w_2,w_3,w_4)+\tilde\phi_2(w_1,w_3,w_2.w_4)$$
$$+\tilde\phi_2(w_2,w_3,w_1,w_4)+\tilde\phi_2(w_1,w_4,w_2,w_3)$$
$$+\tilde\phi_2(w_2,w_4,w_1,w_3)+\tilde\phi_2(w_3,w_4,w_1,w_2)$$
$$+\tilde\phi_3(w_1,w_2,w_3,w_4)+\tilde\phi_3(w_1,w_2,w_4,w_3)$$
$$+\tilde\phi_3(w_1,w_3,w_4,w_2)+\tilde\phi_3(w_2,w_3,w_4,w_1)\Big)$$
$$+384\gamma_2^5 H_2(w_1)H_2(w_2)H_2(w_3)H_2(w_4)H_2^*(\sum_{i=1}^4 w_i) \quad (10)$$

where, $\tilde\phi_2(w_1,w_2,w_3,w_4) =$

$$= H_1(w_1)H_1(w_2)H_2(w_3)H_2(w_4)H_2^*(w_1+w_2+w_3+w_4)$$

and $\tilde\phi_3(w_1,w_2,w_3,w_4) =$

$$= H_2(w_1)H_2(w_2)H_2(w_3)H_1(w_4)H_1^*(w_1+w_2+w_3+w_4)$$

3 NON PARAMETRIC BLIND IDENTIFICATION

We shall assume that the first degree kernel $h_1(\cdot)$ is minimum phase and that $H_1(0) = \sum_k h_1(k) \neq 0$. We evaluate eqs. (7)- (10) at suitable slices and we combine the resulting expressions to determine the kernels magnitude spectra.

Eq. (7) at frequency 0 gives

$$C_{2y}(0) = \gamma_2 H_1^2(0) + 2\gamma_2^2 H_2^2(0) \quad (11)$$

Taking into account eq. (2) we obtain

$$\gamma_2 H_1^2(0) = C_{2y}(0) - 2c_{1y}^2 \quad (12)$$

Eqs. (8)- (9) give

$$C_{3y}(w_1,0) = 2\gamma_2^2 H_1(w_1)H_1(0)H_2^*(w_1)$$
$$+2\gamma_2^2|H_1(w_1)|^2 H_2(0) + 2\gamma_2^2 H_2(w_1)H_1(0)H_1^*(w_1)$$
$$+8\gamma_2^3|H_2(w_1)|^2 H_2(0) \quad (13)$$

$$C_{4y}(w_1,0,0) = 8\gamma_2^3 H_1(w_1)H_1(0)H_2(0)H_2^*(w_1)$$
$$+8\gamma_2^3 H_1(w_1)H_2(0)H_1(0)H_2^*(w_1) + 8\gamma_2^3|H_2(w_1)|^2 H_1^2(0)$$
$$+8\gamma_2^3|H_1(w_1)|^2 H_2^2(0) + 8\gamma_2^3 H_2(w_1)H_1(0)H_2(0)H_1^*(w_1)$$
$$+8\gamma_2^3 H_2(w_1)H_2(0)H_1(0)H_1^*(w_1)$$
$$+48\gamma_2^4|H_2(w_1)|^2 H_2^2(0) \quad (14)$$

Using eqs. (14), (13), (7) and (2) we have that

$$C_{4y}(w_1,0,0) - 8C_{3y}(w_1,0)c_{1y} + 8C_{2y}(w_1)c_{1y}^2 =$$
$$= 8\gamma_2^3 H_1^2(0)|H_2(w_1)|^2$$

This equation becomes using eq. (12)

$$8\gamma_2^2|H_2(w_1)|^2 =$$

$$= \frac{C_{4y}(w_1,0,0) - 8C_{3y}(w_1,0)c_{1y} + 8C_{2y}(w_1)c_{1y}^2}{C_{2y}(0) - 2c_{1y}^2} \quad (15)$$

Thus

$$\gamma_2|H_1(w_1)|^2 = C_{2y}(w_1) - 2\gamma_2^2|H_2(w_1)|^2 \quad (16)$$

From eqs. (15)- (16) we estimate the magnitude spectrum of H_1 and H_2. Since h_1 is a minimum phase system H_1 is uniqely determined. It remains to estimate the phase spectrum of h_2. Using eqs. (9)- (10) we get

$$\begin{aligned} C_{4y}(w_1,w_2,0) &= 8\gamma_2^3 H_1(w_1)H_1(w_2)H_2(0)H_2^*(w_1+w_2) \\ &+8\gamma_2^3 H_1(w_1)H_2(w_2)H_1(0)H_2^*(w_1+w_2) \\ &+8\gamma_2^3 H_2(w_1)H_1(w_2)H_1(0)H_2^*(w_1+w_2) \\ &+8\gamma_2^3 H_1(w_1)H_2(w_2)H_2(0)H_1^*(w_1+w_2) \\ &+8\gamma_2^3 H_2(w_1)H_1(w_2)H_2(0)H_1^*(w_1+w_2) \\ &+8\gamma_2^3 H_2(w_1)H_2(w_2)H_1(0)H_1^*(w_1+w_2) \\ &+48\gamma_2^4 H_2(w_1)H_2(w_2)H_2(0)H_2^*(w_1+w_2) \end{aligned} \quad (17)$$

$$\begin{aligned} C_{5y}(w_1,w_2,0,0) &= \\ = 48\gamma_2^4 & H_1(w_1)H_1(w_2)H_2(0)H_2(0)H_2^*(w_1+w_2) \\ +48\gamma_2^4 & H_1(w_1)H_2(w_2)H_1(0)H_2(0)H_2^*(w_1+w_2) \\ +48\gamma_2^4 & H_2(w_1)H_1(w_2)H_1(0)H_2(0)H_2^*(w_1+w_2) \\ +48\gamma_2^4 & H_1(w_1)H_2(w_2)H_2(0)H_1(0)H_2^*(w_1+w_2) \\ +48\gamma_2^4 & H_2(w_1)H_1(w_2)H_2(0)H_1(0)H_2^*(w_1+w_2) \\ +48\gamma_2^4 & H_2(w_1)H_2(w_2)H_1(0)H_1(0)H_2^*(w_1+w_2) \\ +48\gamma_2^4 & H_1(w_1)H_2(w_2)H_2(0)H_2(0)H_1^*(w_1+w_2) \\ +48\gamma_2^4 & H_2(w_1)H_1(w_2)H_2(0)H_2(0)H_1^*(w_1+w_2) \\ +48\gamma_2^4 & H_2(w_1)H_2(w_2)H_1(0)H_2(0)H_1^*(w_1+w_2) \\ +48\gamma_2^4 & H_2(w_1)H_2(w_2)H_2(0)H_1(0)H_1^*(w_1+w_2) \\ +384\gamma_2^5 & H_2(w_1)H_2(w_2)H_2(0)H_2(0)H_2^*(w_1+w_2) \end{aligned} \quad (18)$$

Eqs. (18), (17), (8) and (2) give

$$\begin{aligned} C_{5y}(w_1,w_2,0,0) &- 12C_{4y}(w_1,w_2,0)c_{1y} \\ &+24C_{3y}(w_1,w_2)c_{1y}^2 = \\ &= 48\gamma_2^4 H_1^2(0)H_2(w_1)H_2(w_2)H_2^*(w_1+w_2) \end{aligned}$$

which takes the form

$$48\gamma_2^3 H_2(w_1)H_2(w_2)H_2^*(w_1+w_2) = \frac{A}{B} \quad (19)$$

where

$$A = C_{5y}(w_1,w_2,0,0) - 12C_{4y}(w_1,w_2,0)c_{1y} + 24C_{3y}(w_1,w_2)c_{1y}^2$$

and

$$B = C_{2y}(0) - 2c_{1y}^2$$

Eq. (19) states that H_2 is the frequency response of a linear system whose output cumulant spectrum of order 3 is given by the right hand side. Using well known techniques of phase specrum estimation for linear systems [8] we obtain $h_2(k)$.

4 BLIND IDENTIFICATION OF FINITE EXTEND HAMMERSTEIN SERIES

Finite extent second order Hammerstein series is a special case of (1) with finite extent kernels:

$$\begin{aligned} y(n) &= \sum_{k=0}^{K_1} h_1(k)u(n-k) + \sum_{k=1}^{K_2} h_2(k)u^2(n-k), \\ & h_1(0) = 1 \end{aligned} \quad (20)$$

In a manner reminiscent of moving average identification, we proceed working with the time domain formulas (2)- (5). It turns out that cumulants of order 5 are needed only if $K_2 < K_1$.

Let us first assume that $K_1 = K_2 = K$. Then

$$c_{2y}(K) = \gamma_2 h_1(K) \quad (21)$$

$$\begin{aligned} c_{3y}(K,i) &= 2\gamma_2^2 h_1(i)h_2(K) + 2\gamma_2^2 h_1(K)h_2(i), \\ & 0 \le i \le K \end{aligned} \quad (22)$$

$$c_{4y}(K,K,i) = 8\gamma_2^3 h_1(i)h_2^2(K) + 16\gamma_2^2 h_1(K)h_2(K)h_2(i),$$

$$0 \le i \le K \quad (23)$$

From eqs. (21)- (23) we find

$$\gamma_2 = \frac{2c_{3y}(K,0)c_{2y}(K)}{c_{3y}(K,K)} \quad (24)$$

$$h_1(i) = \frac{2c_{3y}(K,i)}{c_{3y}(K,0)} - \frac{c_{4y}(K,K,i)}{c_{4y}(K,K,0)}, \qquad 0 \le i \le K \quad (25)$$

$$\gamma_2 h_2(i) = \frac{c_{4y}(K,K,i)}{2c_{3y}(K,K)} - \frac{c_{3y}(K,i)}{2c_{2y}(K)}, \qquad 0 \le i \le K \quad (26)$$

Suppose next that $K_2 > K_1$. Then

$$c_{3y}(K_2, i) = \begin{cases} 2\gamma_2^2 h_1(i) h_2(K_2), & \text{if } 0 \le i \le K_1 \\ 0, & \text{if } K_1 < i \le K_2 \end{cases} \quad (27)$$

$$c_{4y}(K_2, K_2, i) =$$

$$= \begin{cases} 8\gamma_2^3 h_1(i) h_2^2(K_2), & \text{if } 0 \le i \le K_1 \\ 0, & \text{if } K_1 < i \le K_2 \end{cases} \quad (28)$$

$$c_{4y}(K_2, K_1, i) =$$

$$= \begin{cases} 8\gamma_2^3 h_1(i) h_2(K_1) h_2(K_2) \\ +8\gamma_2^3 h_1(K_1) h_2(K_2) h_2(i), & \text{if } 0 \le i \le K_1 \\ 8\gamma_2^3 h_1(K_1) h_2(K_2) h_2(i), & \text{if } K_1 < i \le K_2 \end{cases} \quad (29)$$

Hence

$$\gamma_2 = \frac{2c_{3y}(K_2, 0) c_{3y}(K_2, K_1)}{c_{4y}(K_2, K_2, K_1)} \quad (30)$$

$$h_1(i) = \frac{c_{3y}(K_2, i)}{c_{3y}(K_2, 0)}, \qquad 0 \le i \le K_1 \quad (31)$$

$$\gamma_2 h_2(i) = \frac{c_{4y}(K_2, K_1, i)}{4c_{3y}(K_2, K_1)} - \frac{c_{3y}(K_2, i) c_{4y}(K_2, K_1, K_1)}{8c_{3y}^2(K_2, K_1)},$$

$$0 \le i \le K_2 \quad (32)$$

Likewise if $K_1 > K_2$, then

$$c_{2y}(K_1) = \gamma_2 h_1(K_1) \quad (33)$$

$$c_{3y}(K_1, i) = \begin{cases} 2\gamma_2^2 h_1(K_1) h_2(i), & \text{if } 0 \le i \le K_2 \\ 0, & \text{if } K_2 < i \le K_1 \end{cases} \quad (34)$$

$$c_{4y}(K_1, K_2, i) =$$

$$= \begin{cases} 8\gamma_2^3 h_1(K_1) h_2(K_2) h_2(i), & \text{if } 0 \le i \le K_2 \\ 0, & \text{if } K_2 < i \le K_1 \end{cases} \quad (35)$$

$$c_{5y}(K_2, K_2, K_2, i) =$$

$$= \begin{cases} 48\gamma_2^4 h_1(i) h_2^3(K_2) \\ +144\gamma_2^4 h_1(K_2) h_2^2(K_2) h_2(i), & \text{if } 0 \le i \le K_2 \\ 48\gamma_2^4 h_1(i) h_2^3(K_2), & \text{if } K_2 < i \le K_1 \end{cases} \quad (36)$$

From eqs. (33)- (36) we find

$$\gamma_2 = \frac{c_{5y}(K_2, K_2, K_2, 0) c_{2y}^2(K_1)}{3c_{4y}(K_1, K_2, K_2) c_{3y}(K_1, K_2)} \quad (37)$$

$$h_1(i) = \frac{c_{5y}(K_2, K_2, K_2, i)}{c_{5y}(K_2, K_2, K_2, 0)}$$

$$-\frac{3c_{5y}(K_2, K_2, K_2, K_2) c_{3y}(K_1, i)}{4c_{5y}(K_2, K_2, K_2, 0) c_{3y}(K_1, K_2)}, \quad 0 \le i \le K_1 \quad (38)$$

$$\gamma_2 h_2(i) = \frac{c_{3y}(K_1, i)}{2c_{2y}(K_1)}, \qquad 0 \le i \le K_2 \quad (39)$$

5 CONCLUSIONS

Blind identification of second order Hammerstein series has been considered. The Volterra kernels are calculated using output cumulants of order up to 5. Extensions to more general Volterra systems including higher order Hammerstein series are examined.

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