TWO NEW MODELS FOR DESCRIPTOR SYSTEMS: TIME AND FREQUENCY BEHAVIOR

Amel B. H. Adamou-Mitiche (amelmitiche@yahoo.fr)
Lahcène Mitiche (l_mitiche@yahoo.fr)
Centre Universitaire de Djelfa, Algeria.

***Abstract*— We present in this paper, a new algorithm to construct different types of lower degree approximates of a high-degree singular system based onto SVD-approach, that had the main advantage to preserve the key properties of the original system, such as stability, and give a quantization of approximation error. By the use of *Schur* method, model reduction is performed either on the proper and improper parts of a high-degree original system. A numerical example is provided to illustrate the time and the frequency behavior of approximates vis--vis original system.**

***Index Terms*— High-degree system, modeling, singular systems, descriptor systems, singular values, model reduction, *Weierstrass* canonical forms.**

I. Introduction

Since they gave a more complete class of dynamical models than the state variable systems (also called standard systems), singular systems (or descriptor systems [1]) are of great importance. Great number of notions and results for state space systems, like stability, have been successfully extended to descriptor systems and more recently, several works in the field of model reduction of descriptor systems have been done [2,3]. We suggest, in this paper, an extension of the SVD-based model reduction to the continuous-time, linear, c-stable singular systems: this tool permits to construct two stable low- order approximates by truncation of the smallest singular values; the first one is obtained by reducing only the proper part, while improper part is copied entirely, the second one is the result of reduction performed in proper and improper parts. In the next, preliminary results and a procedure for constructing reduced-order models are stated, followed by an illustrative example. Comments and concluding notes highlight the main idea of our work.

II. Main results

Consider a linear continuous-time, c-stable descriptor system (1)

$$\begin{array}{rcl} E\dot{x}(t) & = & Ax(t)+Bu(t)\,, \\ y(t) & = & Cx(t)+Du(t)\,, \end{array} \tag{1}$$

where $x(t) \in R^n$ is the state-vector, $u(t) \in R^m$ is the control-vector and $y(t) \in R^p$ is the output-vector, and $E, A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}$ are constant matrices with E singular. We assume that the pencil $\lambda E - A$ is regular. The proper procedure of any descriptor model reduction scheme is to replace the n-order system (1) by a reduced r-order system (2),

$$\begin{array}{rcl} E_r\dot{x}_r(t) & = & A_rx_r(t)+B_ru(t)\,, \\ y_r(t) & = & C_rx_r(t)+D_ru(t)\,, \end{array} \tag{2}$$

with $E_r, A_r \in R^{r\times r}$, $B_r \in R^{r\times m}$, $C_r \in R^{p\times r}$, and $r < n$.
The system (2) must keep the same key properties of the initial system (1), like regularity and stability, and the approximation error is wished to be small.

The Gramians

The proper (improper) gramians of the descriptor system (1) are solutions of the *projected generalized continuous-time (discrete-time) Lyapunov equations* [5].
The proper controllability gramian G_{pc} and proper observability gramian G_{po} are the unique symmetric, positive semidefinite solutions of the *projected generalized continuous-time Lyapunov equations*

$$EG_{pc}A^T + AG_{pc}E^T = -P_lBB^TP_l^T\,, \tag{3}$$

$$E^TG_{po}A + A^TG_{po}E = -P_r^TC^TCP_r\,, \tag{4}$$

where P_l and P_r are the *spectral projectors* onto the left and right deflating subspaces of the pencil $\lambda E - A$ corresponding to the finite eigenvalues. Furthermore, the improper controllability gramian G_{ic} and improper observability gramian G_{io} are the unique symmetric, positive semidefinite solutions of the *projected generalized discrete-time Lyapunov equations*

$$AG_{ic}A^T - EG_{ic}E^T = (I-P_l)BB^T(I-P_l)^T\,, \tag{5}$$

$$A^TG_{io}A - E^TG_{io}E = (I-P_r^T)C^TC(I-P_r)\,. \tag{6}$$

While transforming the pencil $\lambda E - A$ in *Weierstrass* canonical form, there exist two nonsingular matrices W and T such that

$$E = W\begin{pmatrix} I_{n_f} & 0 \\ 0 & N \end{pmatrix}T\,, \tag{7}$$

$$A = W\begin{pmatrix} J & 0 \\ 0 & I_{n_\infty} \end{pmatrix}T\,, \tag{8}$$

where I_n denotes the n-order identity matrix, N is nilpotent with index of nilpotency ν, and J is a Jordan matrix. This representation defines the decomposition of the system (1) into two deflating subspaces of dimensions n_f and n_∞ corresponding to the finite (proper) and infinite (improper) eigenvalues of the pencil $\lambda E - A$. In this case, the projection matrices will have the following expressions

$$P_r = T^{-1} \begin{pmatrix} I_{n_f} & 0 \\ 0 & 0 \end{pmatrix} T\,, \tag{9}$$

$$P_l = W \begin{pmatrix} I_{n_f} & 0 \\ 0 & 0 \end{pmatrix} W^{-1}\,. \tag{10}$$

The set of the proper and improper gramians forms the gramians of the descriptor system (1).

III. Descriptor model reduction

In order to reduce the descriptor system (1), we have to compute the full rank factors L and R of the proper observability and controllability gramians. Initially, the pencil is transformed in the GUPTRI (Generalized Upper Triangular) form [6], i.e.,

$$E = V \begin{bmatrix} E_{n_f} & E_u \\ 0 & E_{n_\infty} \end{bmatrix} U^T\,, \tag{11}$$

$$A = V \begin{bmatrix} A_{n_f} & A_u \\ 0 & A_{n_\infty} \end{bmatrix} U^T\,, \tag{12}$$

where E_{n_f} is upper triangular, nonsingular and E_{n_∞} is upper triangular with zeros on the diagonal, A_{n_f} is upper quasi-triangular and A_{n_∞} is upper triangular, nonsingular. Then W_∞ and T_∞ are computed as

$$W_\infty = V \begin{bmatrix} 0 \\ I_{n_\infty} \end{bmatrix}, T_\infty = U \begin{bmatrix} Y \\ I_{n_\infty} \end{bmatrix}, \tag{13}$$

where Y satisfies the *generalized Sylvester equations*

$$E_{n_f} Y - Z E_{n_\infty} = -E_u\,, \tag{14}$$
$$A_{n_f} Y - Z A_{n_\infty} = -A_u\,, \tag{15}$$

and the transformations associating with the Y and Z are given by (11) and (12).
V and U are defined by equations

$$V^T B = \begin{bmatrix} B_{n_f} \\ B_{n_\infty} \end{bmatrix}, \tag{16}$$

$$CU = \begin{bmatrix} C_{n_f} & C_{n_\infty} \end{bmatrix}. \tag{17}$$

The algorithm that we suggest gives out two types of state-space reduced systems. The first reduced model $M1$, given by $(E_{r1}, A_{r1}, B_{r1}, C_{r1}, D_{r1})$, results of reduction of only the proper part of the descriptor system, the improper part is copied, while the approximate model $M2$ obtained by the second approach, and given by $(E_{r2}, A_{r2}, B_{r2}, C_{r2}, D_{r2})$, resulting of the reduction carried out at once on the proper and improper parts.

The Algorithm:
The two reduced models $M1$ and $M2$ will be calculated by following step-by-step procedure.

Input: A realization (E, A, B, C, D) of the descriptor system (1) such that $\lambda E - A$ is regular and c-stable.

Step 1 Reduce E and A to the GUPTRI form (11) and (12).

Step 2 Compute the solutions Y and Z of the *generalized Sylvester equations* (14,15)

Step 3 Form and partition the matrices in (16,17), according to the partition of V and U induced by (11) and (12).

Step 4 Compute the *Cholesky* factors R_f and L_f of the solutions $X_c = R_f R_f^T$ and $X_o = L_f^T L_f$ of the *generalized Lyapunov equations*

$$E_{n_f} X_c A_{n_f}^T + A_{n_f} X_c E_{n_f}^T = -(B_{n_f} - Z B_{n_\infty})(B_{n_f} - Z B_{n_\infty})^T\,, \tag{18}$$

$$E_{n_f}^T X_o A_{n_f} + A_{n_f}^T X_o E_{n_f} = -C_{n_f}^T C_{n_f}\,. \tag{19}$$

Step 5 Test: If rank(R_f)¡n_f (rank(L_f)¡n_f), compute the full column (row) rank matrix R_1 (L_1) from the QR decompositions with column (row) pivoting

$$R_f^T = Q_R \begin{bmatrix} R_1^T \\ 0 \end{bmatrix}, \tag{20}$$

$$L_f = Q_L \begin{bmatrix} L_1 & 0 \end{bmatrix}. \tag{21}$$

Otherwise $R_1 = R_f$ ($L_1 = L_f$).

Step 6 Form the matrices

$$R = U \begin{bmatrix} R_1 \\ 0 \end{bmatrix}, \tag{22}$$

$$L = \begin{bmatrix} L_1 & -L_1 Z \end{bmatrix} V^T\,. \tag{23}$$

Step 7 Compute the matrices

$$W_\infty = V \begin{bmatrix} 0 \\ I_{n_\infty} \end{bmatrix}, \tag{24}$$

$$T_\infty = U \begin{bmatrix} Y \\ I_{n_\infty} \end{bmatrix}. \tag{25}$$

Step 8 Compute and partition for the singular values decomposition of the matrix LER,

$$LER = \begin{bmatrix} U_1 & U_0 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_0 \end{bmatrix} \begin{bmatrix} V_1 & V_0 \end{bmatrix}^T, \tag{26}$$

where $\Sigma_1 = [\sigma_1, \sigma_2, \cdots, \sigma_{l_f}]$ is the preserved part corresponding to the l_f largest proper singular values of the proper system.

Step 9 Construct the matrices

$$W_l = \begin{bmatrix} L^T U_1 \Sigma_1^{-\frac{1}{2}}, & W_\infty \end{bmatrix}, \tag{27}$$

$$T_l = \begin{bmatrix} R V_1 \Sigma_1^{-\frac{1}{2}}, & T_\infty \end{bmatrix}. \tag{28}$$

Step 10 Reduce the order of the improper subsystem $(E_{n_\infty}, A_{n_\infty}, B_{n_\infty}, C_{n_\infty})$, or equivalently, reduce the order of the regular discrete-time system, assuming, according to the GUPTRI decomposition, that A_{n_∞} is nonsingular,

$$\begin{array}{lcl} A_{n_\infty} z(k+1) & = & E_{n_\infty} z(k) + B_{n_\infty} \nu(k) \\ \psi(k) & = & C_{n_\infty} z(k)\,. \end{array} \tag{29}$$

The classical discrete-time *Schur* method [7] gives a reduced model $(E_{\infty r}, A_{\infty r}, B_{\infty r}, C_{\infty r})$ of the improper model, and the singular values of the system (29) are called the improper singular values of the system (1), and are given by $\Phi = [\phi_1, \phi_2, \cdots, \phi_{l_\infty}, 0, \cdots, 0]$.

Output: The approximates $M1$ and $M2$ are given by

$$[E_{r1}, A_{r1}, B_{r1}, C_{r1}, D_{r1}] = [W_l^T E T_l, W_l^T A T_l, W_l^T B, C T_l, D]\,, \tag{30}$$

$$[sE_{r2} - A_{r2}] = diag[sE_{fr} - A_{fr}, sE_{\infty r} - A_{\infty r}] \quad (31)$$

$$B_{r2} = [B_{fr}; B_{\infty r}],\; C_{r2} = [C_{fr}, C_{\infty r}],\; D_{r2} = D, \quad (32)$$

where the matrices $E_{fr}, A_{fr} \in R^{l_f \times l_f}$, $B_{fr} \in R^{l_f \times m}$, $C_{fr} \in R^{p \times l_f}$, are obtained by truncation of the l_f rows and columns of A_{r1}, l_f rows of B_{r1}, and l_f columns of C_{r1} respectively. The matrices $E_{\infty r} = I_{l_\infty}, A_{\infty r} \in R^{l_\infty \times l_\infty}$, $B_{\infty r} \in R^{l_\infty \times m}$, $C_{\infty r} \in R^{p \times l_\infty}$, are the state space matrices of the l_∞ reduced-order system coming from the order reduction on the improper part (Step 10). Note that the system $M1$ will have $(l_f + n_\infty)$-order, while $M2$ will have a smaller order, equal to $(l_f + l_\infty)$.
Remarks The H_∞-norm of the error system transfer function $\Delta H(s) = H(s) - H_r(s)$ verifies the following upper bound [4]

$$||H(s) - H_r(s)||_{H_\infty} = \sup_{\omega \in R} ||H(j\omega) - H_r(j\omega)||_2 \leq 2 \sum_{i=l_f+1}^{n_f} \sigma_i \,, \quad (33)$$

where $||.||_2$ denotes the spectral matrix norm.

IV. Simulation results

Consider a 120-order descriptor MIMO system (2 Inputs/3 Outputs) [8], where $n_f = 100$ and $n_\infty = 20$. The proposed algorithm [1] compute two reduced-order models $M1$ and $M2$, for $l_f = 1$, and $l_\infty = 1$. The two approximates $M1$ of order $r = 21$, singular, and $M2$, of order $r = 2$, regular are c-stable. In Fig. 1, we show how well the error estimate (33), for the model $M1$, is tight for each I/O combination, and the *Hankel* upper bound always guaranteed. The proper singular values is shown in Fig. 2, when only the proper sub-system is considered, and the improper singular values when improper sub-system is considered see Fig. 3. We traced also magnitude of approximation error of approach 2 in Fig. 8 (the plots are same for other Inputs/Outputs channels). For the sake of comparison, in addition to the frequency responses, we plot the time responses (step and impulse responses) of original descriptor and its two approximates $M1$ and $M2$ (see Fig. 4 and Fig. 4), where only one channel is considered ($1stI/1stO$); note that the time responses of the reduced-order models $M1$ and $M2$ come closer of the one of the full-order model. Similar time responses plots are obtained for the other $Input/Output$ channels.

[1]To solve the *generalized Sylvester and Lyapunov equations*, the package SLICOT [9] was used.

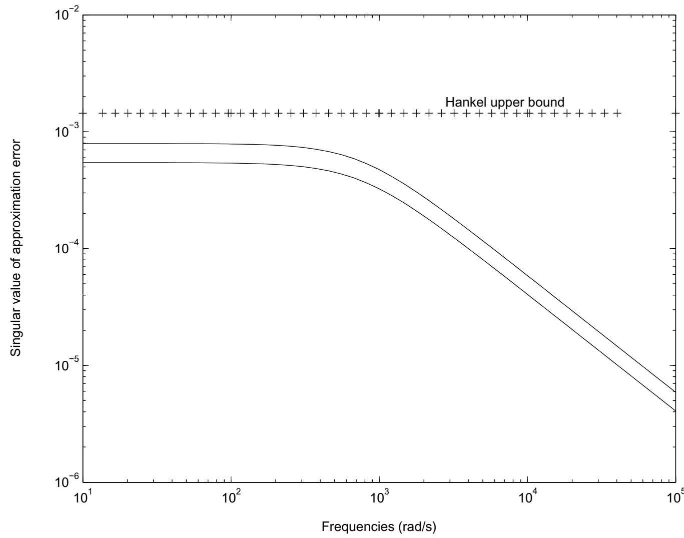

Fig. 1. Singular value of approximation error and upper bound for reduced-order model $M1$, $l_f = 1$, $n_f = 100$

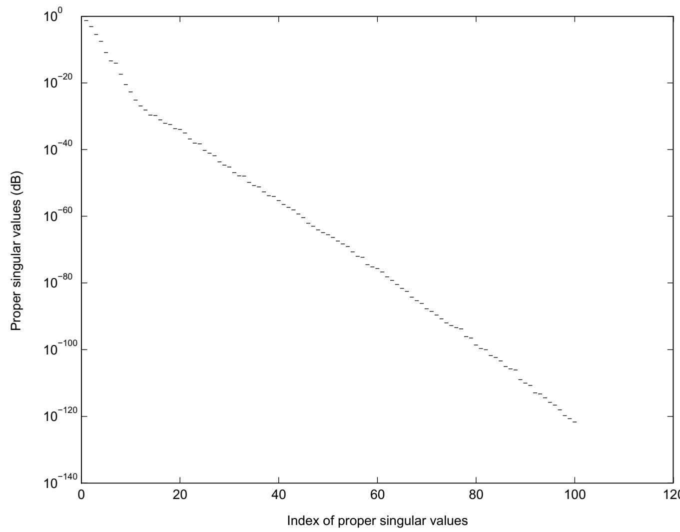

Fig. 2. Proper singular values distribution

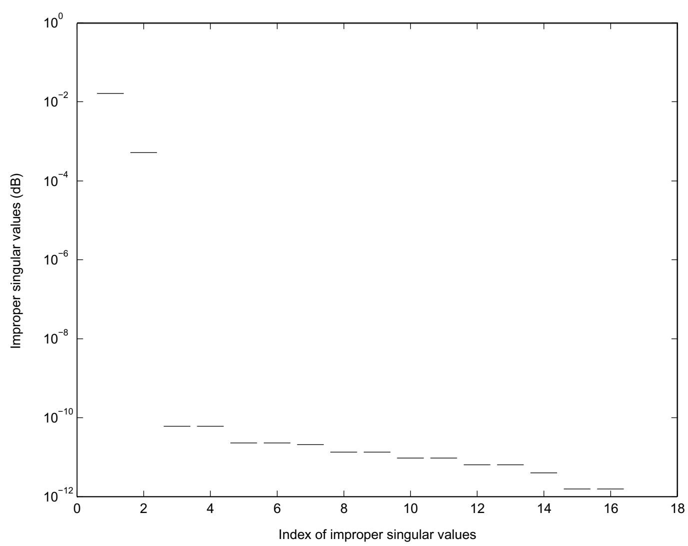

Fig. 3. Improper singular values distribution

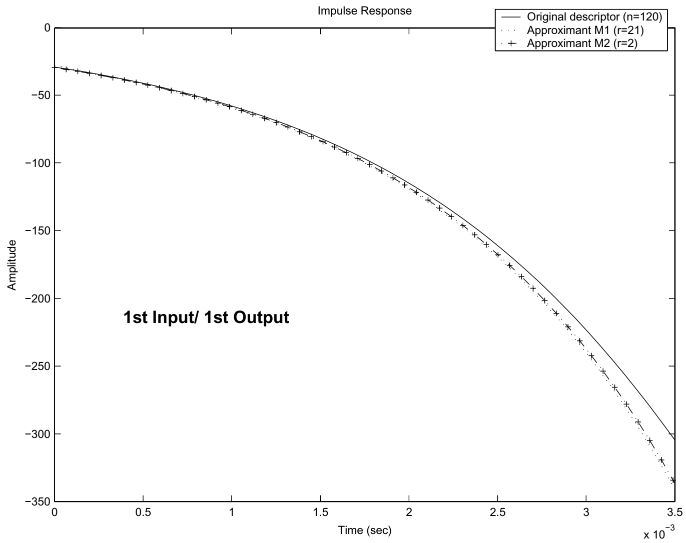

Fig. 4. Impulse responses of original and the two reduced-order models

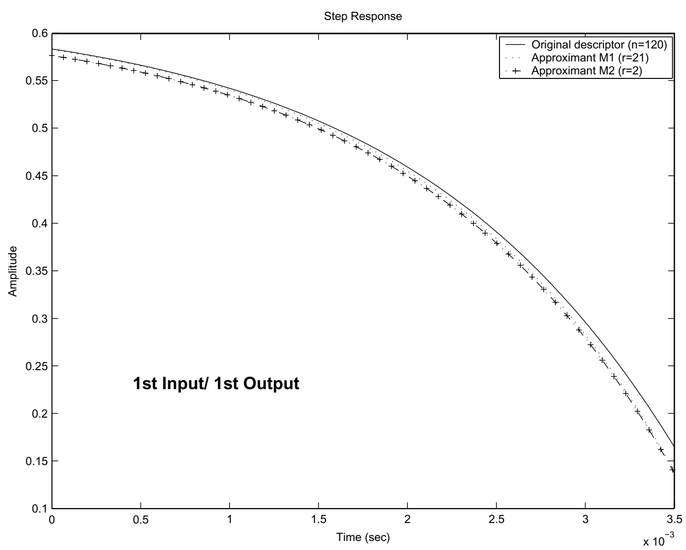

Fig. 5. Step responses of original and the two reduced-order models

V. Conclusion

In this paper, model reduction of descriptor systems is investigated. So, SVD-based model reduction techniques of descriptor systems are presented and new model reduction algorithm is proposed. To built it, at first we examined the model reduction algorithm reported in [5] and we extended it to either the proper and improper part of descriptor system by the use of the regular discrete-time *Schur*-based model reduction for the improper part of the descriptor system. As a result, two different stable approximates: in the first one, only the proper part (corresponding to the finite eigenvalues of the pencil $\lambda E - A$) is reduced, and the improper part (corresponding to the infinite eigenvalues of the pencil) is fully preserved, while in the second approach, the twice of parts are reduced offering an interesting characterization of the resulting low-order model since it is described by a regular state-space realization. It is worth noting that they are very close to the original system behavior.

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