AN ALGEBRAIC DERIVATIVE-BASED APPROACH FOR THE ZERO-CROSSINGS ESTIMATION

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ABSTRACT

A new approach to the design of a zero-crossings estimation algorithm is proposed. The approach uses elementary differential algebraic operations in the frequency domain for accurate derivative estimation. Such estimates are composed of iterated integrals of noisy observed signal. A detector-signal, which is exactly equal to zero when there is no intersection between the observed signal and the real axis and is greater than zero when a zero-crossing occurs, is obtained. To justify the theoretical analysis and to investigate the performances of the developed method, simulated experiments are performed.

1. INTRODUCTION

Given a piecewise continuous signal, the purpose of this paper is to detect its zero-crossings and estimate their locations. The problem is especially challenging for applications requiring on-line detection: the main difficulties stem from corrupting noises which blur the zero-crossings, and the combined need of fast calculations for real-time implementation and of reliable detection. A large amount of literature is devoted to these questions in fields such as signal processing [1] [2] [3], industrial electronics [4] [5], fluid mechanics, speech processing [6] [7], biomedical engineering, optics, neurophysiology, structural dynamics, communications, image processing, to name just a few. The use of the zero-crossing detection and calculation of the number of cycles that occur in a predetermined time interval is a simple and well-known methodology, especially in signal processing [1]. The importance of the zero-crossings is well documented in [8] where the fruitful connection between zero-crossing counts and time-invariant linear filtering is investigated. This connection leads to interesting properties for the fast analysis of random signals. Moreover, the zero-crossing counts in random signals and their filtered versions essentially constitute a domain which is equivalent to the spectral domain. In this paper a new method to estimate the zero-crossings of a generic signal is proposed. The main idea is to create a function with no defined first order derivative and the related discontinuity in the second order derivative in the zero-crossing points. These discontinuities can be effectively detected by using reliable techniques of parameters estimation [9]. The paper is organized as follows. Section 2 describes the proposed approach for zero-crossings estimation; in Section 3 the main aspects of the algebraic identification method and its application to the derivatives estimation are discussed; Section 4 contains some considerations about the implementation and the robustness of the proposed approach; Section 5 illustrates the performances obtained by simulated experiments and it is followed by final conclusions.

2. PROPOSED APPROACH

The logic scheme of the proposed zero-crossings detector is depicted in Fig. 1. The piecewise smooth signal $y(t)$, which has a zero-crossing at $t = t_s$, is the input signal. Based on the observation $y_n(t) = y(t) + n(t)$, where $n(t)$ is an additive noise corruption, it is desirable to estimate the zero-crossing location. Clearly both $y(t)$ and $|y(t)|$ have the same zero-crossing, moreover in t_s the first order derivative of $|y(t)|$ is not defined as shown in Fig. 2.

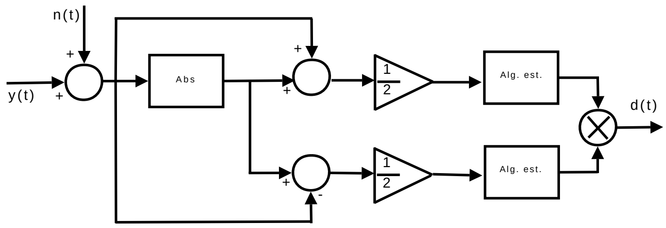

Figure 1: Block diagram of the zero-crossing system detector.

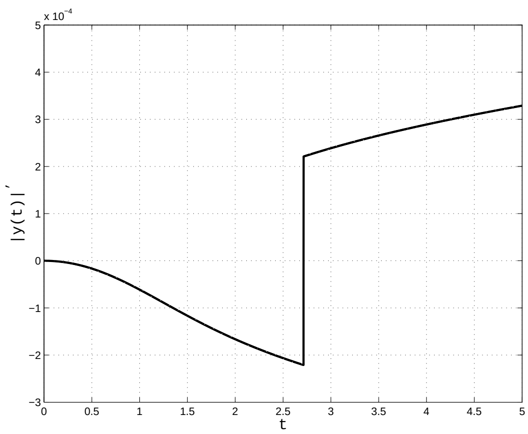

Figure 2: First order derivative of $|y(t)|$.

Even though the first order derivative of $|y(t)|$ is not defined in $t = t_s$, from a numerical point of view, the second order derivative of the function $|y(t)|$ presents a positive peak around the zero-crossing t_s (see Fig. 3).

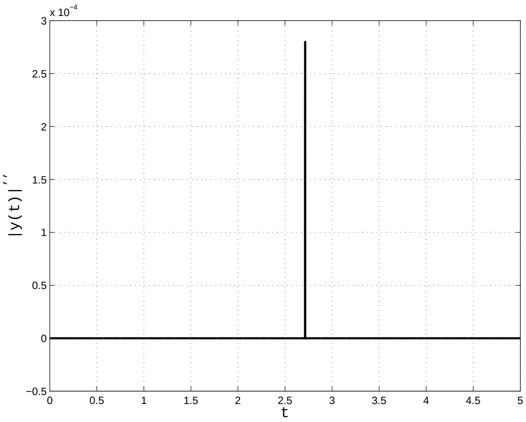

Figure 3: Second order derivative of $|y(t)|$.

The proposed scheme is capable to obtain a detector-signal, namely $d(t)$, which is exactly equal to zero when there is no intersection with real axis and is greater than zero when there is a zero-crossing. The idea is to compute the second order derivative of the signals

$$y_1(t) = \frac{|y_n(t)| + y_n(t)}{2}, \tag{1}$$

and

$$y_2(t) = \frac{|y_n(t)| - y_n(t)}{2}, \tag{2}$$

denoted as $d_1(t)$ and $d_2(t)$ respectively. Let us ignore the noise for a moment and suppose, without loss of generality, that $y(t_s^-) < 0$, $y(t_s^+) > 0$; in this case the signals $y_1(t)$ and $y_2(t)$ become respectively

$$y_1(t) = \begin{cases} 0, & t \leq t_s, \\ y(t), & t > t_s, \end{cases} \quad y_2(t) = \begin{cases} -y(t), & t < t_s, \\ 0, & t \geq t_s. \end{cases}$$

Therefore signals $d_1(t)$ and $d_2(t)$ assume the following expressions

$$d_1(t) = \begin{cases} 0, & t \in [0, t_s), \\ \frac{d^2 y(t)}{dt^2}, & t > t_s, \end{cases} \quad d_2(t) = \begin{cases} -\frac{d^2 y(t)}{dt^2}, & t \in [0, t_s), \\ 0, & t > t_s, \end{cases}$$

and the detector signal defined as

$$d(t) = d_1(t) d_2(t) \tag{3}$$

has a peak in the instant $t = t_s$ and is zero for $t \neq t_s$.

3. NUMERICAL DIFFERENTIATION: A SHORT SUMMARY

Derivative estimation of noisy time signals is a longstanding difficult ill-posed problem. Here the numerical differentiation problem is dealt with the algebraic parameter estimation initially presented in [9], [10]. Given a smooth signal, a key point of this approach is to consider its second order derivative in $t = \tau$, for each fixed $\tau \geq 0$, as a single parameter to be estimated from a noisy observation of the signal. A pointwise derivative estimation therefore follows by varying τ. The main aspect of the algebraic parameter estimation is to operate in the operational calculus domain [11], [12], [13], where an extensive use of differential elimination and a series of algebraic manipulations yield, back in the time domain, an explicit expression for the estimate of the second order derivative in $t = \tau$ as an integral operator of the noisy observation within a short time interval $[\tau, \tau + T]$. Let us consider the estimation of $x^{(2)}(t)$, the second order derivative of a smooth signal $x(t)$ defined on an interval $\mathscr{I} \in \mathbb{R}_+ = [0, +\infty)$. Assume that $x(t)$ is analytic on $\mathscr{I}$ so that it is possible to consider the approximation of the signal $x(t)$ with a second order polynomial

$$x(t) \approx a_0 + a_1 t + a_2 t^2. \tag{4}$$

By considering classic operational calculus operators, $x(t)$ can be rewritten as

$$X(s) = \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{2a_2}{s^3}. \tag{5}$$

Multiply both sides of (5) by s^3, the following expression holds:

$$s^3 X(s) = a_0 s^2 + a_1 s + 2a_2. \tag{6}$$

Taking the derivative of both sides of (6) with respect to s, one and two times respectively, expressions (7) and (8) are obtained

$$3s^2 X(s) + s^3 \frac{dX(s)}{ds} = 2a_0 s + a_1, \tag{7}$$

$$6sX(s) + 6s^2 \frac{dX(s)}{ds} + s^3 \frac{d^2 X(s)}{ds^2} = 2a_0. \tag{8}$$

The coefficients a_0, a_1 and a_2 are obtained via the triangular system of Eqs. (6)-(8). Derivative operations in time domain are avoided by multiplying both sides of Eqs. (6)-(8) by s^{-n}, $n \geq 3$. To express such equations back in the time domain, let us recall that for a given signal $u(t)$, and a positive integer α, the time domain analog of $V(s) = \frac{1}{s^\alpha} \frac{d^\beta}{ds^\beta} U(s)$ is the iterated integral, of order α, of $(-1)^\beta t^\beta u(t)$. Using the Cauchy formula, this leads to a single integral

$$v(t) = \frac{1}{(\alpha - 1)!} \int_0^t (t - \xi)^{\alpha - 1} (-1)^\beta \xi^\beta u(\xi) d\xi. \tag{9}$$

The corresponding iterated time integrals are low pass filters which attenuate the corrupting noises. A quite short time window is sufficient for obtaining accurate values of a_0, a_1 and a_2. The extension to polynomial functions of higher degree is straightforward. For the second order derivative estimation, assume that the second order polynomial be the truncated Taylor expansion around a given time instant τ, and apply the previous computations to obtain the coefficient a_2. Resetting and utilizing sliding time window, an estimation of the derivative at any sampled time instant can be performed. Note also that unstructured noises, which can be considered as high frequency perturbations, are attenuated by the iterated integrals, which are simple examples of low-pass filters. Of course it is possible to estimate all the coefficients a_i, $i = 0, 1, 2$, simultaneously. However, not only the coefficients a_i, $i = 0, 1$ are not necessary for the estimation of

$x^{(2)}(t)$, but also simultaneous estimation is more sensitive to noise and numerical computation errors. In the proposed approach, all the terms $a_i,\ i=0,1$ are consequently considered as undesired perturbations to annihilate. To this aim it suffices to find a differential operator, i.e.

$$\Pi = \sum_{\text{finite}} \rho_l(s)\frac{d^l}{ds^l}, \quad \rho_l(s) \in \mathbb{C}(s), \tag{10}$$

satisfying

$$\Pi X(s) = \rho(s)a_2, \tag{11}$$

for some rational function $\rho(s) \in \mathbb{C}(s)$. Such a linear differential operator, called an *annihilator* for a_2, obviously exists and is not unique. It is also clear that to each annihilator Π, there is a unique $\rho(s) \in \mathbb{C}(s)$ such that (11) holds. In this case Π and $\rho(s)$ are said *associated*.

Lemma 1 *The linear differential operator* $\Pi = \Pi_1\Pi_0$, *where* $\Pi_k = \frac{d}{ds}s^{k+1},\ k=0,1$, *is an annihilator of* a_2 *associated to the function* $\rho(s) = \frac{4}{s^2}$.

Proof. Applying the operator Π_0 to (5) it is easy to obtain that

$$\Pi_0 X(s) = -\frac{a_1}{s^2} - 4\frac{a_2}{s^3}. \tag{12}$$

Finally

$$\Pi_1\Pi_0 Y(s) = \frac{d}{ds}\left[s^2\Pi_0 Y(s)\right] = \frac{4a_2}{s^2}, \tag{13}$$

from which the proof follows. ■

According to Lemma 1 it is easy to verify that the following relation holds:

$$2sX(s) + 4s^2\frac{dX(s)}{ds} + s^3\frac{d^2X(s)}{ds^2} = 4\frac{a_2}{s^2}. \tag{14}$$

To eliminate the time derivations, which can amplify the noise effects on the signal $x(t)$, Eq. (14) is divided by s^4 thus introducing at least an integral effect on each term which contains the signal $x(t)$:

$$2\frac{X(s)}{s^3} + 4\frac{\frac{dX(s)}{ds}}{s^2} + \frac{\frac{d^2X(s)}{ds^2}}{s} = 4\frac{a_2}{s^6}. \tag{15}$$

Therefore, Eq. (15) can be expressed in time domain as:

$$a_2 = 30\frac{\int_0^T\left[(T-\xi)^2 - 4(T-\xi)\xi + \xi^2\right]x(\xi)}{T^5}d\xi, \tag{16}$$

where T denotes the estimation time.

In the same way, considering the terms $a_i,\ i=0,2$ as undesired perturbations to annihilate, the following lemma gives an annihilator of a_1.

Lemma 2 *The linear differential operator* $\Pi = \Pi_1\Pi_0$, *where* $\Pi_k = \frac{d}{ds}s^{2k+1},\ k=0,1$, *is an annihilator of* a_1 *associated to the function* $\rho(s) = -1$.

By considering Lemma 2 it follows that

$$a_1 = -\frac{24\int_0^T\left[\frac{3}{2}(T-\xi)^2 - 5(T-\xi)\xi + \xi^2\right]x(\xi)d\xi}{T^4}. \tag{17}$$

Since the noisy signal $x_n(t) = x(t) + n(t)$ is available then, after a change of variable to reduce the estimation interval from $[0,T]$ to $[0,1]$, the estimations of a_1 and a_2, namely $\hat{a}_1$ and $\hat{a}_2$ respectively, are computed as

$$\hat{a}_1 = -12\frac{\int_0^1\left[15\xi^2 - 16\xi + 3\right]x_n(T\xi)}{T}d\xi, \tag{18}$$

$$\hat{a}_2 = 30\frac{\int_0^1\left[6\xi^2 - 6\xi + 1\right]x_n(T\xi)}{T^2}d\xi. \tag{19}$$

4. DETECTOR SYNTHESIS AND IMPLEMENTATION

To implement the detector signal $d(t)$, a moving window of length T is used. The samples of the current window are then used to compute the value of d_1 and d_2 referred to its end-point. In this section it is shown that, when a peak is detected in $d(t)$, the corresponding t being the mid-point of the analyzing window is declared to be the zero-crossing location. Let us consider, for example, the signal $y_1(t)$. The same approach can be used for the signal $y_2(t)$.

According to the previous assumptions, $y_1(t)$ is equal to zero from 0 to t_s, if t_s is the position of the zero-crossing. Suppose also that in the interval $[t_s, T]$, with T sufficiently small, such a signal can be approximated with a second order polynomial which, obviously, must have the same zero-crossing in t_s

$$y_1(t) \approx a(t-t_s)^2 + b(t-t_s). \tag{20}$$

Note that, by Eq. (4), $a_2 = a$ and $a_1 = b - 2at_s$.

Let $op[y_1(t)]$ be the operator consisting in the computation of a_2 for the signal $y_1(t)$, i.e.

$$op[y_1(t)] = \frac{(T-t_s)^2\left[15bt_s^2 + a(T-t_s)(T^2+3Tt_s+6t_s^2)\right]}{T^5}. \tag{21}$$

Eq. (21) represents the value of the operator in $t=T$ as function of t_s. To find the position of the maximum of $op[y_1(t)]$, corresponding to the peak which is desirable to detect, the stationary points of $op[y_1(t)]$ are needed. Such points can be computed as solutions of the following equation

$$\frac{\partial op[y_1(t)]}{\partial t_s} = 0 \rightarrow t_s(T-t_s)\left[(T-t_s)t_s - \frac{b}{a}(T-2t_s)\right] = 0. \tag{22}$$

Therefore two stationary points are in $t=t_s$ and $t=T$. The others are solutions of

$$(T-t_s)t_s + K(T-2t_s) = 0, \tag{23}$$

where $K = -\frac{b}{a}$. As the value of K increases, which means that a decreases, one roots tends to $t = \frac{T}{2}$ while the other one is outside the interval $[0,T]$, as it can be easily observed by the root-locus in Fig. 4 [14].

Since

$$\frac{\partial^2 op[y_1(t)]}{\partial t_s^2} = -\frac{30}{T^5}\left[2at_s(T^2-3Tt_s+2t_s^2) - b(T^2-6Tt_s+6t_s^2)\right], \tag{24}$$

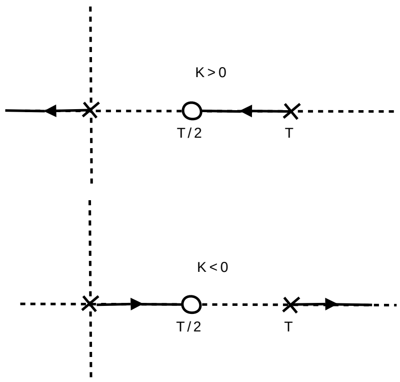

Figure 4: Roots-locus of (23).

then

$$\lim_{t_s\to 0}\frac{\partial^2 op[y_1(t)]}{\partial t_s^2}=\frac{30b}{T^3},$$
$$\lim_{t_s\to T}\frac{\partial^2 op[y_1(t)]}{\partial t_s^2}=\frac{30b}{T^3},$$
$$\lim_{t_s\to \frac{T}{2}}\frac{\partial^2 op[y_1(t)]}{\partial t_s^2}=-\frac{15b}{T^3}.$$

and the function $op[y_1(t)]$ presents a positive peak in $t=T$ of amplitude

$$A_{peak}=\lim_{t_s\to\frac{T}{2}} op(\rho)=\frac{a}{2}+\frac{15b}{16T} \tag{25}$$

when the function has a zero-crossing for $t=\frac{T}{2}$, i.e. there is a delay in the identification of the zero-crossing of about $\frac{T}{2}$ seconds.

Remark 1 *The analysis based on a local quadratic approximation of the signal indicates the possibility of a bias (i.e. $t_s \neq T/2$) if the quadratic coefficient is not small enough respect to the linear one. This drawback can be overcome by an appropriate choice of the window length, i.e. T should be sufficiently small to have a linear approximation of the signal.*

In the case of noisy signals, as it will be shown by numerical experiments in the next section, the method gives satisfactory results, i.e. the zero-crossing is effectively detected with a delay of $T/2$. A demonstration approach of this behavior is reported below. Suppose that

- $a \ll b$, i.e. the window length T is opportunely chosen in order to have an acceptable linear approximation of the signal;
- $Prob\{y_1(t)\neq 0,\ t<t_s\}<\varepsilon$.
  With such hypothesis

$$y_1(t)=\begin{cases} b(t-t_s)+n(t), & t\geq t_s,\\ 0, & t<t_s.\end{cases} \tag{26}$$

To compute $op[n(t)]$ starting from samples of white noise $\hat{n}$, let us consider the continuous signal

$$n(t)=\sum_{k=0}^{n-1}\hat{n}(kT_c)e^{-\frac{(t-kT_c)^2}{2\sigma^2}}, \tag{27}$$

with $T_c = T/(n-1)$ and $\sigma\to 0$.

In this case by using the above approach

$$\frac{\partial}{\partial t_s}op[n(t)]= \frac{30}{T^5}\lim_{\sigma\to 0}\frac{\partial}{\partial t_s}\int_{t_s}^{T}\left[(T-\xi)^2-4(T-\xi)\xi+\xi^2\right]n(\xi)d\xi= \tag{28}$$

$$-\frac{30}{T^5}\left[(T-t_s)^2-4(T-t_s)t_s+t_s^2\right]\hat{n}(t_s). \tag{29}$$

Therefore the stationary points of $op[y_1(t)]$ are the solutions of the following equation

$$t_s(T-t_s)(T-2t_s)+K\left(t_s-\frac{3+\sqrt{3}}{6}T\right)\left(t_s-\frac{3-\sqrt{3}}{6}T\right)=0, \tag{30}$$

with $K=-\frac{\hat{n}(t_s)}{b}$.

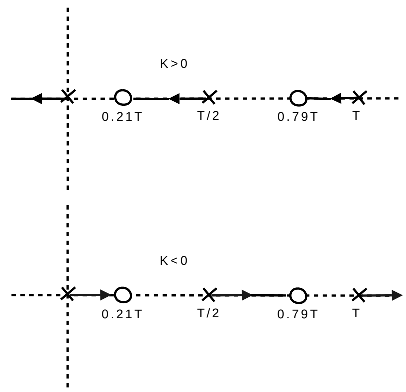

Figure 5: Roots-locus of (30).

Fig. 5 shows the root-locus of Eq. (30) varying K. As it can be observed, for small values of K, i.e. $\hat{n}\ll b$, the zero-crossing remains localized in $\frac{T}{2}$. Moreover for reasonable SNR, the error in the location of the zero-crossing is approximatively bounded by $\frac{T}{2\sqrt{3}}$.

5. NUMERICAL RESULTS

This section includes some numerical results that highlight and point out the advantages and the strengths of the proposed method, in particular it will be devoted to simulated experiments on several signals having a zero-crossing in the interval $[0,4]$. For each signal one hundred experiments have been performed adding a zero-mean gaussian white-noise. In all the experiments the SNR, measured in decibels as the logarithm of the average power of the signal's samples and the noise's samples, over the time of the experiment, is equal to 40dB; the sampling time is $T_s = 4\times 10^{-4}$. The goodness of the proposed method will be measured in terms of the mean value and the variance of the error $\hat{e}$ between the true zero-crossing and the estimated one. The proposed method, namely (**FCJ**), is compared with a zero-crossing detection method by interpolation [1], [15], [16], namely (**ZCI**). The ZCI implementation identifies two points of the signal: the first just before the positive going zero-crossing and the second just after the same zero-crossing. The hypothesis is that the shape of the signal is very close the straight line near the zero-crossing. The true zero-crossing is then computed by linear interpolation between these two points. Table 1 resumes the results obtained in the performed experiments.

Test function	FCJ / ZCI	
	$\mu(\hat{e})$	$\sigma(\hat{e})$
$\sin(t\pi/3+\pi/7)$	-2.24×10^{-4}	1.62×10^{-5}
	9.70×10^{-3}	1.16×10^{-5}
$5-\sqrt{t^3+5}$	1.10×10^{-3}	2.37×10^{-5}
	1.43×10^{-2}	1.76×10^{-5}
$1-t+\sin(3t)$	1.49×10^{-4}	5.50×10^{-6}
	5.90×10^{-3}	6.77×10^{-6}

Table 1: Mean and variance of the index $\hat{e}$ over 100 tests.

A severe test for the zero-crossing methods is when the signal is tangent to the real axis. In this case noise affecting the signal could create false zero-crossings. For example the signal $y(t)=\sin(2t)\cos(t)$ is tangent to the real axis in $t=\pi/2$ and has a zero-crossing in $t=\pi$. Interpolation based methods fail to estimate zero-crossings while the proposed one is able to correctly detect zero-crossing as it is shown in Fig. 6.

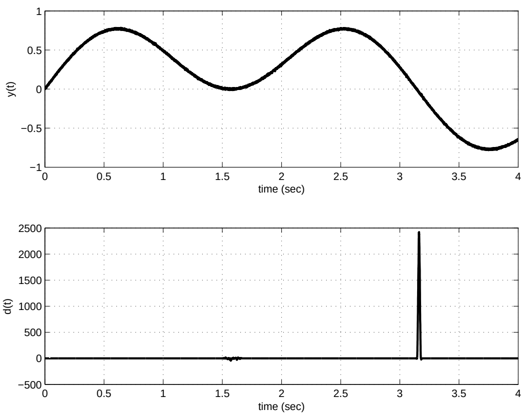

Figure 6: Test function $y(t)=\sin(2t)\cos(t)$ and the detector signal.

6. CONCLUSIONS

A method for estimating the zero-crossing instants of a sampled signal using algebraic derivative approach in the frequency domain was presented. This method provides a detector signal which is equal to zero when the signal has no intersection with real axis and has a peak when a zero-crossing occurs. The main idea was to numerically compute the second order derivative of two signals, built on the available one, which have a discontinuity in the zero-crossing instant. According to a local polynomial model of the signal, the second order derivative was estimated by the iterated integrals of the signal, which behave as low pass filter and mitigate the noise effects. The analysis of the estimator in terms of error on the zero-crossing was discussed. Numerical comparisons with other techniques show a quite satisfactory performance of the proposed method. It seems to be a promising technique for the detection of zero-crossing rates and locations since it is also able to discriminate false zero-crossings, for example in case of signal tangent to the real axis, and true ones.

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