BLOCK DISCRETE-TIME SCHWARZ FORM OF MULTIVARIABLE RATIONAL INTERPOLANT AND POSITIVITY BY LINEAR MATRIX INEQUALITY

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ABSTRACT

The state-space realization of the multivariable rational interpolant with bounded McMillan degree is given by the block discrete-time Schwarz form. A characterization of the positive realness of the block discrete-time Schwarz form is given by a linear matrix inequality.

1. INTRODUCTION

Given covariance matrices

$$\begin{bmatrix} R_0 & R_1 & \cdots & R_n \end{bmatrix},$$

consider the class of infinite extensions

$$R_{n+1}, R_{n+2}, R_{n+3}, \ldots$$

of the first $n+1$ covariance matrices such that

$$f(z) := \frac{1}{2}R_0 + R_1 z^{-1} + R_2 z^{-2} + \cdots,$$

is positive real. This is an well-known covariance extension problem [2, 10].

For applications to the spectral estimation, it is common that the spectral density is a rational model [11], and that less complexity of the rational model is required for some applications [2]. Therefore, it is desirable to incorporate a degree constraint on a parameterization of the positive rational extensions of the covariance sequence. If the positive realness constraint is removed, a parameterization of all solutions to the scalar interpolation problem with bounded degree is given in [2, 5]. The generalization of [2, 5] to the multivariable case is given in this paper, where the coprime factorizations by matrix orthogonal polynomials [1, 12] describe the parameterization of the interpolants with bounded McMillan degree.

In this paper, we show the state-space realization of the parameterization of the multivariable interpolants with bounded McMillan degree by the block discrete-time Schwarz form [7, 4]. The state-space realization is the generalization of the result of the scalar case [5]. We also present a characterization of the positive realness of this parameterization by a linear matrix inequality.

Notations

$\mathbb{R}$ and $\mathbb{C}$ denote real numbers and complex numbers. Denote by $\mathbb{R}^{j\times k}$ $j\times k$ real matrices. A^* denotes the transpose of matrix A. I denotes $m\times m$ identity matrix, and 0 denotes $m\times m$ zero matrix. We use the notations $A>0$ and $A\geq 0$ to denote that the matrix A is Hermitian positive definite and Hermitian positive semidefinite. The matrix square root $A^{\frac{1}{2}}$ of the Hermitian positive definite matrix A is defined by $A = A^{\frac{1}{2}}A^{\frac{1}{2}}$. Let us define $f(z)^* := \overline{f(\bar{z}^{-1})^T}$. The function $f(z)$ is called positive real if it is analytic in the the outside of the closed unit disc, and it satisfies

$$f(z) + f(z)^* \geq 0. \tag{1}$$

on the unit circle.

The state-space realization of a transfer function $G(z)$ with m inputs, m outputs and n states is denoted by

$$G(z) = C(zI-A)^{-1}B + D.$$

The McMillan degree of the transfer function $G(z)$ is defined by the size of the matrix A. We also use a notation

$$G(z) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

to denote $G(z)$. We note useful identities

$$G(z) = \left[\begin{array}{c|c} A-BD^{-1}C & -BD^{-1} \\ \hline D^{-1}C & D^{-1} \end{array}\right]$$

$$G_1(z)G_2(z) = \left[\begin{array}{cc|c} A_1 & B_1C_2 & B_1D_2 \\ 0 & A_2 & B_2 \\ \hline C_1 & D_1C_2 & D_1D_2 \end{array}\right].$$

2. PRELIMINARIES

We review the Whittle-Wiggins-Robinson algorithm (WWRA) [9], the theory of the matrix orthogonal polynomials [1, 12], some results of the block discrete-time Schwarz form [7, 4]. We show a parameterization of the multivariable rational interpolants with bounded McMillan degree.

2.1 WWRA and Matrix Orthogonal Polynomials

Assume that the Toeplitz matrix

$$\Gamma_{n+1} = \begin{bmatrix} R_0 & R_1 & \cdots & R_n \\ R_1^* & \ddots & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ R_n^* & \cdots & \cdots & R_0 \end{bmatrix}$$

is positive definite, and also assume that $R_0 = I$. Consider the upper Cholesky factorization of the Toeplitz matrix Γ_{n+1}

$$\Gamma_{n+1} = U_{n+1}\Sigma_{n+1}U_{n+1}^*,$$

where

$$\Sigma_{n+1} := \begin{bmatrix} Q_n & 0 & \cdots & 0 \\ 0 & Q_{n-1} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & Q_0 \end{bmatrix}$$

and $Q_0 = I$ since $R_0 = I$. We denote the inverse of U_{n+1} by

$$U_{n+1}^{-1} = \begin{bmatrix} I & A_{n,1} & \cdots & A_{n,n} \\ 0 & I & \cdots & A_{n-1,n-1} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix}.$$

Similarly, let us consider the lower Cholesky factorization of the Toeplitz matrix Γ_{n+1}

$$\Gamma_{n+1} = V_{n+1}\Lambda_{n+1}V_{n+1}^*,$$

where

$$\Lambda_{n+1} = \begin{bmatrix} S_0 & 0 & \cdots & 0 \\ 0 & S_1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & S_n \end{bmatrix}$$

and $S_0 = I$. We denote the inverse of V_{n+1} by

$$V_{n+1}^{-1} = \begin{bmatrix} I & 0 & \cdots & 0 \\ B_{n,1} & I & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ B_{n,n} & B_{n-1,n-1} & \cdots & I \end{bmatrix}.$$

Then, it is well-known that the WWRA gives the solution to the Yule-Walker (YW) equation in a recursive way [9]. The solution to the YW equation

$$\begin{bmatrix} I & A_{n+1,1} & \cdots & A_{n+1,n} & A_{n+1,n+1} \\ B_{n+1,n+1} & B_{n+1,n} & \cdots & B_{n+1,1} & I \end{bmatrix}\Gamma_{n+2}$$
$$= \begin{bmatrix} Q_{n+1} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & S_{n+1} \end{bmatrix}$$

is given by

$$\begin{bmatrix} I & A_{n+1,1} & \cdots & A_{n+1,n} & A_{n+1,n+1} \\ B_{n+1,n+1} & B_{n+1,n} & \cdots & B_{n+1,1} & I \end{bmatrix}$$
$$= \begin{bmatrix} I & -P_nS_n^{-1} \\ -P_n^*Q_n^{-1} & I \end{bmatrix}\begin{bmatrix} I & A_{n,1} & \cdots & A_{n,n} & 0 \\ 0 & B_{n,n} & \cdots & B_{n,1} & I \end{bmatrix}$$
$$Q_{n+1} = Q_n - P_nS_n^{-1}P_n^*$$
$$S_{n+1} = Q_n - P_n^*Q_n^{-1}P_n$$
$$P_n = R_{n+1} + A_{n,1}R_n + \cdots + A_{n,n}R_1.$$

The initial values for the recursion are the following

$$\begin{aligned} A_{1,1} &= -R_1 \\ B_{1,1} &= -R_1^* \\ Q_1 &= I + A_{1,1}R_1^* \\ S_1 &= I + B_{1,1}R_1. \end{aligned}$$

The WWRA is equivalent to the theory of the matrix orthogonal polynomials [1, 12]. The left matrix orthogonal polynomials of the first kind are given by

$$\begin{bmatrix} A_n(z) \\ A_{n-1}(z) \\ \vdots \\ I \end{bmatrix} = \begin{bmatrix} I & A_{n,1} & \cdots & A_{n,n} \\ 0 & I & \cdots & A_{n-1,n-1} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix}\begin{bmatrix} z^nI \\ z^{n-1}I \\ \vdots \\ I \end{bmatrix},$$

and the right matrix orthogonal polynomials of the first kind are given by

$$\begin{bmatrix} I & B_1(z) & \cdots & B_n(z) \end{bmatrix}$$
$$= \begin{bmatrix} I & zI & \cdots & z^nI \end{bmatrix}\begin{bmatrix} I & B_{1,1}^* & \cdots & B_{n,n}^* \\ 0 & I & \cdots & B_{n,n-1}^* \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix}.$$

Let us define

$$\begin{aligned} \bar{\Gamma}_{n+1} &:= \frac{1}{2}(M_{n+1}^{-1} + M_{n+1}^{-*}) \\ &= \begin{bmatrix} I & \bar{R}_1 & \cdots & \bar{R}_n \\ \bar{R}_1^* & I & \cdots & \vdots \\ \vdots & \vdots & & \vdots \\ \bar{R}_n^* & \bar{R}_{n-1}^* & \cdots & I \end{bmatrix} \\ M_{n+1} &:= \begin{bmatrix} I & 2R_1 & \cdots & 2R_n \\ 0 & I & \cdots & 2R_{n-1} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix}. \end{aligned}$$

Then, the left matrix orthogonal polynomials of the second kind are given by

$$\begin{bmatrix} C_n(z) \\ C_{n-1}(z) \\ \vdots \\ I \end{bmatrix} = \begin{bmatrix} I & C_{n,1} & \cdots & C_{n,n} \\ 0 & I & \cdots & C_{n-1,n-1} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix}\begin{bmatrix} z^nI \\ z^{n-1}I \\ \vdots \\ I \end{bmatrix},$$

where

$$\begin{bmatrix} I & C_{n,1} & \cdots & C_{n,n} \\ 0 & I & \cdots & C_{n-1,n-1} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix} = U_{n+1}^{-1}M_{n+1}$$
$$=: \bar{U}_{n+1}^{-1}.$$

Similarly, the right matrix orthogonal polynomials of the second kind are given by

$$\begin{bmatrix} I & D_1(z) & \cdots & D_n(z) \end{bmatrix}$$
$$= \begin{bmatrix} I & zI & \cdots & z^nI \end{bmatrix}\begin{bmatrix} I & D_{1,1}^* & \cdots & D_{n,n}^* \\ 0 & I & \cdots & D_{n,n-1}^* \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix},$$

where

$$\begin{bmatrix} I & D_{1,1}^* & \cdots & D_{n,n}^* \\ 0 & I & \cdots & D_{n,n-1}^* \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix} = M_{n+1}^{-1}V_{n+1}^{-*}$$
$$=: \bar{V}_{n+1}^{-*}.$$

Then, the upper Cholesky factorization of the Toeplitz matrix $\bar{\Gamma}_{n+1}$ is given by

$$\bar{\Gamma}_{n+1} = \bar{U}_{n+1}\Sigma_{n+1}\bar{U}_{n+1}^*,$$

and the lower Cholesky factorization of $\bar{\Gamma}_{n+1}$ is given by

$$\bar{\Gamma}_{n+1} = \bar{V}_{n+1}\Lambda_{n+1}\bar{V}_{n+1}^*.$$

Those Cholesky factors give the solution to the YW equation of $\bar{\Gamma}_{n+1}$. The solution to the YW equation

$$\begin{bmatrix} I & C_{n+1,1} & \cdots & C_{n+1,n} & C_{n+1,n+1} \\ D_{n+1,n+1} & D_{n+1,n} & \cdots & D_{n+1,1} & I \end{bmatrix}\bar{\Gamma}_{n+2}$$
$$= \begin{bmatrix} Q_{n+1} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & S_{n+1} \end{bmatrix}$$

is given by

$$\begin{bmatrix} I & C_{n+1,1} & \cdots & C_{n+1,n} & C_{n+1,n+1} \\ D_{n+1,n+1} & D_{n+1,n} & \cdots & D_{n+1,1} & I \end{bmatrix}$$

$$= \begin{bmatrix} I & -T_n S_n^{-1} \\ -T_n^* Q_n^{-1} & I \end{bmatrix} \begin{bmatrix} I & C_{n,1} & \cdots & C_{n,n} & 0 \\ 0 & D_{n,n} & \cdots & D_{n,1} & I \end{bmatrix}$$

$$Q_{n+1} = Q_n - P_n S_n^{-1} P_n^*$$
$$S_{n+1} = Q_n - P_n^* Q_n^{-1} P_n$$
$$T_n = \bar{R}_{n+1} + C_{n,1}\bar{R}_n + \cdots + C_{n,n}\bar{R}_1.$$

The initial values for the recursion are the following,

$$\begin{aligned} C_{1,1} &= -\bar{R}_1 \\ D_{1,1} &= -\bar{R}_1^* \\ Q_1 &= I + C_{1,1}\bar{R}_1^* \\ S_1 &= I + D_{1,1}\bar{R}_1. \end{aligned}$$

We shall use *Lemma* below.

Lemma 1. *[9]: $T_n = -P_n$ holds.*

2.2 Block Discrete-time Schwarz Form

We give a brief review of the block discrete-time Schwarz form in [7, 4]. Consider the YW equation of Γ_{n+1}

$$\begin{bmatrix} \Gamma_n & \rho_n \\ \rho_n^* & I \end{bmatrix} \begin{bmatrix} u_n \\ I \end{bmatrix} = \begin{bmatrix} 0 \\ S_n \end{bmatrix},$$

where

$$\begin{aligned} \rho_n &:= \begin{bmatrix} R_n^* & \cdots & R_1^* \end{bmatrix}^* \\ u_n &:= \begin{bmatrix} B_{n,n} & \cdots & B_{n,1} \end{bmatrix}^*. \end{aligned}$$

It gives

$$\begin{aligned} u_n &= -\Gamma_n^{-1}\rho_n \\ S_n &= I - \rho_n^*\Gamma_n^{-1}\rho_n. \end{aligned} \qquad (2)$$

Let us define

$$\begin{aligned} F_n &:= \Lambda_n^{\frac{1}{2}} V_n^* (Z_n - u_n e_n^*) V_n^{-*} \Lambda_n^{-\frac{*}{2}} \\ &= \Lambda_n^{\frac{*}{2}} V_n^* (Z_n + \Gamma_n^{-1}\rho_n e_n^*) V_n^{-*} \Lambda_n^{-\frac{*}{2}} \\ K_{n+1} &:= Q_n^{-\frac{1}{2}} P_n S_n^{-\frac{*}{2}}, \end{aligned} \qquad (3)$$

where

$$\begin{aligned} Z_n &:= \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{bmatrix} \\ e_n &:= \begin{bmatrix} 0 & 0 & \cdots & 0 & I \end{bmatrix}^*. \end{aligned}$$

We can verify

$$I - F_n^* F_n = \Lambda_n^{-\frac{1}{2}} e_n [I - \rho_n^* \Gamma_n^{-1}\rho_n] e_n^* \Lambda_n^{-\frac{*}{2}} \qquad (4)$$

and

$$\begin{aligned} e_n^*(I - F_n^* F_n) e_n &= e_n^* \Lambda_n^{-\frac{1}{2}} e_n [I - \rho_n^* \Gamma_n^{-1}\rho_n] e_n^* \Lambda_n^{-\frac{*}{2}} e_n \\ &= S_{n-1}^{-\frac{1}{2}} S_n S_{n-1}^{-\frac{*}{2}} \\ &= I - K_n^* K_n. \end{aligned}$$

The equation (4) implies that the matrix F_n is almost orthogonal, i.e., its first $n-1$ block columns form an orthogonal set and its last block column is orthogonal to this set, but it is not normalized. This and Hessenberg property force a particular structure on F_n. Let us define

$$\bar{K}_n := \begin{bmatrix} K_n & K_n^c \\ K_n^{Tc*} & -K_n^s \end{bmatrix},$$

where the matrices K_n^c, K_n^{Tc} and K_n^s are given by

$$\begin{aligned} K_n^c K_n^{c*} &= I - K_n K_n^* \\ K_n^{Tc*} K_n^{Tc} &= I - K_n^* K_n \\ K_n^s &= (K_n^{Tc})^{-1} K_n^* K_n^c. \end{aligned}$$

The matrix $\bar{K}_n$ satisfies $\bar{K}_n^* \bar{K}_n = I$.

Lemma 2. *[4]: The block upper-Hessenberg matrix F_n satisfying* (4) *can be expressed as*

$$F_n = \begin{bmatrix} K_1 & K_1^c K_2 & K_1^c K_2^c K_3 & \cdots & K_1^c K_2^c \cdots K_{n-1}^c K_n \\ K_1^{Tc*} & -K_1^s K_2 & -K_1^s K_2^c K_3 & \cdots & -K_1^s K_2^c \cdots K_{n-1}^c K_n \\ 0 & K_2^{Tc*} & -K_2^s K_3 & \cdots & -K_2^s K_3^c \cdots K_{n-1}^c K_n \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & -K_{n-1}^s K_n \end{bmatrix}.$$

We call this form of matrix the block discrete-time Schwarz form as the natural generalization of the scalar case [7, 4]. The salient feature of F_n is the nesting property, i.e., F_{n+1} has F_n in the upper block. Let us define

$$g_n := \Lambda_n^{\frac{1}{2}} V_n^* e_1, \qquad e_1 := \begin{bmatrix} I & 0 & \cdots & 0 & 0 \end{bmatrix}^*.$$

Then, the covariance matrices are given by

$$R_k = g_n^* F_n^k g_n, \quad k = 0, \ldots, n-1.$$

2.3 Multivariable Rational Interpolation with McMillan Degree Constraint

We review the result of a parameterization of the multivariable rational interpolants with bounded McMillan degree, which is the generalization of the scalar case [2, 5]. A class of rational functions to be considered here is

$$f(z) = \frac{1}{2} N(z) M(z)^{-1}. \qquad (5)$$

where $M(z)$ and $N(z)$ are $m \times m$ matrix polynomials of degree n. We denote this class of rational functions by $\mathscr{R}_r$.

Lemma 3. *All functions in $\mathscr{R}_r$, of which power series expansion begins with*

$$\frac{1}{2}R_0 + R_1 z^{-1} + \cdots + R_n z^{-n},$$

admit the right coprime factorization

$$f(z) = \frac{1}{2} N(z) M(z)^{-1}, \qquad (6)$$

and the right coprime factors are parameterized by

$$\begin{aligned} M(z) &= B_n(z) + B_{n-1}(z)\alpha_1 + \cdots + \alpha_n \\ N(z) &= D_n(z) + D_{n-1}(z)\alpha_1 + \cdots + \alpha_n, \end{aligned}$$

where the matrices $\alpha_k \in \mathbb{R}^{m \times m}$, $k = 1, \ldots, n$, are free parameters.

The proof is omitted due to the space limitation. The choice of the free parameter, $\alpha_k = 0$, $k = 1, \ldots, n$, yields the so-called maximum entropy interpolant [1],

$$f(z) = \frac{1}{2} D_n(z) B_n(z)^{-1}.$$

It is positive real, and maximize the entropy rate of the spectral density

$$\mathbb{I}(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log\det[f(e^{i\theta}) + f(e^{i\theta})^*] d\theta.$$

3. MAIN RESULTS

We give the state-space realization of the parameterization in *Lemma* 3 by the block discrete-time Schwarz form. We also give a characterization of the positive realness of the parameterization by a linear matrix inequality.

3.1 State-space Realization by Block Discrete-time Schwarz Form

Let us define the normalized block discrete-time Schwarz form

$$\hat{F}_n := \Lambda_n^{-\frac{*}{2}} F_n \Lambda_n^{\frac{*}{2}},$$

and

$$\begin{aligned} \alpha &:= \begin{bmatrix} \alpha_n \\ \vdots \\ \alpha_1 \end{bmatrix} \\ e_1 &= \begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix}^*. \end{aligned}$$

Theorem 1. *The state-space realization of* (6) *is given by*

$$f(z) = \frac{I}{2} + e_1^* \hat{F}_n (zI - \hat{F}_n + \alpha e_n^*)^{-1} e_1. \tag{7}$$

Proof. For the right matrix orthogonal polynomials of the first kind, consider the identity [3],

$$\begin{bmatrix} B_n(z)^{-1} & 0 & \cdots & 0 \end{bmatrix} T = e_n^* (zI - F_c)^{-1}$$

where

$$\begin{aligned} F_c &:= Z_n - u_n e_n^* \\ T &:= \begin{bmatrix} I & zI & \cdots & z^{n-1}I \\ 0 & I & \cdots & z^{n-2}I \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix}. \end{aligned}$$

By (3), we obtain

$$\begin{bmatrix} B_n(z)^{-1} & 0 & \cdots & 0 \end{bmatrix} T V_n^{-*} \Lambda_n^{-\frac{*}{2}} = e_n^* V_n^{-*} \Lambda_n^{-\frac{*}{2}} (zI - F_n)^{-1}. \tag{8}$$

The left hand side of (8) is

$$\begin{aligned} &\begin{bmatrix} B_n(z)^{-1} & 0 & \cdots & 0 \end{bmatrix} T V_n^{-*} \Lambda_n^{-\frac{*}{2}} \\ &= B_n(z)^{-1} \begin{bmatrix} I & B_1(z) & \cdots & B_{n-1}(z) \end{bmatrix} \Lambda_n^{-\frac{*}{2}}. \end{aligned}$$

The right hand side of (8) is

$$e_n^* V_n^{-*} \Lambda_n^{-\frac{*}{2}} (zI - F_n)^{-1} = e_n^* \Lambda_n^{-\frac{*}{2}} (zI - F_n)^{-1}.$$

Thus, we obtain

$$B_n(z)^{-1} \begin{bmatrix} I & B_1(z) & \cdots & B_{n-1}(z) \end{bmatrix} \Lambda_n^{-\frac{*}{2}} = e_n^* \Lambda_n^{-\frac{*}{2}} (zI - F_n)^{-1}.$$

It is equivalent to

$$B_n(z)^{-1} \begin{bmatrix} I & B_1(z) & \cdots & B_{n-1}(z) \end{bmatrix} = e_n^* (zI - \hat{F}_n)^{-1}, \tag{9}$$

and (9) implies

$$B_n(z)^{-1} = e_n^* (zI - \hat{F}_n)^{-1} e_1. \tag{10}$$

Similarly, for the right matrix orthogonal polynomials of the second kind, we obtain

$$D_n(z)^{-1} \begin{bmatrix} I & D_1(z) & \cdots & D_{n-1}(z) \end{bmatrix} \Lambda_n^{-\frac{*}{2}} = e_n^* \Lambda_n^{-\frac{*}{2}} (zI - \bar{F}_n)^{-1}.$$

The matrix $\bar{F}_n$ is obtained by replacing K_i by $-K_i$ of F_n by *Lemma* 1. Due to the structure of F_n, this replacement does not affect any element of F_n except for those in the first block row, which change their signs. Thus, we conclude

$$\bar{F}_n = F_n - 2 e_1 e_1^* F_n.$$

Let us define the normalized discrete-time Schwarz form of $\bar{F}_n$

$$\begin{aligned} \tilde{F}_n &:= \Lambda_n^{-\frac{*}{2}} \bar{F}_n \Lambda_n^{\frac{*}{2}} \\ &= \hat{F}_n - 2 e_1 e_1^* \hat{F}_n. \end{aligned} \tag{11}$$

By multiplying α to (9), we obtain

$$B_n(z)^{-1} [\alpha_n + B_1(z)\alpha_{n-1} + \cdots + B_{n-1}(z)\alpha_1] = e_n^* (zI - \hat{F}_n) \alpha.$$

Thus,

$$\begin{aligned} M(z) &= B_n(z) + B_{n-1}(z)\alpha_1 + \cdots B_1(z)\alpha_{n-1} + \alpha_n \\ &= B_n(z)[I + e_n^*(zI - \hat{F}_n)^{-1}\alpha]. \end{aligned} \tag{12}$$

Similarly,

$$\begin{aligned} N(z) &= D_n(z) + D_{n-1}(z)\alpha_1 + \cdots D_1(z)\alpha_{n-1} + \alpha_n \\ &= D_n(z)[I + e_n^*(zI - \tilde{F}_n)^{-1}\alpha]. \end{aligned} \tag{13}$$

By (10), the inverse of (12) is given by

$$\begin{aligned} M(z)^{-1} &= [I + e_n^*(zI - \hat{F}_n)^{-1}\alpha]^{-1} B_n(z)^{-1} \\ &= \left[\begin{array}{c|c} \hat{F}_n - \alpha e_n^* & -\alpha \\ \hline e_n^* & I \end{array}\right] \left[\begin{array}{c|c} \hat{F}_n & e_1 \\ \hline e_n^* & 0 \end{array}\right] \\ &= \left[\begin{array}{cc|c} \hat{F}_n - \alpha e_n^* & -\alpha e_n^* & 0 \\ 0 & \hat{F}_n & e_1 \\ \hline e_n^* & e_n^* & 0 \end{array}\right] \\ &= \left[\begin{array}{cc|c} \hat{F}_n - \alpha e_n^* & 0 & e_1 \\ 0 & \hat{F}_n & e_1 \\ \hline e_n^* & 0 & 0 \end{array}\right] \\ &= \left[\begin{array}{c|c} \hat{F}_n - \alpha e_n^* & e_1 \\ \hline e_n^* & 0 \end{array}\right] \end{aligned}$$

where we changed the coordinate of the states by $\begin{bmatrix} I & I \\ 0 & I \end{bmatrix}$. Similarly, by (11), the inverse of (13) is given by

$$N(z)^{-1} = \left[\begin{array}{c|c} \hat{F}_n - 2e_1 e_1^* \hat{F}_n - \alpha e_n^* & e_1 \\ \hline e_n^* & 0 \end{array}\right].$$

We can verify that

$$\begin{aligned} 2N(z)^{-1} f(z) &= \left[\begin{array}{cc|c} \hat{F}_n - 2e_1 e_1^* \hat{F}_n - \alpha e_n^* & 2e_1 e_1^* & e_1 \\ 0 & \hat{F}_n - \alpha e_n^* & e_1 \\ \hline e_n^* & 0 & 0 \end{array}\right] \\ &= \left[\begin{array}{cc|c} \hat{F}_n - 2e_1 e_1^* \hat{F}_n - \alpha e_n^* & 0 & 0 \\ 0 & \hat{F}_n - \alpha e_n^* & e_1 \\ \hline e_n^* & e_n^* & 0 \end{array}\right] \\ &= \left[\begin{array}{c|c} \hat{F}_n - \alpha e_n^* & e_1 \\ \hline e_n^* & 0 \end{array}\right] \\ &= M(z)^{-1}, \end{aligned}$$

where we changed the coordinate of the states by $\begin{bmatrix} I & -I \\ 0 & I \end{bmatrix}$. □

3.2 Characterization of Positivity by Linear Matrix Inequality

A condition, which makes (7) positive real, is given by the linear matrix inequality below. In [6], a similar result of another parameterization of the solution to the Nevanlinna-Pick interpolation problem is found.

Theorem 2. *$f(z)$, given by (7), is positive real if there exist $P > 0$ and $\beta \in \mathbb{R}^{mn\times m}$ such that*

$$\begin{bmatrix} P & Pe_1 & P\hat{F}_n - \beta e_n^* \\ e_1^* P & I & e_1^*\hat{F}_n \\ \hat{F}_n^* P - e_n\beta^* & \hat{F}_n^* e_1 & P \end{bmatrix} \geq 0. \quad (14)$$

Moreover, for given P and β, we obtain α by

$$\beta = P\alpha. \quad (15)$$

Proof. By KYP lemma [8], (7) is strictly positive real if and only if there exists $P > 0$ and α such that

$$\begin{bmatrix} P^{-1} & e_1 & \hat{F}_n - \alpha e_n^* \\ e_1^* & I & e_1^*\hat{F}_n \\ \hat{F}_n^* - e_n\alpha^* & \hat{F}_n^* e_1 & P \end{bmatrix} > 0.$$

Multiply $\begin{bmatrix} P & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$ to both sides, we obtain

$$\begin{bmatrix} P & Pe_1 & P\hat{F}_n - P\alpha e_n^* \\ e_1^* P & I & e_1^*\hat{F}_n \\ \hat{F}_n^* P - e_n\alpha^* P & \hat{F}_n^* e_1 & P \end{bmatrix} > 0.$$

By changing $P\alpha$ as (15) and by extending the inequality to the boundary by continuity, we obtain $P \geq 0$ and (14). The change of the variable (15) is nonsingular if P is invertible. Thus, it is necessary that $P > 0$. □

4. NUMERICAL EXAMPLE

Consider a positive real function

$$\begin{aligned} g(z) &= \frac{1}{2}\frac{z-1}{z-\frac{1}{2}} \\ &= \frac{1}{2} - \frac{1}{4}z^{-1} - \frac{1}{8}z^{-1} + \cdots . \end{aligned}$$

We consider the parameterization of the McMillan degree one. The Toeplitz matrix is given by

$$\Gamma_2 = \begin{bmatrix} 1 & -\frac{1}{4} \\ -\frac{1}{4} & 1 \end{bmatrix}.$$

The discrete-time Schwarz form is given by

$$\frac{1}{2} + \hat{F}_1(z - \hat{F}_1 + \alpha)^{-1} = \frac{1}{2}\frac{z - \frac{1}{4} + \alpha}{z + \frac{1}{4} + \alpha}, \quad (16)$$

where $\hat{F}_1 = -\frac{1}{4}$. The linear matrix inequality for the positive realness is given by

$$\begin{aligned} \begin{bmatrix} p & p & p\hat{F}_1 - \beta \\ p & 1 & \hat{F}_1 \\ \hat{F}_1^* p - \beta & \hat{F}_1 & p \end{bmatrix} &= \begin{bmatrix} p & p & -\frac{1}{4}p - \beta \\ p & 1 & -\frac{1}{4} \\ -\frac{1}{4}p - \beta & -\frac{1}{4} & p \end{bmatrix} \\ &\geq 0. \end{aligned}$$

The linear matrix inequality gives

$$\begin{aligned} &0 \leq p \leq 1 \\ &-p(p-1)(p - \frac{1}{16}) - \beta^2 \geq 0. \end{aligned}$$

If $p \neq 0$, then, the second inequality gives

$$\frac{\beta^2}{p^2} = \alpha^2 \leq -\frac{(p-1)(p-\frac{1}{16})}{p}.$$

The right hand side takes the maximum at $p = \frac{1}{4}$, and, we obtain

$$\alpha^2 \leq \frac{9}{16}.$$

At each boundary value of α, the parameterization (16) gives

$$\begin{aligned} f(z) &= \frac{1}{2}\frac{z-1}{z-\frac{1}{2}}, \quad \alpha = -\frac{3}{4} \\ f(z) &= \frac{1}{2}\frac{z+\frac{1}{2}}{z+1}, \quad \alpha = \frac{3}{4}. \end{aligned}$$

At $\alpha = -\frac{3}{4}$, we realize the corresponding positive real function.

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