STAIRCASING REDUCTION MODEL APPLIED TO TOTAL VARIATION BASED IMAGE RECONSTRUCTION

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ABSTRACT

Total variation based image restoration method was first proposed by Rudin Osher and Fatermi in 1992. The images resulting from its application are usually piecewise constant, and have sometimes undesirable staircasing effect. To reduce this effect, we propose an improved model by combining the advantages of total variation and H^1 regularization. The new model substantially reduces the staircase effect, while preserving sharp edges. This model can be used in image reconstruction, it has advantages of keeping edges and recovering smooth region's value. We give 1D and 2D experimental results to show the efficiency of the proposed model.

1. INTRODUCTION

Image processing refers to the analysis and extraction of information from images, including restoration, compression and segmentation. Applications can be found in many areas like medical diagnosis, satellite surveying and computer techniques.

The aim of image restoration is to estimate the ideal true image from the recorded one. The direct problem is the computing of blurred image from a given image. The usual model for it is the convolution by a given kernel or point spread function. In many cases, the inverse problem of computing the true image from the observation is ill-posed. A general method to dealing with inverse problem is that of regularization. The choice of regularization will be essential for a satisfactory image restoration process. The solution of regularization based on least squares criteria is usually continuous, therefore, the image edges can not be well restored. To overcome this difficulty, a technique based on the minimization of total variation norm subject to some noise constraints is proposed by Rudin, Osher and Fatemi [1], that is, to seek solutions in BV space. The space of functions of bounded total variation plays an important role when accurate estimation of discontinuities in solutions is required. The total variation (TV) denoising method preserves edges well, but has sometimes undesirable staircase effect, namely the transformation of smooth regions into piecewise constant regions (stairs), which implied that the finer details in the original image may not be recovered satisfactorily. To solve this problem, Chan, Marquina and Mulet [2] proposed an improved model, constructed by adding a nonlinear fourth order diffusive term to the Euler-Lagrange equations of the variational TV model. Marquina and Osher [3] preconditioned the right hand side of the parabolic equation with $|\nabla u|$ which had a staircase reducing effect. Another popular way to reduce staircasing is to introduce in some way higher order derivatives into the regularization term. Chambolle and Lions [4] do this by minimizing the inf-convolution of the TV norm and a second order functional. Instead of combing TV norm and second order derivatives within one regularization functional, Lysaker and Tai [5] use two regularization functionals. In [6], Blomgren, Chan and Mulet propose a "TV-H^1 interpolation" approach to address the staircase problem of the TV technique. The approach is performed by redefining the Total Variation functional $R(u)$ in view of the properties of TV-norm and H^1-seminorm. However, it is not completely clear how to choose a function Φ, which makes the regularizing functional $R(u)$ being convex. In this paper, we give a choice of function Φ, and the corresponding regularizing functional $R(u)$ verifies the sufficient conditions for convexity. This is mathematically desirable, for then the constrained optimization problem will have some kind of uniqueness.

The paper is organized as follows: in section 2, we introduce the image restoration problem using the Total Variation norm as regularization functional. In section 3, we describe the staircase effect caused by the TV model and briefly review some techniques proposed in literature to deal with it. In section 4, we construct an improved regularizing functional to reduce the staircase effect. We then analysis our model and give its Euler-Lagrange equation as well as its discretization method. In section 5,we give numerical examples to test the efficiency of our new model. The final part is our conclusion.

2. TOTAL VARIATION IMAGE RESTORATION

An image can be interpreted as either a real function defined on Ω, a bounded and open domain of R^2, or as a suitable discretization of this continuous image. Our aim is to restore an image which is contaminated with noise and blur. The restoration process includes the recovery of edges and smooth regions. Let us denote by z the observed image and u the real image. We assume that the degradation model is $Ku+n=z$, where K is a known linear blur operator, and n is a Gaussian white noise, i.e. the values n_i of n at pixels i are independent random variables, each with a Gaussian distribution of zero mean and variance σ^2. Our objective is to estimate u from given z. The inverse problem has many solutions and is ill-posed. If we impose a certain regularity condition on the solution u, then it may become well-posed [7]. In [1], it is proposed to use as regularization functional the so-called Total Variation norm or TV-norm :

$$TV(u) = \int_\Omega |\nabla u| dxdy = \int_\Omega \sqrt{u_x^2 + u_y^2} dxdy. \qquad (2.1)$$

Since TV norm does not penalize discontinuities in u, thus we can recover the edges of the original image. The restoration problem can be written as:

$$\min_u \int_\Omega |\nabla u| dxdy, \quad (2.2a)$$

$$\text{subject to} \quad \|Ku - z\|_{L^2}^2 = |\Omega|\sigma^2. \quad (2.2b)$$

Using known techniques, the solution of problem (2.2) can be achieved by solving the equivalent unconstrained problem:

$$\min_u \int_\Omega (\alpha|\nabla u| + \frac{1}{2}(Ku - z)^2) dxdy. \quad (2.3)$$

where α represents the tradeoff between smoothness and fidelity to the original data. Assuming homogeneous Neumann boundary conditions, the Euler-Lagrange equation of (2.3) is:

$$0 = -\alpha\nabla \cdot (\frac{\nabla u}{|\nabla u|}) + K^*(Ku - z). \quad (2.4)$$

The above equation (2.4) is not well defined at locations where $|\nabla u| = 0$, due to the presence of the term $1/|\nabla u|$. The common method to overcome this technical difficulty is to slightly perturb the total variation functional to become:

$$\int_\Omega \sqrt{|\nabla u|^2 + \beta} dxdy,$$

where β is a small positive number.

In [9] it is shown that the solutions of the perturbed problems

$$\min_u \int_\Omega (\alpha\sqrt{|\nabla u|^2 + \beta} + \frac{1}{2}(Ku - z)^2) dxdy \quad (2.5)$$

converge to the solutions of (2.3) when $\beta \to 0$. The Euler-Lagrange equation of (2.5) is

$$0 = -\alpha\nabla \cdot (\frac{\nabla u}{\sqrt{|\nabla u|^2 + \beta}}) + K^*(Ku - z), \quad (2.6)$$

with homogeneous Neumann boundary conditions.

3. THE STAIRCASE EFFECT

The image restoration model based on total variation regularization tends to yield piecewise constant images. This is 'staircasing effect'. Smooth regions in original image are recovered as piecewise smooth regions. In order to overcome this difficulty, some works focus on introducing higher order derivatives into the regularization term. Some starts from the parabolic equation and reform the right hand side of the equation to get reduced effect of staircasing. A popular approach to reducing staircasing is to combine the ability of TV denoising to preserve edges with the ability of H^1 to preserve smooth regions. Blomgren, Chan and Mulet [6] proposed to use as regularizing functionals the interpolation of TV-norm and H^1-seminorm, because staircase effect is partly due to the fact that the TV-norm is not biased against discontinuous nor continuous functions. On the other hand, the functional

$$H^1(u) = \int_\Omega |\nabla u|^2 dxdy,$$

has a strong bias against discontinuous functions.

Consider functionals of the type:

$$R(u) = \int_\Omega |\nabla u|^p dxdy, \quad (3.1)$$

where $p \in [1,2]$. TV-norm and H^1 functionals can be obtained by Eq. (3.1) with $p = 1,2$, respectively. In [6], numerical evidence show that sharp edges are obtained for $p = 1, 1.1$, and the staircase effect does exist. With the increasing of p, for instance $p = 1.5, 2$, those sharp edges are smeared, but the staircase effect is alleviated. In view of these results, the criterion of constructing regularization functionals should be that obtain TV behavior at sharp gradients (edges) and H^1 behavior away from edges. The approach which is proposed by Blomgren, Chan and Mulet is to consider regularizing functionals of the type:

$$R(u) = \int_\Omega \Phi(|\nabla u|) dxdy, \quad (3.2)$$

$\Phi(|\nabla u|)$ could be a "convex combination" of x and x^2, with variable weight $\alpha(x) \in [0,1]$:

$$\Phi(x) = \alpha(x)x + (1 - \alpha(x))x^2,$$

with $\alpha(x) \to 1$ when $x \to \infty$ and $\alpha(x) \to 0$ when $x \to 0$. That is, at edges where $|\nabla u|$ is very large, $\Phi(x)$ is close to x, the result of using functional $R(u)$ is approximately equal to that of TV-norm. At smoother region where $|\nabla u|$ is very small, $\Phi(x)$ is close to x^2, the result of using functional $R(u)$ is approximately equal to that of H^1-seminorm.

4. A CONVEX REGULARIZING FUNCTIONAL FOR STAIRCASE REDUCTION

As stated in section 3, we consider regularizing functional $R(u)$,

$$R(u) = \int_\Omega \Phi(|\nabla u|) dxdy,$$

$$\Phi(x) = \alpha(x)x + (1 - \alpha(x))x^2 \quad (4.1)$$

where $\alpha(x) = \frac{x}{1+x}$, which satisfies $\alpha(x) \to 1$ when $x \to \infty$ and $\alpha(x) \to 0$ when $x \to 0$. Thus we get regularizing functional

$$R(u) = \int_\Omega \frac{2|\nabla u|^2}{1 + |\nabla u|} dxdy \quad (4.2)$$

Therefore, the new model for total variation denoising is

$$\min \alpha \int_\Omega \frac{2|\nabla u|^2}{1 + |\nabla u|} + \frac{1}{2}\|Ku - z\|_{L^2}^2 \quad (4.3)$$

The Euler-Lagrange equation of (4.3) is

$$0 = -\nabla \cdot (\frac{2 + |\nabla u|}{(1 + |\nabla u|)^2}\nabla u) + \lambda K^*(Ku - z) \quad (4.4)$$

We calculate the derivatives of functional $R(u)$,

$$R'(u) = -\nabla \cdot (\frac{\Phi'(|\nabla u|)}{|\nabla u|}\nabla u) \quad (4.5)$$

$$R''(u)v = -\nabla \cdot (\frac{\Phi'(|\nabla u|)}{|\nabla u|}(\nabla v - \frac{(\nabla u, \nabla v)}{|\nabla u|^2}\nabla u)$$

$$+\Phi''(|\nabla u|)\frac{(\nabla u,\nabla v)}{|\nabla u|^2}\nabla u). \quad (4.6)$$

From (4.6) we deduce that:

$$(R''(u)v,v) = \int_\Omega (\frac{\Phi'(|\nabla u|)}{|\nabla u|}(|\nabla v|^2 - \frac{(\nabla u,\nabla v)^2}{|\nabla u|^2})$$

$$+\Phi''(|\nabla u|)\frac{(\nabla u,\nabla v)^2}{|\nabla u|^2})dxdy. \quad (4.7)$$

The Cauchy-Schwartz inequality implies that

$$|\nabla v|^2 - \frac{(\nabla u,\nabla v)^2}{|\nabla u|^2} \geq 0,$$

therefore $\Phi'(x) \geq 0$ and $\Phi''(x) \geq 0$, $x \geq 0$, that is , Φ is an increasing convex function in $[0,\infty)$, are sufficient conditions for the functional R of (4.2) being convex. It's easy to get the expression of $\Phi'(x)$ and $\Phi''(x)$:

$$\Phi'(x) = \frac{x(x+2)}{(1+x)^2}$$

$$\Phi''(x) = \frac{2}{(1+x)^3}$$

Obviously, $\Phi'(x) \geq 0$, $\Phi''(x) > 0$ when $x \geq 0$. Therefore, the functional R of (4.2) is convex.

There are many methods to solve Euler-Lagrange equation (4.4). L. Rudin, S. Orsher and E. Fatemi [1] use a time marching scheme to reach a steady state of a parabolic equation; C. Vogel and M. Oman [8] propose the fixed point iteration method, which results in the lagged diffusivity fixed point algorithm. Chan and Mulet [9] give the convergence of the lagged diffusivity fixed point method. Considering the presence of highly nonlinear and non-differentiable term in Euler-Lagrange equation, Chan, Golub and Mulet proposed a nonlinear primal-dual method [10], Chan and Chen [11] intrduced the nonlinear multigrid method. Further works about fast total variation minimization method and algorithm can be seen in literature [12, 13]. In our computation, we referenced Vogel and Oman's fast, robust total variation-based image reconstruction method [14]. To solve Euler-Lagrange equation (4.4), fixed point iteration technique is adopted:

$u^0 = z$, solve for u^{k+1}:

$$-\nabla\cdot(\frac{2+|\nabla u^k|}{(1+|\nabla u^k|)^2}\nabla u^{k+1}) + \lambda K^*(Ku^{k+1} - z) = 0. \quad (4.8)$$

The new model (4.3) has some advantages: First, because of the convexity of the regularizing functional $R(u)$, the solution to problem (4.3) has some kind of uniqueness. Second, our model has no non-differentiable locations. It is not necessary to do numerical regularization, namely, to replace the term $|\nabla u|$ by $\sqrt{|\nabla u|^2+\beta}$ for a small enough positive artificial parameter β. Third, the new model can efficiently reduce the staircase effect in smooth regions while keep sharp edges behaving like total variation based image restoration model.

5. NUMERICAL EXAMPLES

In this section,we perform numerical experiments in 1D and 2D images. In the first experiment, we use a synthetic 1D image which includes piecewise constant, piecewise linear and piecewise parabolic regions. The original image, shown in Figure 1, is added random noise and blurred by Gaussian kernel. The kernel is defined as

$$g(x) = \frac{1}{\sqrt{2\pi\sigma}}e^{\frac{-x^2}{2\sigma^2}}.$$

31 points of the discrete kernel with $\sigma = 4.5$ is used to get the contaminated image Figure 2. From Figure 3 to Figure 5 we give three kinds of restoration of the corrupted image. Restoration by total variation regularization is shown in Figure 3, restoration by BCM (Blomgren, Chan and Mulet) model [6] in Figure 4 and our proposed model in Figure 5. We observe that in Figure 5 the staircase effect of smooth regions is improved and edges are correctly reserved. Figure 4 has better staircasing reduction in some region, but worse edge location retaining and the 'Gibbs' phenomenon exists. To evaluate the quality of restored images, we calculated signal to noise ratio (SNR) for each image. The SNR values of images in Figure 4 and Figure 5 are 14.5dB and 15.5dB respectively. Before restoration, the image SNR is 11.8dB. We can see that the reconstructed image with proposed model has better SNR value. In the computation, we found that the staircase effect depends on the choice of regularizing parameter λ in (4.8), therefore, we use the same value ($\lambda = 0.005$) when doing comparison.

In the second experiment, we perform 2D image restoration with different staircasing reduction models. The original image is created by a two values function, 1 inside and 18 outside (Figure 6). We contaminate the image with random noise and Gaussian kernel (Figure 7). The kernel takes the value of $\sigma = 4.5$. We discretize the gaussian function by step size $h = 0.02$ both in x and y directions. Similar to the 1D case, we use 31 by 31 points to blur the original image. In Figure 8, the noisy and blurred image is restored using TV technique. Figure 9 and Figure 10 are reconstructed images using BCM model and our model respectively. We can see that both TV and BCM model have problems on recovering curve edges. The resulted edges do not look so smooth as it should be. This is suffering from the staircasing effect. Using our model curve edges can be better recovered. Notice also that how the adjoint parts of the two edge circles are recovered. Our model retains corner better than BCM model does. BCM model uses a third order polynomial interpolating between 0 and sg_{max}. g_{max} is the maximum reliable gradient on the discrete grid and $0 < s \leq 1$. The recovered image is sensitive to the choice of s. We have tried different s, Figure 9 gives best image recovery among all other images we have obtained by BCM model. The SNR values corresponding to images from Figure 8 to Figure 10 are, respectively, 13.7dB, 14.4dB and 14.1dB. The contaminated image SNR value is 4.2dB. For the 2D image our model has very close SNR value improvement with BCM model, but the advantage of curve edge recovery is obvious. In Figure 11 we plot three cross lines which respectively correspond to the original, the blurred and the restored images. We can observe that the proposed model is efficient in recovering image edges and pixel values. In application fields, it's necessary for both pixel values and edge locations be recovered.

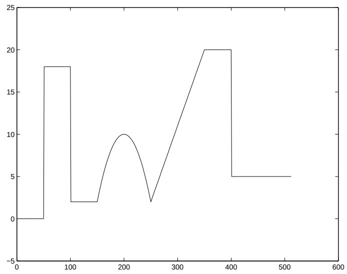

Figure 1: Original image.

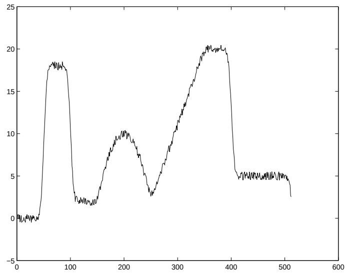

Figure 2: Noisy and blurred image.

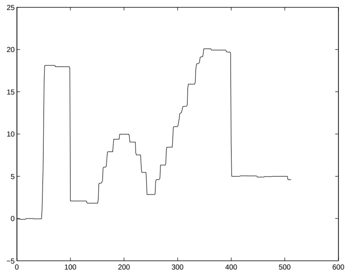

Figure 3: Total Variation restoration.

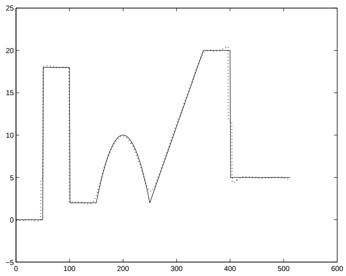

Figure 4: BCM model restoration. Dotted line is reconstructed image, solid line is original image

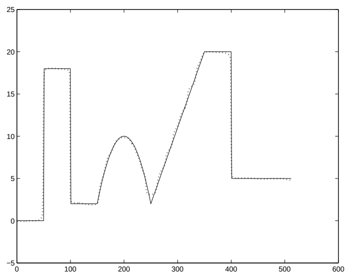

Figure 5: Our proposed model restoration. Dotted line is reconstructed image, solid line is original image

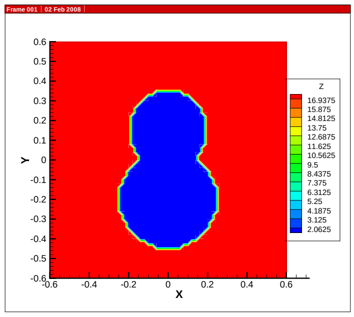

Figure 6: Original image with value 1 inside and 18 outside.

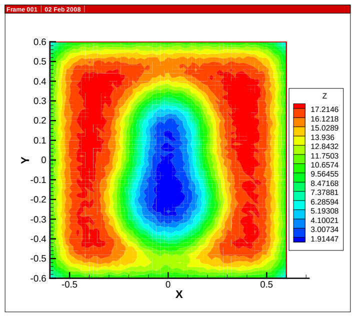

Figure 7: Blurred image with random noise and gaussian kernel.

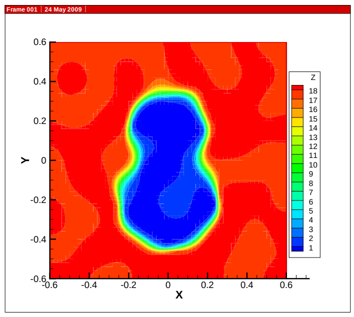

Figure 8: Restored image by Total Variation regularization.

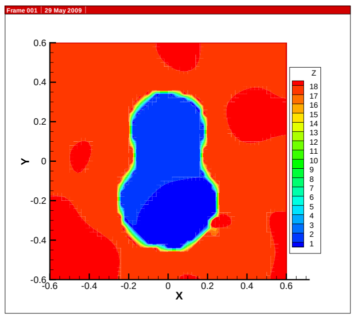

Figure 9: Restored image by BCM model.

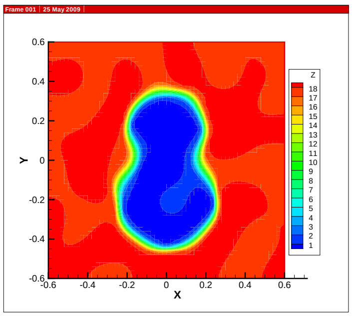

Figure 10: Restored image by our proposed model.

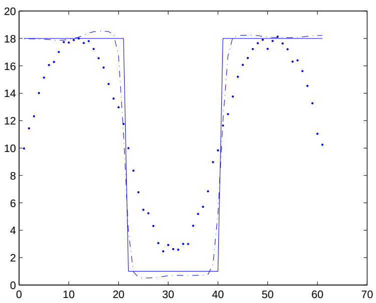

Figure 11: Comparison by cross lines.The solid line, dotted line and dashed line respectively corresponds to the original image,blurred image and restored image by our model.

6. CONCLUSION

TV based image restoration method is widely used in image processing area. Its disadvantage is the staircase effect in smooth regions. We proposed an improved model which combines the advantage of TV and H^1. It can reduce the staircase effect and recover both the pixel values and correct edge locations. We have used it to reconstruct object densities from real data of x-ray radiograph tomography in nuclear field.

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