CLOSED-FORM DESIGN OF VARIABLE FRACTIONAL DELAY FILTER USING DISCRETE FOURIER TRANSFORM

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ABSTRACT

In this paper, the design of variable fractional delay filter is investigated. First, the discrete Fourier transform (DFT) interpolation approach is described. Then, the Taylor series expansion and DFT interpolation formula are used to design the variable fractional delay filter which can be implemented by using Farrow structure. Finally, numerical comparison with conventional Lagrange-type variable fractional delay filter is made to demonstrate the effectiveness of this new design approach.

1. INTRODUCTION

In many applications of signal processing, there is a need for a delay which is a fraction of the sampling period. These applications include time adjustment in digital receivers, beam steering of antenna array, speech coding and synthesis, modelling of music instruments, sampling rate conversion, time delay estimation and fractional order differentiator design etc [1]-[8]. An excellent survey of fractional delay filter design is presented in tutorial paper [1]. Generally speaking, the design methods can be classified into two categories. One is the fixed fractional delay (FFD) filter design, the other is variable fractional delay (VFD) filter design. In FFD case, the delay value is fixed, so conventional FIR and IIR filter design techniques can be used to design FFD filter directly. In VFD case, the delay is adjustable or tuneable, so variable filter design method must be developed to design this type filters, In this paper, we will study the design of VFD filter. On the other hand, the discrete Fourier transform (DFT) is defined as follows:

$$X(k) = \sum_{n=0}^{N-1} x(n)W_N^{kn} \qquad 0 \le k \le N-1$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-kn} \qquad 0 \le n \le N-1 \qquad (1)$$

where $W_N = e^{-j\frac{2\pi}{N}}$. It is well known that zero padding in the frequency domain provides interpolation in the time domain. Therefore, we should be able to use zero padding in the high frequency range of the DFT domain as a means of interpolating finite-duration, discrete-time signals. The details can be found in the literature [9]-[14]. So far, the DFT interpolation and variable fractional delay filter design are two independent research topics in signal processing area.

The relation between them has not been investigated. The purpose of this paper is to study this relation such that DFT interpolation concept can be directly used to design variable fractional delay FIR filters. The main advantage is that the filter coefficients are easily computed because a closed-form design is obtained.

2. DFT INTERPOLATION METHOD

In this section, the DFT interpolation formula of even-length sequence will be derived. Given DFT $X(k)$ of an even-length real-valued sequence $x(n)$, let us define the zero-padded DFT as

$$X_d(k) = \begin{cases} LX(k) & 0 \le k \le \frac{N}{2}-1 \\ \frac{1}{2}LX(\frac{N}{2}) & k = \frac{N}{2} \\ 0 & \frac{N}{2}+1 \le k \le M-\frac{N}{2}-1 \\ \frac{1}{2}LX(\frac{N}{2}) & k = M-\frac{N}{2} \\ LX(k-M+N) & M-\frac{N}{2}+1 \le k \le M-1 \end{cases} \qquad (2)$$

Also, we assume M is an integer multiple of N, say $M = LN$, where L is called interpolation factor. The above DFT has zero values in the high frequency range and satisfies the following conjugate symmetry condition:

$$X_d(M-k) = X_d(k)^* \qquad 1 \le k \le \frac{N}{2} \qquad (3)$$

Now, the interpolated sequence $x_d(n)$ is chosen as the length-M inverse DFT of $X_d(k)$, i.e.,

$$\begin{aligned} x_d(n) &= \frac{1}{M}\sum_{k=0}^{M-1} X_d(k)W_M^{-kn} \\ &= \frac{1}{N}\left(\sum_{k=0}^{\frac{N}{2}-1} X(k)W_M^{-kn} + \sum_{k=M-\frac{N}{2}+1}^{M-1} X(k)W_M^{-kn} \right) \\ &\quad + \frac{1}{2N}\left(X(\tfrac{N}{2})W_M^{-\frac{N}{2}n} + X(\tfrac{N}{2})W_M^{-(M-\frac{N}{2})n} \right) \end{aligned} \qquad (4)$$

Let $k' = M-k$ and using the equality $W_M^{-Mn} = 1$ and Eq.(3), the second term in Eq.(4) can be written as

$$\begin{aligned} \sum_{k=M-\frac{N}{2}+1}^{M-1} X(k)W_M^{-kn} &= \sum_{k'=1}^{\frac{N}{2}-1} X(M-k')W_M^{-(M-k')n} \\ &= \sum_{k'=1}^{\frac{N}{2}-1} X(k')^* W_M^{k'n} \end{aligned} \qquad (5)$$

Substituting Eq.(5) into Eq.(4) and using the equality $W_M^{-(M-\frac{N}{2})n} = W_M^{\frac{Nn}{2}}$, we have

$$x_d(n) = \frac{1}{N}\left[X(0) + \sum_{k=1}^{\frac{N}{2}-1}\left(X(k)W_M^{-kn} + X(k)^* W_M^{kn}\right)\right]$$
$$+ \frac{1}{2N}X(\tfrac{N}{2})\left(W_M^{-\frac{N}{2}n} + W_M^{\frac{N}{2}n}\right) \quad (6)$$

Using the DFT defined in Eq.(1), the first term in Eq.(6) can be written as

$$T_1 = \frac{1}{N}\sum_{m=0}^{N-1}x(m)\left[1 + \sum_{k=1}^{\frac{N}{2}-1}\left(W_N^{mk}W_M^{-kn} + W_N^{-mk}W_M^{kn}\right)\right]$$
$$= \frac{1}{N}\sum_{m=0}^{N-1}x(m)\left[1 + 2\sum_{k=1}^{\frac{N}{2}-1}\cos\left(\frac{2\pi nk}{M} - \frac{2\pi mk}{N}\right)\right] \quad (7)$$

Moreover, using the equality $W_N^{\frac{N}{2}} = -1$, the second term in Eq.(6) can be written as

$$T_2 = \frac{1}{2N}X(\tfrac{N}{2})\left(W_M^{-\frac{N}{2}n} + W_M^{\frac{N}{2}n}\right)$$
$$= \frac{1}{2N}\sum_{m=0}^{N-1}x(m)W_N^{\frac{N}{2}m} 2\cos\left(\frac{n\pi N}{M}\right) \quad (8)$$
$$= \frac{1}{N}\sum_{m=0}^{N-1}(-1)^m x(m)\cos\left(\frac{n\pi N}{M}\right)$$

Substituting Eq.(7) and Eq.(8) into Eq.(6), the sequence $x_d(n)$ becomes the form:

$$x_d(n) = T_1 + T_2$$
$$= \frac{1}{N}\sum_{m=0}^{N-1}x(m)\left\{ \begin{array}{l} 1 + 2\sum_{k=1}^{\frac{N}{2}-1}\cos\left(\dfrac{2\pi nk}{M} - \dfrac{2\pi mk}{N}\right) \\ + (-1)^m \cos\left(\dfrac{n\pi N}{M}\right) \end{array}\right\} \quad (9)$$

Using the identities $\cos(\theta_1 - \theta_2) = \cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2)$, $\cos(m\pi) = (-1)^m$ and $\sin(m\pi) = 0$, we have

$$(-1)^m \cos\left(\tfrac{n\pi N}{M}\right)$$
$$= \cos(m\pi)\cos\left(\tfrac{2\pi n\frac{N}{2}}{M}\right) \quad (10)$$
$$= \cos\left(\tfrac{2\pi n\frac{N}{2}}{M} - \tfrac{2\pi m\frac{N}{2}}{N}\right)$$

Substituting Eq.(10) into Eq.(9), it yields

$$x_d(n) = \sum_{m=0}^{N-1}x(m)\sum_{k=0}^{\frac{N}{2}}\beta_k \cos\left(\frac{2\pi nk}{M} - \frac{2\pi mk}{N}\right) \quad (11)$$

where

$$\beta_k = \begin{cases} \dfrac{1}{N} & k = 0, \dfrac{N}{2} \\ \dfrac{2}{N} & k = 1, 2, \cdots, \dfrac{N}{2}-1 \end{cases} \quad (12)$$

Because the sequence $x_d(n)$ is the interpolated version of the sequence $x(n)$ with factor L, we have the following relation:

$$x_d(iL + p) \approx x(i + \tfrac{p}{L}) \quad (13)$$

for $0 \le p \le L-1$ and $0 \le i \le N-1$. Combining Eq.(11) and Eq.(13) and using $M = LN$, we have

$$x(i + \tfrac{p}{L}) \approx x_d(iL + p)$$
$$= \sum_{m=0}^{N-1}x(m)\left\{\sum_{k=0}^{\frac{N}{2}}\beta_k \cos\left(\frac{2\pi(iL+p)k}{M} - \frac{2\pi mk}{N}\right)\right\} \quad (14)$$
$$= \sum_{m=0}^{N-1}x(m)b_1(i + \tfrac{p}{L} - m)$$

where

$$b_1(i + \tfrac{p}{L} - m) = \sum_{k=0}^{\frac{N}{2}}\beta_k \cos\left(\frac{2\pi(i + \frac{p}{L} - m)k}{N}\right) \quad (15)$$

Let $t = i + \frac{p}{L}$, then the value of t can be any real number in $[0,N)$ if interpolation factor L approaches infinity. Substituting $t = i + \frac{p}{L}$ into Eq.(14), we get

$$x(t) \approx \sum_{m=0}^{N-1}x(m)b_1(t - m) \quad (16)$$

where the interpolation basis is given by

$$b_1(t) = \sum_{k=0}^{\frac{N}{2}}\beta_k \cos\left(\frac{2\pi kt}{N}\right) \quad (17)$$

This means that the continuous-time signal $x(t)$ can be approximately reconstructed from its samples $x(0)$, $x(1)$, ..., $x(N$-$1)$ in the range $[0,N)$ by using continuous-time interpolation basis $b_1(t)$. Finally, one remark is made. From sampling theorem, it is well-known that the discrete-time signal can be reconstructed by using sinc function below:

$$b_2(t) = \frac{\sin(\pi t)}{\pi t} \quad (18)$$

Thus, it is interesting to compare DFT interpolation basis $b_1(t)$ with sinc basis $b_2(t)$. Fig.1 shows the basis $b_1(t)$ (solid line) for $N = 30$. The dashed line is sinc basis $b_2(t)$. It is clear that $b_1(t)$ is similar to a sinc function $b_2(t)$.

3. DESIGN OF VARIABLE FRACTIONAL DELAY FILTER

In this section, the DFT interpolation formula in Eq.(16) and Taylor series expansion will be used to design variable fractional delay filter which can be implemented by using Farrow structure. The details are described below: The ideal frequency response of a fractional delay filter is given by

$$D(\omega) = e^{-j\omega(I+d)} \quad (19)$$

where I is a positive integer and d is a fractional number in the interval $[-0.5, 0.5]$. The transfer function of the FIR filter of length N used to approximate this specification is given by

$$H(z,d) = \sum_{r=0}^{N-1}h_r(d)z^{-r} \quad (20)$$

Thus, the problem is how to design an FIR filter $H(z,d)$ to fit fractional delay specification $D(\omega)$ as closely as possible. Here, the fractional delay d is adjustable, so this is one of the variable digital filter design problems. As shown in Fig.2, when a signal $s(n)$ passes through the FIR filter $H(z,d)$

in Eq.(20), its output is the weighted average of the integer delayed samples $s(n)$, $s(n-1)$, $s(n-2)$,...., $s(n-N+1)$, i.e.,

$$y(n) = \sum_{r=0}^{N-1} h_r(d)s(n-r) \qquad (21)$$

If the frequency response $H(e^{j\omega}, d)$ approximates specification $D(\omega)$ in Eq.(19) well, then this weighted average output $y(n)$ is almost the same as the fractional delayed sample $s(n-I-d)$, i.e.,

$$y(n) \approx s(n-I-d) \qquad (22)$$

Combining Eq.(21) with Eq.(22), it yields

$$s(n-I-d) \approx \sum_{r=0}^{N-1} h_r(d)s(n-r) \qquad (23)$$

Now, the problem is how to link two expressions in Eq.(16) and Eq.(23) to obtain filter coefficients $h_r(d)$ because basis $b_1(t)$ has been known. In the following, the index mapping will be used to solve this problem. After we choose $x(t) = s(n-(N-1)+t)$, then Eq.(16) can be rewritten as

$$\begin{aligned} &s(n-(N-1)+t) \\ &\approx \sum_{m=0}^{N-1} s(n-(N-1)+m)b_1(t-m) \end{aligned} \qquad (24)$$

Let $m = N-1-r$ and $t = N-1-I-d$, Eq.(24) reduces to the following form:

$$s(n-I-d) \approx \sum_{r=0}^{N-1} s(n-r)b_1(r-I-d) \qquad (25)$$

Comparing Eq.(23) with Eq.(25), it is clear that the filter coefficients $h_r(d)$ are

$$h_r(d) = b_1(r-I-d) \qquad (26)$$

Based on this result and Eq.(17), the coefficients $h_r(d)$ can be rewritten as

$$\begin{aligned} h_r(d) &= b_1(r-I-d) \\ &= \sum_{k=0}^{\frac{N}{2}} \beta_k \cos\left(\frac{2\pi(r-I-d)k}{N}\right) \end{aligned} \qquad (27)$$

From the above results, the procedure for fixed fractional delay filter design is summarized below:

Step 1: Specify the design parameters N, I and d.

Step 2: Using Eq.(27) to compute filter coefficients $h_r(d)$.

Step 3: The designed fixed fractional delay filter is given by $H(z,d) = \sum_{r=0}^{N-1} h_r(d)z^{-r}$.

Now, one example is illustrated. The parameters are chosen as $N=60$, $I=30$ and $d=0.5$. Fig.3(a)(b) shows the magnitude response and group delay of the proposed DFT-based fractional delay filter. Obviously, the specification is fitted well. Because the purpose of this paper is to design variable fractional delay filter, we will use the Taylor series expansion to transform the above fixed filter to a variable filter below.

Using the Taylor series expansion at $d=0$, the coefficients $h_r(d)$ can be written as the following polynomial form:

$$h_r(d) = \sum_{n=0}^{\infty} a_r(n)d^n \qquad (28)$$

where

$$a_r(n) = \frac{1}{n!} \frac{\partial^n h_r(d)}{\partial^n d}\bigg|_{d=0} \qquad (29)$$

In order to use Eq.(27) to compute the coefficients $a_r(n)$ easily, Eq.(27) is rewritten as

$$h_r(d) = \mathrm{Re}\left\{\sum_{k=0}^{\frac{N}{2}} \beta_k e^{j\left(\frac{2\pi(r-I-d)k}{N}\right)}\right\} \qquad (30)$$

where $\mathrm{Re}\{.\}$ denotes the real part of a complex number. Based on Eq.(30), the partial derivatives can be computed as

$$\begin{aligned} \frac{\partial^n h_r(d)}{\partial^n d} &= \mathrm{Re}\left\{\sum_{k=0}^{\frac{N}{2}} \beta_k \left(\frac{-j2\pi k}{N}\right)^n e^{j\left(\frac{2\pi(r-I-d)k}{N}\right)}\right\} \\ &= \mathrm{Re}\left\{\left(\frac{2\pi}{N}\right)^n \sum_{k=0}^{\frac{N}{2}} k^n \beta_k e^{j\left(\frac{2\pi(r-I-d)k}{N}-\frac{\pi}{2}n\right)}\right\} \\ &= \left(\frac{2\pi}{N}\right)^n \sum_{k=0}^{\frac{N}{2}} k^n \beta_k \cos\left(\frac{2\pi(r-I-d)k}{N}-\frac{n\pi}{2}\right) \end{aligned} \qquad (31)$$

Combining Eq.(31) with Eq.(29), we obtain $a_r(n)$ below:

$$a_r(n) = \frac{1}{n!}\left(\frac{2\pi}{N}\right)^n \sum_{k=0}^{\frac{N}{2}} k^n \beta_k \cos\left(\frac{2\pi(r-I)k}{N}-\frac{n\pi}{2}\right) \qquad (32)$$

Moreover, because fractional delay d is in the range [-0.5,0.5], the d^n is very small for large n. Thus, the coefficients $h_r(d)$ in Eq.(28) can be approximated by

$$h_r(d) \approx \sum_{n=0}^{K} a_r(n)d^n \qquad (33)$$

where K is called truncation order. The larger K is, the better approximation has. Substituting Eq.(33) into Eq.(20), the transfer function can be rewritten as

$$\begin{aligned} H(z,d) &= \sum_{r=0}^{N-1}\left(\sum_{n=0}^{K} a_r(n)d^n\right)z^{-r} \\ &= \sum_{n=0}^{K}\left(\sum_{r=0}^{N-1} a_r(n)z^{-r}\right)d^n \\ &= \sum_{n=0}^{K} A_n(z)d^n \end{aligned} \qquad (34)$$

where sub-filters $A_n(z)$ are

$$A_n(z) = \sum_{r=0}^{N-1} a_r(n)z^{-r} \qquad (35)$$

Based on Eq.(34), the fractional delay filter $H(z,d)$ can be implemented by using Farrow structure shown in Fig.4 where fractional delay d can be easily adjusted without redesigning the filter. Finally, the procedure for variable fractional delay filter design is summarized below:

Step 1: Specify the design parameters N, I and K.

Step 2: Using Eq.(32) to compute filter coefficients $a_r(n)$ and obtain sub-filters $A_n(z)$.

Step 3: The designed variable fractional delay filter is given by $H(z,d)=\sum_{n=0}^{K}A_n(z)d^n$.

The main advantage of the above DFT-based method is that the computation of coefficients $a_r(n)$ has closed-form formula in Eq.(32) without performing any optimization.

4. EXAMPLE AND COMPARISON

In this section, we will compare DFT-based design with the conventional Lagrange method because both methods have closed-form designs. For completeness, the Lagrange design is first reviewed briefly. Given the discrete-time sequence $x(0)$, $x(1)$, ..., $x(N-1)$, the Lagrange method to reconstruct the continuous-time signal $x(t)$ is given by

$$x(t)=\sum_{m=0}^{N-1}x(m)b_3(m,t) \qquad (36)$$

where the interpolation basis is

$$b_3(m,t)=\prod_{k=0,k\neq m}^{N-1}\frac{t-k}{m-k} \qquad (37)$$

Based on this polynomial basis, the filter coefficients $h_r(d)$ in Eq.(20) can be obtained as

$$h_r(d)=\prod_{k=0,k\neq r}^{N-1}\frac{I+d-k}{r-k} \qquad (38)$$

The above formula can be found in [1]. To get variable fractional delay filter, Eq.(38) is rewritten as

$$h_r(d)=\prod_{k=0,k\neq r}^{N-1}\left(\frac{I-k}{r-k}+\frac{1}{r-k}d\right)=\sum_{n=0}^{N-1}b_r(n)d^n \quad (39)$$

where $b_r(n)$ is easily computed by using polynomial multiplication. Because fractional delay d is in the range [-0.5,0.5], the d^n is very small for large n. Thus, truncating high-degree terms, the $h_r(d)$ in Eq.(39) can be approximated by

$$h_r(d)\approx\sum_{n=0}^{K}b_r(n)d^n \qquad (40)$$

Substituting Eq.(40) into Eq.(20), the transfer function of Lagrange-type design is given by

$$
\begin{aligned}
H(z,d)&=\sum_{r=0}^{N-1}\left(\sum_{n=0}^{K}b_r(n)d^n\right)z^{-r} \\
&=\sum_{n=0}^{K}\left(\sum_{r=0}^{N-1}b_r(n)z^{-r}\right)d^n \qquad (41) \\
&=\sum_{n=0}^{K}B_n(z)d^n
\end{aligned}
$$

where $B_n(z)=\sum_{r=0}^{N-1}b_r(n)z^{-r}$. Now, let us compare DFT and Lagrange methods. To evaluate the performance, the root mean squares (RMS) error is defined by

$$E=\left(\int_0^{\alpha\pi}\int_{-0.5}^{0.5}\left|e(\omega,d)\right|^2\mathrm{d}d\,\mathrm{d}\omega\right)^{\frac{1}{2}} \qquad (42)$$

where $e(\omega,d)=D(\omega)-H(e^{j\omega},d)$. Obviously, the smaller the error E is, the better the performance of the design method is. Now, one example is illustrated. The parameters are chosen as $N=60$, $K=7$, $I=30$ and $\alpha=0.9$. Fig.5 shows the absolute frequency response error $|e(\omega,d)|$ of the designed variable fractional delay filters in the frequency range $[0,0.9\pi]$ and $d\in[-0.5,0.5]$ for DFT and Lagrange methods. The RMS error E of the proposed DFT-based method is 0.0029. The RMS error E of Lagrange method is 0.0379. It is clear that DFT method has smaller error than Lagrange method. Finally, one remark is made. If the coefficient $a_r(n)$ in Eq.(32) is replaced by $a_r(n)w(r)$ where $w(r)$ is the Hamming window, the RMS error E of DFT design is reduced to 0.002. Thus, design accuracy can be slightly improved by using window. Fig.6 shows the magnitude response and group delay of the designed DFT-based variable fractional delay filter after windowing. Obviously, the specification is fitted well.

5. CONCLUSIONS

In this paper, the design of variable fractional delay filter has been presented. The DFT interpolation formula and Taylor series expansion are applied to design the variable fractional delay filter which can be implemented by using Farrow structure. And, numerical comparison with conventional Lagrange-type variable fractional delay filter is made to demonstrate the effectiveness of this new design approach. However, only one-dimensional design is studied here. Thus, it is interesting to extend DFT interpolation method to design two-dimensional fractional delay filter in the future.

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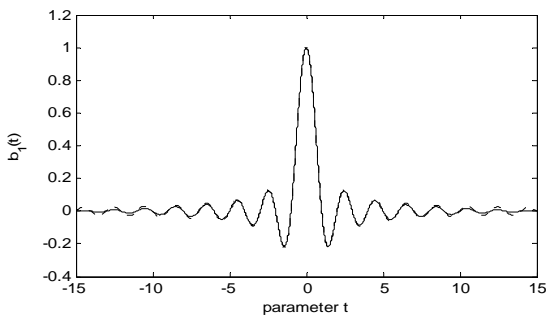
Fig.1 The interpolation basis $b_1(t)$ (solid line) for *N*=30. The dashed line is sinc basis $b_2(t)$.
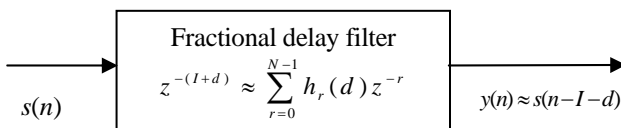

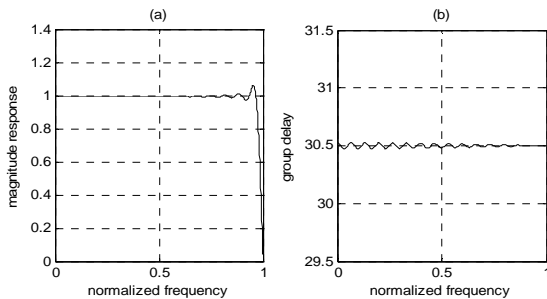Fig.2 The input/output relation of fractional delay filter.

Fig.3 The designed results of a fractional delay filter. (a) Magnitude response. (b) Group delay response.
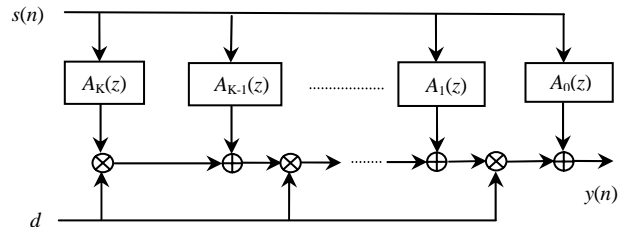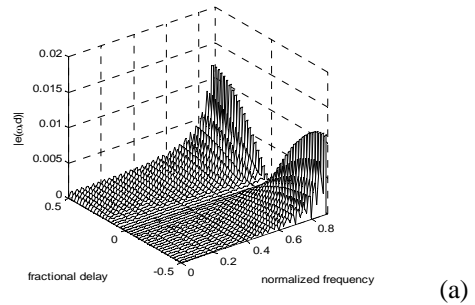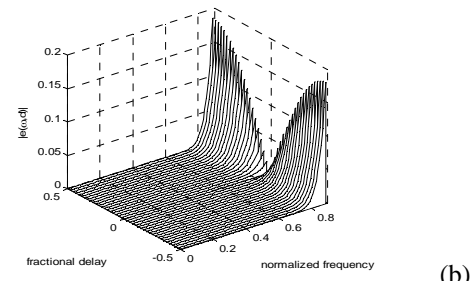

Fig.4 The Farrow structure for implementing variable fractional delay filter with adjustable delay *d*.
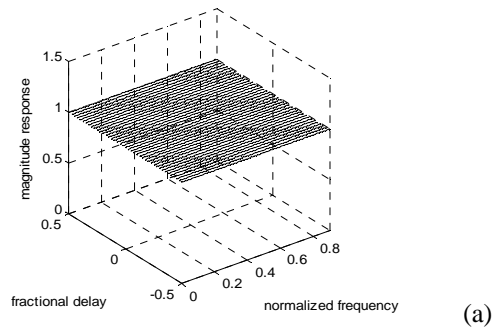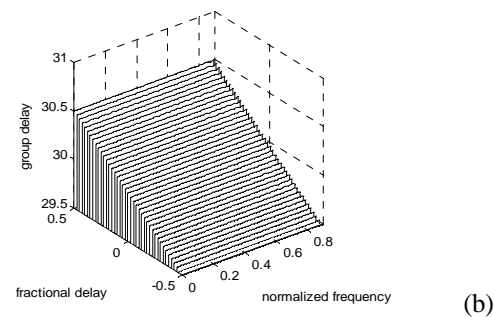

Fig.5 The absolute frequency response error $|e(\omega,d)|$ of the designed variable fractional delay filters. (a) DFT design (b) Lagrange design.

Fig.6 The designed results of DFT-based variable fractional delay filter. (a) Magnitude response. (b) Group delay.

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