ACCURATE NORMALIZED FREQUENCY ESTIMATION BY THE THREE-POINT INTERPOLATED DFT METHOD WITH RECTANGULAR WINDOW

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ABSTRACT

This paper focuses on the estimation of a sine-wave normalized frequency by the three-point interpolated discrete Fourier transform (IpDFT) method with rectangular window. A constraint on the integer number of recorded sine-wave cycles is derived to ensure that the spectral interferences from the image components of the spectrum on the normalized frequency estimation are practically neglected. The statistically efficiency of this method, which applies to a sine-wave corrupted by a stationary white noise is investigated with respect to the unbiased Cramer-Rao (CR) lower bound. Moreover, a constraint on the number of acquired samples which ensures accurate estimates with a high confidence level is derived. The derived expressions are validated by means of computer simulations.

1. INTRODUCTION

In many engineering applications the normalized frequency of a sine-wave must be estimated with high accuracy. Methods used for this purpose can be classified either in parametric or in nonparametric methods [1]. The latter are often used since they do not require a rigid signal model and can be easily implemented due to the availability of Fourier analysis package.

In practice the sampling process is often noncoherent with the sine wave (noncoherent mode). One of the nonparametric methods often used in this case to estimate the normalized frequency of a sine-wave is the Interpolated Discrete Fourier transform (IpDFT) method [2]-[6]. Its most simple implementation is achieved when the rectangular window is used [2]. In this case the normalized frequency estimates are affected by the spectral interferences from the image components of the spectrum and by the wide band noise superimposed to the acquired sine-wave [7]. The influence of the spectral interferences depends on the integer number of recorded sine-wave cycles and the influence of the noise depends on the number of acquired samples [7]. For the values of the integer number of recorded sine-wave cycles used in practice the accuracy of the normalized frequency estimates is mainly affected by the influence of the spectral interferences. To overcome this situation a higher number of interpolation points must be used [8]. The analytical formula for frequency estimation by the three-point IpDFT method with rectangular window is given in [8]. Unfortunately, in the scientific literature the influence of the spectral interferences and of the noise on the estimation of a sine-wave normalized frequency by the IpDFT method with rectangular window is not investigated. Therefore, this paper is focused on this task. A constraint on the integer number of recorded sine-wave cycles which ensures that the influence of the spectral interferences from the image components of the spectrum on the estimation of the normalized frequency is practically neglected is derived. Then, for a sine-wave corrupted by a stationary white noise, the statistically efficiency of the method used with respect to the unbiased Cramer-Rao (CR) lower bound is determined. In addition, a constraint on the number of acquired samples which ensures accurate normalized frequency estimates with a high confidence level is derived. The validity of the derived expressions is verified by means of computer simulations.

2. NORMALIZED FREQUENCY ESTIMATION

Let us consider a sine-wave sampled at a known frequency f_s, i.e.,

$$x(m) = A\sin\left(2\pi \frac{f_{in}}{f_s} m + \phi\right), \quad m = 0,1,\ldots, M-1 \qquad (1)$$

where A, f_{in}, and ϕ are the amplitude, frequency, and phase of the sine-wave; and M is the acquisition length. To satisfy the Nyquist criterion, f_{in} is assumed to be smaller than $f_s/2$. The ratio between f_{in} and f_s is given by:

$$\frac{f_{in}}{f_s} = \frac{\lambda_0}{M} = \frac{l+\delta}{M}, \qquad (2)$$

where l and δ, respectively are the integer and the fractional parts of the number of recorded sine-wave cycles λ_0. δ is related to the noncoherent sampling mode $-0.5 \le \delta < 0.5$. This mode is very common in practical applications.

From (1) it follows that $x(m)$ can be written as following:

$$x(m) = x_w(m) = x(m)\cdot w(m), \qquad (3)$$

where $w(m)$ is the rectangular (uniform) window, defined as:

$$w(m) = \begin{cases} 1, & 0 \le m \le M-1 \\ 0, & otherwise \end{cases}.$$

The discrete-time Fourier transform (DTFT) of $x_w(\cdot)$ is given by:

$$X_w(\lambda)=\frac{A}{2j}\left[W(\lambda-\lambda_0)e^{j\phi}-W(\lambda+\lambda_0)e^{-j\phi}\right],\quad \lambda\in[0,M) \quad (4)$$

where λ is the normalized frequency expressed in bin, $W(\cdot)$ is the DTFT of $w(\cdot)$, given by:

$$W(\lambda)=\frac{M\sin(\pi\lambda)}{\pi\lambda}e^{-j\frac{M-1}{M}\lambda}. \quad (5)$$

Noticed that the second term in (4) represents the image part of the spectrum $X_w(\cdot)$.

The value of l can be easily determined by means of (2) if enough accurate estimates for f_{in} and f_s are available or by using a maximum search routine applied to $|X_w(\lambda)|$ for λ = 1, 2,…, $M/2 - 1$, if the sine-wave frequency signal-to-noise ratio is above threshold [4]. Thus, for estimating the normalized frequency λ_0, it is necessarily the estimation of δ.

To estimate δ by means of the three-point IpDFT method the following ratio α is firstly determined [8]:

$$\alpha=\frac{s|X_w(l+1)|+s|X_w(l-1)|}{2|X_w(l)|+s\left[|X_w(l+1)|-|X_w(l-1)|\right]}, \quad (6)$$

where $s = \text{sgn}(\delta)$, in which $\text{sgn}(\cdot)$ is the sign function.

From (4) the following equations hold:

$$|X_w(l-1)|=\frac{A}{2}\left|\left[W(-1-\delta)e^{j\phi}-W(2l-1+\delta)e^{-j\phi}\right]\right|$$

$$|X_w(l)|=\frac{A}{2}\left|\left[W(-\delta)e^{j\phi}-W(2l+\delta)e^{-j\phi}\right]\right| \quad (7)$$

$$|X_w(l+1)|=\frac{A}{2}\left|\left[W(1-\delta)e^{j\phi}-W(2l+1+\delta)e^{-j\phi}\right]\right|.$$

For larger value of l the image parts of $X_w(l-1)$, $X_w(l)$ and $X_w(l+1)$ are neglected and for (7) α becomes:

$$\alpha\cong\frac{s|W(1+\delta)|+s|W(1-\delta)|}{2|W(\delta)|+s\left[|W(1-\delta)|-|W(1+\delta)|\right]}. \quad (8)$$

Using (5) it follows that δ can be estimated by:

$$\hat{\delta}=\alpha=\frac{s|X_w(l+1)|+s|X_w(l-1)|}{2|X_w(l)|+s\left[|X_w(l+1)|-|X_w(l-1)|\right]}. \quad (9)$$

For small and relative small values of l the $\hat{\delta}$ estimates are affected by the image parts of $X_w(l-1)$, $X_w(l)$, and $X_w(l+1)$ noted in the following as $X_{wi}(l-1)$, $X_{wi}(l)$, and $X_{wi}(l+1)$

3. CONSTRAINT ON THE INTEGER PART l

For a sine-wave, the influence of the spectral interferences from the image components of the spectrum on the δ estimates obtained by the IpDFT with rectangular window are practically neglected if the following constraint on the l, is fulfilled [7]:

$$4l-1>100\cdot 2^n\frac{2A}{FSR}, \quad (10)$$

where A is sine-wave amplitude, n is the resolution of the used Analog-to-Digital Converter (ADC) and FSR is its full-scale range.

The previously condition cannot be fulfilled for the value of l used in practice. For example if A = 1, n = 8 bits, and FSR = 5, the minimum value of l which satisfy (10) is equal to 2561, which is a very high value. For higher value of n, it is obvious that higher values of l are required. Therefore, in practice always the estimation of δ by the IpDFT method with rectangular window is affected by the spectral interferences.

The aim of this section is to derive a constraint on the l which ensures that the spectral interferences from the image components of the spectrum practically not affect the $\hat{\delta}$ estimates obtained by the three-point IpDFT method with rectangular window, given by (9). Afterthen, the minimum values required for l are compared with those required by the IpDFT method.

It can be established that:

$$\begin{aligned}&err_1\\&=s|X_w(l+1)|+s|X_w(l-1)|-\frac{A}{2}s|W(1-\delta)|-\frac{A}{2}s|W(1+\delta)|\\&=a_1\cdot s\cdot e_1+b_1,\\&err_2=s|X_w(l+1)|-s|X_w(l-1)|+2|X_w(l)|\\&-\frac{A}{2}s|W(1-\delta)|+\frac{A}{2}s|W(1+\delta)|-A|W(\delta)|=a_2\cdot s\cdot e_2+b_2,\end{aligned} \quad (11)$$

where:

$$\begin{aligned}e_1&=\frac{A}{2}|W(2l-1+\delta)|-\frac{A}{2}|W(2l+1+\delta)|\\&=|X_{wi}(l-1)|-|X_{wi}(l+1)|,\\e_2&=A|W(2l+\delta)|-\frac{A}{2}|W(2l-1+\delta)|-\frac{A}{2}|W(2l+1+\delta)|\\&=2|X_{wi}(l)|-|X_{wi}(l-1)|-|X_{wi}(l+1)|,\end{aligned}$$

and a_i, b_i, i = 1, 2, are the coefficients of the best line-fit corresponding to the dependences $err_1 = f(e_1)$ and $err_2 = f(e_2)$. They have the following characteristics: $\max_{\varphi,\delta}\{|a_i|\}\cong 1$ and $b_i \cong 0$.

Fig. 1 shows the coefficients a_i and b_i, i = 1, 2, as a function of δ and φ. The sine wave has the amplitude A = 1 and the phase φ variable in the range [0, 2π) rad with a step of $\pi/50$ rad. M is set to 1024. δ varies in the range [–0.5, 0.5) with a step of 1/33 and l varies in the range [3, 50) with an step of 1. Fig. 2 shows the coefficients a_i and b_i, i = 1, 2 as a function of φ for $\delta = -1/6$ (Fig. 2(a)) and as a function of δ for $\varphi = 2\pi/5$ rad (Fig. 2(b)).

It is obvious that the errors due to the spectral interference decrease as $|e_1|$ decreases as well $|e_2|$. Using (5) and (7) it can be obtained:

$$\frac{|e_1|}{|e_2|}=2l+\delta>1. \quad (12)$$

Due to the fact that $|e_1| > |e_2|$, it follows that very small errors due to the spectral interferences are obtained when $|e_1|$ is very small.

The magnitude $|X_{wi}(l-1)|$ can be considered equal to the magnitude corresponding to a sine-wave of amplitude A_1, phase ϕ_1, and frequency f_{in} sampled at f_s frequency. Thus, the following equality holds on:

$$\frac{A}{2}|W(2l+\delta-1)|=\frac{A_1}{2}\left|\left[W(\delta)e^{j\phi_1}-W(2l+\delta)e^{-j\phi_1}\right]\right|. \quad (13)$$

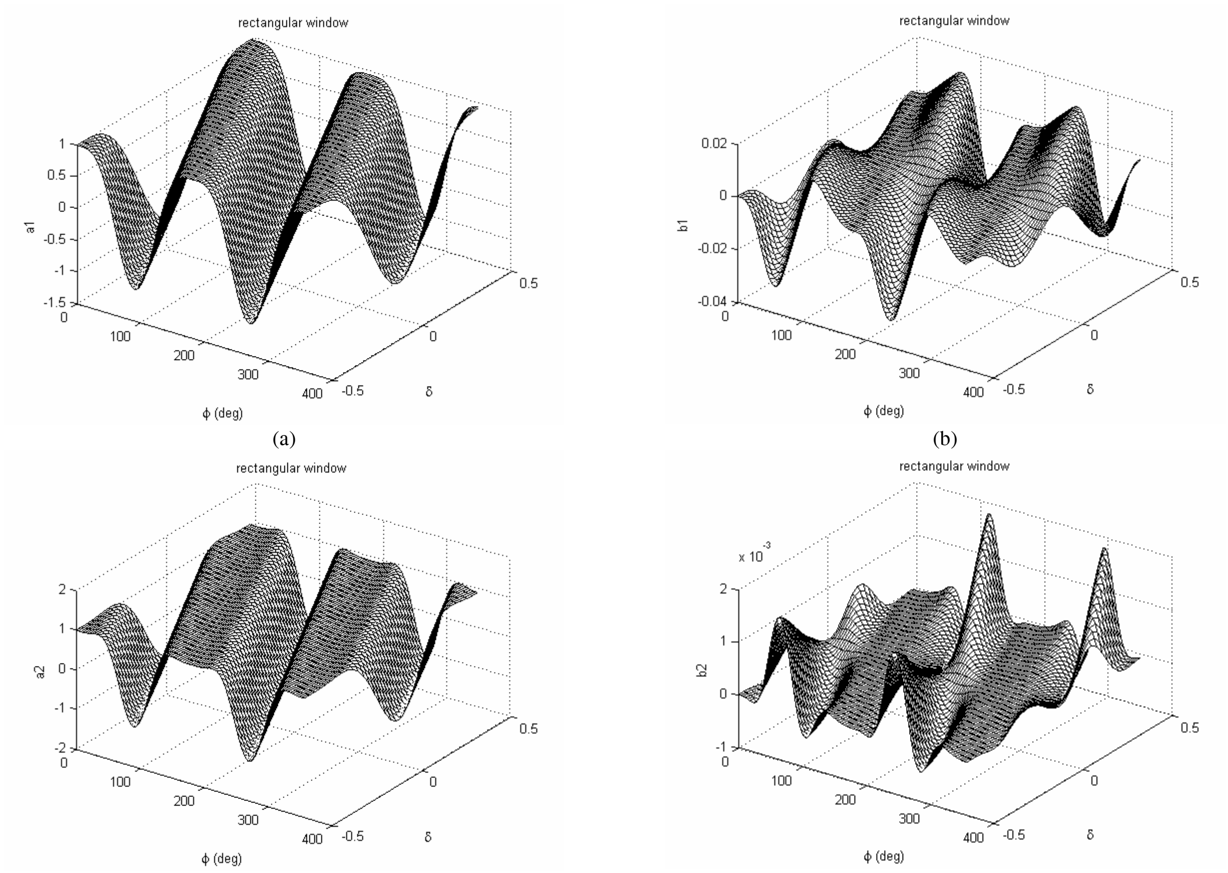


Fig. 1. Coefficients a_i and b_i, i = 1, 2 as a function of δ and ϕ.

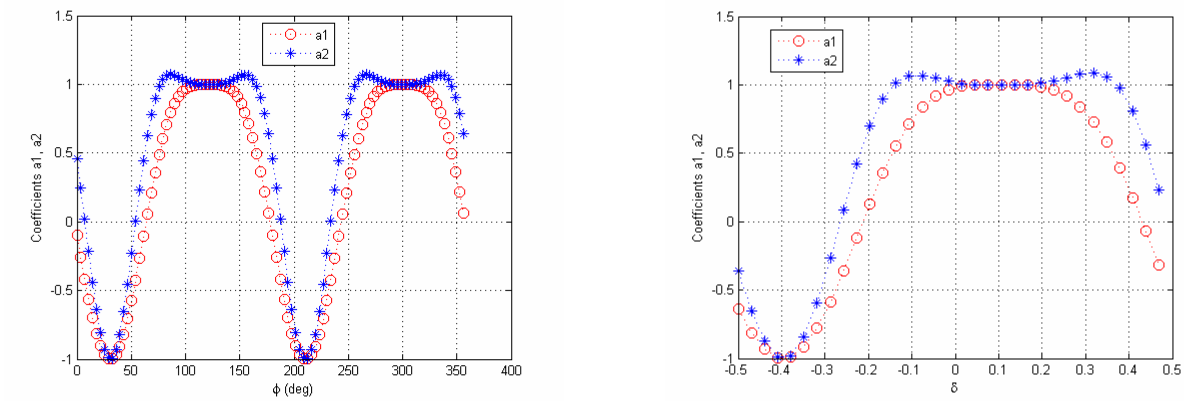


Fig. 2. Coefficients a_i and b_i, i = 1, 2 as a function of: (a) ϕ for $\delta = -1/6$, (b) δ for $\phi = 2\pi/5$ rad.

For large enough l from (5) it follows that $|W(2l+\delta)|$ is very small compared with $|W(-\delta)|$. Thus, we have:

$$A_1 \cong A\frac{|W(2l+\delta-1)|}{|W(\delta)|} = A\frac{|\delta|}{2l+\delta-1}. \qquad (14)$$

Similar the magnitude $|X_{wi}(l+1)|$ can be considered equal to the magnitude corresponding to a sine-wave of amplitude A_2, phase ϕ_2, and frequency f_{in} sampled at f_s frequency:

$$\frac{A}{2}|W(2l+\delta+1)| = \frac{A_2}{2}\left|\left[W(\delta)e^{j\phi_2} - W(2l+\delta)e^{-j\phi_2}\right]\right|. \qquad (15)$$

and

$$A_2 \cong A\frac{|W(2l+\delta+1)|}{|W(\delta)|} = A\frac{|\delta|}{2l+\delta+1}. \qquad (16)$$

The error $|e_1|$ is proportional to the difference between A_1 and A_2, ΔA. The maximum of ΔA, and also of the error $|e_1|$, is obtained for $\delta = -0.5$, and is equal to:

$$(\Delta A)_{\max} = (A_1 - A_2)\big|_{\delta=-0.5} = \frac{4A}{(4l-1)(4l+3)}. \qquad (17)$$

Due to the digitized process the quantization noise is always present in sampled sine-waves. To obtain very small errors due to the spectral interferences A_1 and A_2 should be smaller than the amplitude of the quantization errors, which is equal to $q/2$, in which q is the quantization step of the used ADC, for all values of δ. This condition imply that $(\Delta A)_{\max}$ must be μ-times smaller than $q/2$, where $\mu > 1$, i.e.,

$$\frac{4A}{(4l-1)(4l+3)} > \frac{q}{2\mu}. \qquad (18)$$

If the quantization noise is generated by an n-bit ADC, with full-scale range FSR ($FSR \geq 2A$), then $q = FSR/2^n$, and the following constraint on the l can be established:

$$(4l-1)(4l+3) > 2^{n+2}\mu\frac{2A}{FSR}. \qquad (19)$$

To obtain very high accurate $\hat{\delta}$ estimates, it has been established by means of computer simulation that a good choice is $\mu = 50$ [see section (6)]. In this case the previously relationship becomes:

$$(4l-1)(4l+3) > 50 \cdot 2^{n+2}\frac{2A}{FSR}. \qquad (20)$$

For example if $A = 1$, $n = 8$ bits, and $FSR = 5$, the minimum value of l which satisfy (20) is equal to 37, i.e. 69-times smaller than the minimum value required when the IpDFT method is used. It should be noticed that this methods is well suited when ADCs with low and medium resolutions are employed in the digitizing process. Otherwise, higher values for l are required.

4. STATISTICAL EFFICIENCY OF THE METHOD

In order to accurately model common real-life situations, we assume that a stationary white noise with zero mean and variance σ_n^2 is added to the sine-wave.

By applying the low of uncertainty propagation [9] on (9) after some calculations the following expression for the standard deviation of the $\hat{\delta}$ estimates is obtained:

$$\sigma_{\hat{\delta}} = \frac{\pi\delta}{\sin(\pi\delta)} \cdot \frac{(1-\delta^2)\sqrt{3\delta^2+1}}{A\sqrt{M}} \cdot \sigma_n. \qquad (21)$$

The unbiased CR lower bound for the δ estimates is approximately [4]:

$$(\sigma_{\hat{\delta}})_{CR} = \frac{\sqrt{6}}{\pi} \cdot \frac{\sigma_n}{A\sqrt{M}}. \qquad (22)$$

From (21) and (22) it follows that the statistically efficiency of the three-point IpDFT method is given by:

$$\frac{\sigma_{\hat{\delta}}}{(\sigma_{\hat{\delta}})_{CR}} = \frac{\pi^2\delta(1-\delta^2)\sqrt{18\delta^2+6}}{6\sin(\pi\delta)}. \qquad (23)$$

The maximum of the $\sigma_{\hat{\delta}}$, $\sigma_{\hat{\delta}_{\max}}$, is obtained for $\delta = -0.5$ (see Fig. 4), and it is approximately two-times higher than the $(\sigma_{\hat{\delta}})_{CR}$.

5. CHOICE OF THE NUMBER OF SAMPLES

When the sine-wave is corrupted by a stationary white noise the normalized frequency estimator exhibits an almost normal distribution [4]. Thus, if the maximum acceptable normalized frequency absolute error $|\Delta\delta|$ is V_δ, the following constraint must be satisfied:

$$c \cdot \sigma_{\hat{\delta}_{\max}} < V_\delta, \qquad (24)$$

where c is the suitable coverage factor (i.e. $c = 3$).

From (21) and (24) after some calculations we can obtain the following constraint on the M:

$$M \geq \frac{2.43c^2}{A^2} \cdot \frac{\sigma_n^2}{V_\delta^2}. \qquad (25)$$

To reduce the DFT computational effort, M should be chosen equal to the minimum integer power of two satisfying (25).

6. SIMULATION RESULTS

The aim of this section is to verify the validity of the relationships (20), (23), and (25) by means of the computer simulations.

a) Validation of the constraint (20)

The sine-wave used in simulation is characterized by $A = 1$, and φ variable in the range $[0, 2\pi)$ rad with a step equal to $\pi/50$ rad. M is set to 1024. δ varies in the range $[-0.5, 0.5)$ with a step equal to 1/25. The sine-wave is corrupted by the quantization noise of a bipolar ADC with FSR equal to 5. The quantization noise was modeled by a uniformly distributed additive noise. For each value of δ the maximum of the absolute error of δ, $|\Delta\delta|_{\max}$, occurring during the phase scan is retained. Fig. 3 shows the $|\Delta\delta|_{\max}$ as a function δ for different value of l when the ADC resolution is equal to 8 bits and to 10 bits. The second value used for l is equal to the minimum value of l satisfying (20).

As it can be seen from Fig. 3 when the l values are higher than or equal to the minimum value given by the constraint (20) the $\hat{\delta}$ estimates are very close since they are not practically affected by the spectral interferences.

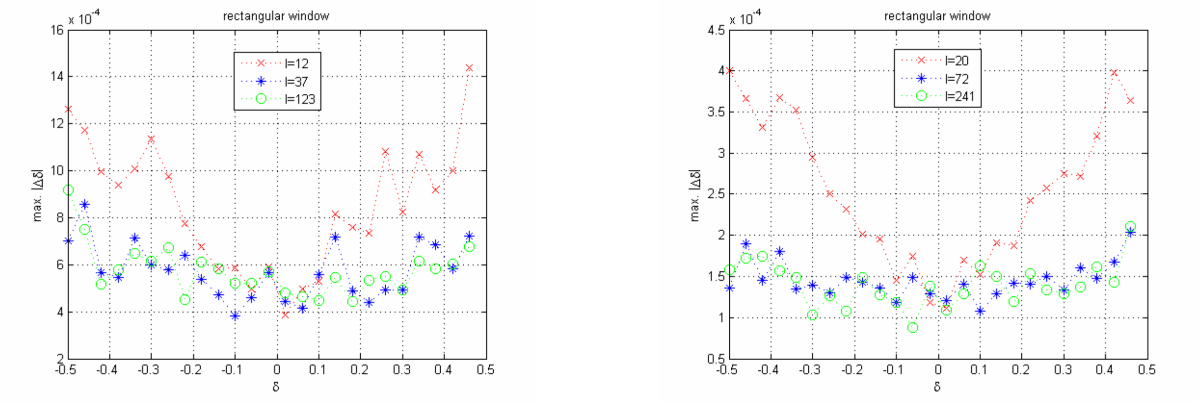
b) Validation of the expression (23)

The sine-wave used in simulation is characterized by $A = 1$, φ uniformly distributed in the range $[0, 2\pi)$ rad, and $l = 123$. M is set to 1024. δ varies in the range $[-0.5, 0.5)$ with a step equal to 1/25. The sine-wave is corrupted by the quantization noise of a bipolar 10-bit ADC with FSR equal to 5. For each value of δ, 5000 runs are done to calculate the standard deviation of the $\hat{\delta}$ estimates. Fig. 4 shows the efficiency of $\hat{\delta}$ estimates as a function of δ obtained by both (23) and computer simulation.

In Fig. 4 it can be seen the good agreement between the simulations and theoretical results obtained by (23).

c) Validation of the constraint (25)

δ varies in the range $[-0.5, 0.5)$ with a step equal to 1/50. In rest the same characteristics as in Fig. 4 are used. The maximum allowed error V_δ is obtained from (25) for $M = 1024$ ($V_\delta \cong 2\cdot10^{-4}$). For each value of δ, 1000 runs are performed and the value $|\Delta\delta|_{\max}$ is retained. Fig. 5 shows the number of occurrences of the errors $|\Delta\delta|$ higher than V_δ as a function of δ. From Fig. 5 it follows that in the worst case the probability to have $\hat{\delta}$ estimates with absolute errors $|\Delta\delta|$ smaller than V_δ is equal to 99.7%, which is very close to the theoretical ones, that is 99.73%.



(a) n = 8 bits.

(b) n = 10 bits

Fig. 3. $|\Delta\delta|_{max}$ as a function δ for different value of l when the ADC resolution is equal to: (a) 8 bits, (b) 10 bits.

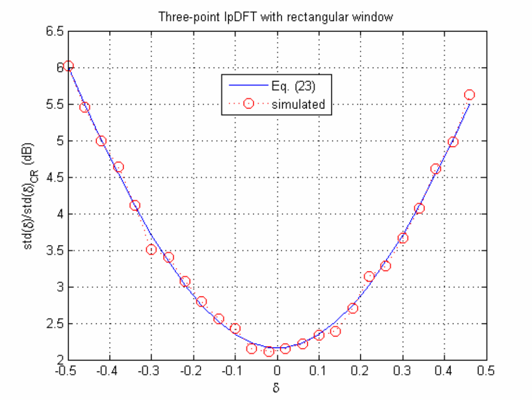


Fig. 4. Standard deviation of the $\hat{\delta}$ estimates normalized to the corresponding CR lower bound.

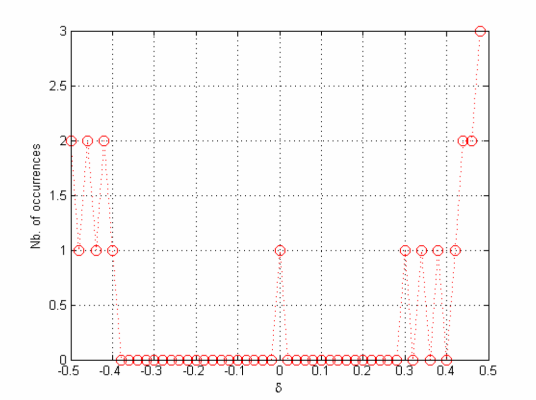


Fig. 5. Number of error occurrences higher than the maximum allowed error V_δ obtained from (25) for M = 1024.

7. CONCLUSION

This paper has been focused on the estimation of the normalized frequency by the three-point IpDFT method with rectangular window. A constraint on the integer number of recorded sine-wave cycles which ensures that the spectral interferences from the image components of the spectrum on the δ estimates (and also on the normalized frequency λ_0 estimates) are practically neglected is derived. Based on this constraint it has been shown that the three-point IpDFT method is more efficient in reducing the systematic errors than the IpDFT method. In addition, it has been demonstrated that the maximum of the standard deviation of the δ estimates is two-times higher than the unbiased CR lower bound. Finally, a constraint on the minimum number of acquired samples that ensures the estimation with a given accuracy of the normalized frequency has been derived. All the derived expressions have been validated by means of computer simulations. The analyzed method is recommended to be used when in the digitizing process ADCs with low and medium resolutions are used. It should be noticed that this method is very simple to implement. Therefore, it is well suited for real-time measurements of the normalized frequency.

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