ON OPTIMAL SENSOR PLACEMENT FOR TIME-DIFFERENCE-OF-ARRIVAL LOCALIZATION UTILIZING UNCERTAINTY MINIMIZATION

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ABSTRACT

In this paper we investigate optimal sensor placement for time-difference-of-arrival localization. The minimization of estimation uncertainty is used as the optimization criterion, which is equivalent to maximization of the determinant of the Fisher information matrix. It is shown that for equal sensor noise variances equiangular sensor separation is an optimal localization geometry irrespective of the emitter range from sensors. For six or more sensors the optimal geometry is not unique. If the sensor noise variances are different, equiangular sensor separation does not always correspond to an optimal configuration. These results are illustrated with numerical examples.

1. INTRODUCTION

In this paper we consider optimal placement of sensors for time-difference-of-arrival (TDOA) emitter localization in the 2D plane. As optimality criterion we adopt minimization of the area of estimation confidence region, which is equivalent to maximization of the determinant of the Fisher information matrix [1].

The objective of emitter localization is to determine the location of an emitter by processing signals originating from the emitter. Multiple sensors or a moving sensor platform is utilized to process the received signals. Passive emitter localization is extensively employed in many civilian and defence applications such as mobile communications, wireless sensor networks and electronic warfare. There are several techniques available for emitter localization, including angle of arrival (AOA), time of arrival (TOA), time difference of arrival, scan time, Doppler shift and received signal strength localization. Hybrid localization techniques combining some of these techniques are also available.

Optimal sensor placement methods for AOA and scan based localization have been developed in [2, 3]. In dynamic sensor placement problems involving moving sensors the Fisher information matrix may be approximated by replacing the unknown true emitter location with its maximum likelihood estimate [2]. In this paper we extend the idea of uncertainty minimization to optimal sensor placement for TDOA localization. When deriving the optimality conditions for TDOA sensor placement we also note some fundamental differences between optimal AOA and TDOA geometries.

The paper is organized as follows. Section 2 provides an overview of the TDOA localization problem. In Section 3 the optimization problem is defined. The main results of the paper relating to optimal sensor placement are obtained in Section 4. Optimal TDOA geometries are discussed in Section 5. The paper concludes in Section 6.

2. OVERVIEW OF TDOA LOCALIZATION

The objective of 2-D TDOA emitter localization is to determine the location of an emitter $\boldsymbol{s} = [x, y]^T$ (where ^{T} denotes matrix transpose) by utilizing $N-1$ TDOA measurements obtained from N sensors at known locations $\boldsymbol{r}_i = [x_i, y_i]^T$, $i = 1, \ldots, N$.

TDOA between signals received at a pair of sensors is defined by

$$t_{ij} = t_j - t_i, \quad i, j \in \{1, \ldots, N\} \tag{1}$$

where t_i is the propagation time for the signal transmitted by the emitter to arrive at sensor i, i.e.,

$$t_i = \frac{\|\boldsymbol{d}_i\|}{c}, \quad i \in \{1, \ldots, N\}. \tag{2}$$

Here c is the speed of propagation for the transmitted signal, $\|\cdot\|$ denotes the Euclidean norm, and $\boldsymbol{d}_i$ is the emitter range vector from the sensor at $\boldsymbol{r}_i$:

$$\boldsymbol{d}_i = \boldsymbol{s} - \boldsymbol{r}_i, \quad i \in \{1, \ldots, N\}. \tag{3}$$

An appealing feature of TDOA localization is that it does not require knowledge of transmission time of a received signal. Using (1) and (2), the range difference of arrival (RDOA), g_{ij}, for signals received at sensors i and j is given by

$$g_{ij} = \|\boldsymbol{d}_j\| - \|\boldsymbol{d}_i\|, \quad i, j \in \{1, \ldots, N\} \tag{4a}$$
$$= c t_{ij}. \tag{4b}$$

Each RDOA defines a hyperbola of possible emitter locations. In TDOA emitter localization, it is common practice to nominate one of the sensors as the reference sensor and take all TDOA measurements with respect to it. We will assume that the sensor at $\boldsymbol{r}_1$ is the reference sensor. Given the RDOAs with respect to the reference sensor, g_{1i}, $i = 2, \ldots, N$, the emitter location $\boldsymbol{s}$ is obtained by solving the following set of nonlinear equations:

$$\begin{aligned} \|\boldsymbol{s} - \boldsymbol{r}_2\| - \|\boldsymbol{s} - \boldsymbol{r}_1\| &= g_{12} \\ \|\boldsymbol{s} - \boldsymbol{r}_3\| - \|\boldsymbol{s} - \boldsymbol{r}_1\| &= g_{13} \\ &\vdots \\ \|\boldsymbol{s} - \boldsymbol{r}_N\| - \|\boldsymbol{s} - \boldsymbol{r}_1\| &= g_{1N}. \end{aligned} \tag{5}$$

To solve the above set of equations requires a minimum of two equations (i.e., $N \geq 3$) since there are two unknowns, viz., the x and y coordinates of the emitter. To guarantee uniqueness of the solution, it may be necessary to have more than three sensors (i.e., $N > 3$).

In practice, RDOAs are estimated from received noisy signals. For continuous-wave signals, the RDOAs can

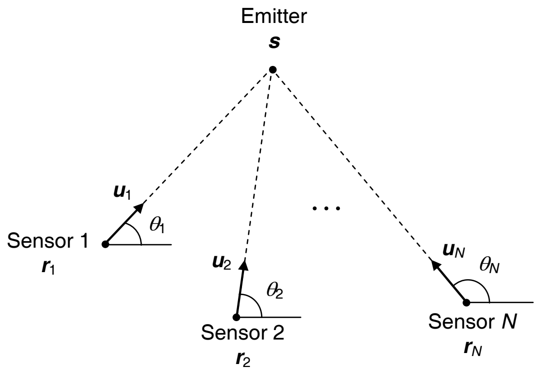

Figure 1: TDOA localization and directional unit vectors $\boldsymbol{u}_i$.

be estimated by using the method of generalized cross-correlation [4]. The resulting RDOA measurements $\hat{g}_{1i}$ can be expressed as

$$\hat{g}_{1i} = g_{1i} + n_{1i}, \quad i = 2, \dots, N \tag{6}$$

where $n_{1i} = c(e_i - e_1)$ is the RDOA noise with e_i denoting the time-of-arrival (TOA) estimation error. We assume that e_i is an independent Gaussian noise with zero mean and variance $E\{e_i^2\} = \sigma_i^2/c^2$. The RDOA noise variance $E\{n_{1i}^2\}$ is usually a function of the emitter range among other things. The covariance matrix of RDOA measurements is given by

$$\boldsymbol{\Sigma} = E\left\{\begin{bmatrix} n_{12} \\ \vdots \\ n_{1N}\end{bmatrix} [n_{12} \quad \cdots \quad n_{1N}]\right\} \tag{7a}$$

$$= \sigma_1^2 \mathbf{1} + \underbrace{\begin{bmatrix} \sigma_2^2 & & & \mathbf{0} \\ & \sigma_3^2 & & \\ & & \ddots & \\ \mathbf{0} & & & \sigma_N^2 \end{bmatrix}}_{\boldsymbol{T}} \tag{7b}$$

where $\mathbf{1}$ is the $(N-1) \times (N-1)$ matrix of ones.

3. PROBLEM STATEMENT

Suppose that the RDOA noise covariance matrix in (7b) is given and constant. Then the Fisher information matrix (FIM) for TDOA localization is given by [5]

$$\boldsymbol{\Phi} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} = \boldsymbol{J}_o^T \boldsymbol{\Sigma}^{-1} \boldsymbol{J}_o \tag{8}$$

where $\boldsymbol{J}_o$ is the $(N-1) \times 2$ Jacobian matrix evaluated at the true emitter location

$$\boldsymbol{J}_o = \begin{bmatrix} (\boldsymbol{u}_2 - \boldsymbol{u}_1)^T \\ (\boldsymbol{u}_3 - \boldsymbol{u}_1)^T \\ \vdots \\ (\boldsymbol{u}_N - \boldsymbol{u}_1)^T \end{bmatrix}. \tag{9}$$

Here

$$\boldsymbol{u}_i = \frac{\boldsymbol{d}_i}{\|\boldsymbol{d}_i\|} = \begin{bmatrix} \cos\theta_i \\ \sin\theta_i \end{bmatrix} \tag{10}$$

is the unit vector pointing to the emitter (directional unit vector) from sensor i as illustrated in Fig. 1.

Using the matrix inversion lemma [6]:

$$(\boldsymbol{A} + \boldsymbol{BCD})^{-1} = \boldsymbol{A}^{-1} - \boldsymbol{A}^{-1}\boldsymbol{B}(\boldsymbol{C}^{-1} + \boldsymbol{DA}^{-1}\boldsymbol{B})^{-1}\boldsymbol{DA}^{-1} \tag{11}$$

with $\boldsymbol{A} = \boldsymbol{T}$, $\boldsymbol{B} = [1, \dots, 1]^T$, $\boldsymbol{C} = \sigma_1^2$ and $\boldsymbol{D} = \boldsymbol{B}^T$, the inverse of $\boldsymbol{\Sigma}$ in (7b) can be written as

$$\boldsymbol{\Sigma}^{-1} = \boldsymbol{T}^{-1} - a\boldsymbol{w}\boldsymbol{w}^T \tag{12}$$

where

$$a = \frac{1}{\sum_{i=1}^N 1/\sigma_i^2} \tag{13a}$$

$$\boldsymbol{w} = [1/\sigma_2^2, 1/\sigma_3^2, \cdots, 1/\sigma_N^2]^T. \tag{13b}$$

Substituting (9) and (12) into (8) we obtain

$$\boldsymbol{\Phi} = \sum_{i=2}^N \frac{1}{\sigma_i^2}(\boldsymbol{u}_i - \boldsymbol{u}_1)(\boldsymbol{u}_i - \boldsymbol{u}_1)^T - a\sum_{i=2}^N \frac{\boldsymbol{u}_i - \boldsymbol{u}_1}{\sigma_i^2} \sum_{i=2}^N \frac{(\boldsymbol{u}_i - \boldsymbol{u}_1)^T}{\sigma_i^2}. \tag{14}$$

We adopt maximization of the determinant of FIM as the optimal sensor placement criterion. Since the area of the 1-σ error ellipse (39.4% uncertainty region) of an efficient TDOA location estimator is given by $A_{1\sigma} = \pi/|\boldsymbol{\Phi}|^{1/2}$, where $|\cdot|$ denotes determinant, maximization of $|\boldsymbol{\Phi}|$ is equivalent to minimization of the area of the uncertainty ellipse [1]. Throughout the paper we assume that the TDOA localization algorithm under consideration is nearly efficient and unbiased so that its error covariance can be approximated by the Cramer-Rao lower bound (CRLB), which is given by the inverse of $\boldsymbol{\Phi}$.

4. THE MAIN RESULT

After some algebraic manipulations the entries of FIM can be written as

$$\phi_{11} = \sum_{i=2}^N \frac{(\cos\theta_i - \cos\theta_1)^2}{\sigma_i^2} - a\left(\sum_{i=2}^N \frac{\cos\theta_i - \cos\theta_1}{\sigma_i^2}\right)^2 \tag{15a}$$

$$= \sum_{i=1}^N \frac{\cos^2\theta_i}{\sigma_i^2} - a\left(\sum_{i=1}^N \frac{\cos\theta_i}{\sigma_i^2}\right)^2 \tag{15b}$$

$$\phi_{12} = \phi_{21} \tag{16a}$$

$$= \sum_{i=2}^N \frac{(\cos\theta_i - \cos\theta_1)(\sin\theta_i - \sin\theta_1)}{\sigma_i^2} - a\sum_{i=2}^N \frac{\cos\theta_i - \cos\theta_1}{\sigma_i^2} \sum_{j=2}^N \frac{\sin\theta_j - \sin\theta_1}{\sigma_j^2} \tag{16b}$$

$$= \frac{1}{2}\sum_{i=1}^N \frac{\sin 2\theta_i}{\sigma_i^2} - a\sum_{i=1}^N \frac{\sin\theta_i}{\sigma_i^2} \sum_{j=1}^N \frac{\cos\theta_j}{\sigma_j^2} \tag{16c}$$

$$\phi_{22} = \sum_{i=2}^N \frac{(\sin\theta_i - \sin\theta_1)^2}{\sigma_i^2} - a\left(\sum_{i=2}^N \frac{\sin\theta_i - \sin\theta_1}{\sigma_i^2}\right)^2 \tag{17a}$$

$$= \sum_{i=1}^N \frac{\sin^2\theta_i}{\sigma_i^2} - a\left(\sum_{i=1}^N \frac{\sin\theta_i}{\sigma_i^2}\right)^2. \tag{17b}$$

The determinant of FIM is

$$
\begin{aligned}
|\boldsymbol{\Phi}| &= \phi_{11}\phi_{22} - \phi_{12}\phi_{21} \\
&= \frac{1}{4a^2} - \frac{1}{4}\left(\sum_{i=1}^{N} \frac{\sin 2\theta_i}{\sigma_i^2}\right)^2 - \frac{1}{4}\left(\sum_{i=1}^{N} \frac{\cos 2\theta_i}{\sigma_i^2}\right)^2 \\
&\quad - a\sum_{i=1}^{N} \frac{1}{\sigma_i^2}\left(\sin\theta_i \sum_{j=1}^{N} \frac{\cos\theta_j}{\sigma_j^2} - \cos\theta_i \sum_{j=1}^{N} \frac{\sin\theta_j}{\sigma_j^2}\right)^2.
\end{aligned} \tag{18}
$$

Considering that the last three terms in the right-hand side of the above equation can only be negative or zero, for fixed receiver noise variances σ_i^2, $i = 1, \ldots, N$ (i.e., fixed $\boldsymbol{\Sigma}$), the determinant of FIM is maximized by directional unit vector angles $\boldsymbol{\theta} = [\theta_1, \ldots, \theta_N]^T$ solving the following minimization problem

$$
\begin{aligned}
\min_{\boldsymbol{\theta}} &\left(\sum_{i=1}^{N} \frac{\sin 2\theta_i}{\sigma_i^2}\right)^2 + \left(\sum_{i=1}^{N} \frac{\cos 2\theta_i}{\sigma_i^2}\right)^2 \\
&+ 4a\sum_{i=1}^{N} \frac{1}{\sigma_i^2}\left(\sin\theta_i \sum_{j=1}^{N} \frac{\cos\theta_j}{\sigma_j^2} - \cos\theta_i \sum_{j=1}^{N} \frac{\sin\theta_j}{\sigma_j^2}\right)^2.
\end{aligned} \tag{19}
$$

Define

$$
\boldsymbol{v}_i = \begin{bmatrix} \cos 2\theta_i \\ \sin 2\theta_i \end{bmatrix}. \tag{20}
$$

Then the minimization problem in (19) can be rewritten as

$$
\begin{aligned}
\min_{\boldsymbol{\theta}} &\left\| \sum_{i=1}^{N} \frac{\boldsymbol{v}_i}{\sigma_i^2} \right\|^2 + 4a \sum_{i=1}^{N} \frac{1}{\sigma_i^2} \\
&\times \left([\sin\theta_i \quad -\cos\theta_i] \sum_{j=1}^{N} \frac{\boldsymbol{u}_j}{\sigma_j^2} \right)^2
\end{aligned} \tag{21}
$$

or more compactly as

$$
\min_{\boldsymbol{\theta}} \left\| \sum_{i=1}^{N} \frac{\boldsymbol{v}_i}{\sigma_i^2} \right\|^2 + 4a \left\| \sum_{i=1}^{N} \frac{\boldsymbol{u}_i}{\sigma_i^2} \right\|_{\boldsymbol{H}}^2 \tag{22}
$$

where $\|\boldsymbol{x}\|_{\boldsymbol{H}}^2 = \boldsymbol{x}^T \boldsymbol{H} \boldsymbol{x}$ and

$$
\boldsymbol{H} = \sum_{i=1}^{N} \frac{1}{\sigma_i^2} \begin{bmatrix} \sin\theta_i \\ -\cos\theta_i \end{bmatrix} [\sin\theta_i \quad -\cos\theta_i]. \tag{23}
$$

We formally have the following result.

Theorem 1. *For fixed* $\boldsymbol{\Sigma}$ *the determinant of FIM is maximized by*

$$
\min_{\boldsymbol{\theta}} J(\boldsymbol{\theta}) \tag{24}
$$

where

$$
J(\boldsymbol{\theta}) = \left\| \sum_{i=1}^{N} \frac{\boldsymbol{v}_i}{\sigma_i^2} \right\|^2 + 4a \left\| \sum_{i=1}^{N} \frac{\boldsymbol{u}_i}{\sigma_i^2} \right\|_{\boldsymbol{H}}^2 \geq 0 \tag{25}
$$

and $|\boldsymbol{\Phi}| = \frac{1}{4}\left(\frac{1}{a^2} - J(\boldsymbol{\theta})\right)$.

Note that the emitter range does not appear in $J(\boldsymbol{\theta})$ explicitly, and for given σ_i^2 it is dependent on angular sensor separation only. This implies that optimal sensor placement is independent of how far the sensors are from the emitter. This observation is in sharp contrast to optimal AOA sensor placement in which the emitter range plays an important role [3]. However the absence of emitter range in $J(\boldsymbol{\theta})$ has limited practical use because of the close coupling between the sensor noise variance and emitter range as governed by signal-to-noise ratio (SNR).

CRLB is related to FIM via

$$
\text{CRLB} = \boldsymbol{\Phi}^{-1} = \frac{1}{|\boldsymbol{\Phi}|} \begin{bmatrix} \phi_{22} & -\phi_{12} \\ -\phi_{21} & \phi_{11} \end{bmatrix}. \tag{26}
$$

The optimal MSE is given by the trace of CRLB:

$$
\begin{aligned}
\text{MSE} &= \frac{\phi_{11} + \phi_{22}}{|\boldsymbol{\Phi}|} \\
&= \frac{\sum_{i=1}^{N} \frac{1}{\sigma_i^2} - a\left(\left(\sum_{i=1}^{N} \frac{\sin\theta_i}{\sigma_i^2}\right)^2 + \left(\sum_{i=1}^{N} \frac{\cos\theta_i}{\sigma_i^2}\right)^2 \right)}{|\boldsymbol{\Phi}|} \\
&= \frac{\sum_{i=1}^{N} \frac{1}{\sigma_i^2} - a \left\| \sum_{i=1}^{N} \frac{\boldsymbol{u}_i}{\sigma_i^2} \right\|^2}{|\boldsymbol{\Phi}|}.
\end{aligned} \tag{27}
$$

It has been shown that in AOA geometry optimization the maximization of the determinant of FIM is equivalent to minimization of MSE [3]. The equivalence holds whether or not sensor noise variances are identical. We will show by way of an example that this equivalence does not hold for TDOA localization when sensor noise variances are different.

Consider the TDOA localization scenario involving three sensors with receiver noise variances $\sigma_1^2 = 0.5$, $\sigma_2^2 = 0.1$ and $\sigma_3^2 = 0.2$. The emitter is located at the origin $\boldsymbol{s} = [0, 0]^T$. The first sensor is assumed to have directional angle of $\theta_1 = 0$. Fig. 2(a) shows the plot of $J(\boldsymbol{\theta})$ as a function of θ_2 and θ_3. The optimal angles minimizing $J(\boldsymbol{\theta})$ and maximizing the determinant of FIM are $\theta_2 = 120^\circ$ and $\theta_3 = 240^\circ$. This corresponds to equiangular sensor separation. In Fig. 2(b) a contour plot of MSE is provided. For the simulated scenario with unequal sensor noise variances MSE is minimized by $\theta_2 = 126^\circ$ and $\theta_3 = 226^\circ$. We observe that uncertainty minimization and MSE minimization yield different sensor placements in this case.

If each sensor is subject to identical noise variances, i.e., $\sigma_1^2 = \sigma_2^2 = \cdots = \sigma_N^2 = \sigma^2$, the optimal sensor placements minimizing estimation uncertainty are defined by the following theorem:

Theorem 2. *For fixed* $\boldsymbol{\Sigma}$ *with* $\sigma_1^2 = \sigma_2^2 = \cdots = \sigma_N^2 = \sigma^2$ *the determinant of FIM is maximized at*

$$
|\boldsymbol{\Phi}_{max}| = \frac{N^2}{4\sigma^4}. \tag{28}
$$

by

$$
J(\boldsymbol{\theta}) = 0 \tag{29}
$$

which is satisfied if and only if

$$
\sum_{i=1}^{N} \sin 2\theta_i = 0, \quad \sum_{i=1}^{N} \cos 2\theta_i = 0 \tag{30a}
$$

$$
\sum_{i=1}^{N} \sin\theta_i = 0, \quad \sum_{i=1}^{N} \cos\theta_i = 0. \tag{30b}
$$

Proof. For identical noise variances (25) becomes

$$
J(\boldsymbol{\theta}) = \frac{1}{\sigma^4} \left\| \sum_{i=1}^{N} \boldsymbol{v}_i \right\|^2 + \frac{4}{N\sigma^4} \left\| \sum_{i=1}^{N} \boldsymbol{u}_i \right\|_{\tilde{\boldsymbol{H}}}^2 \geq 0 \tag{31}
$$

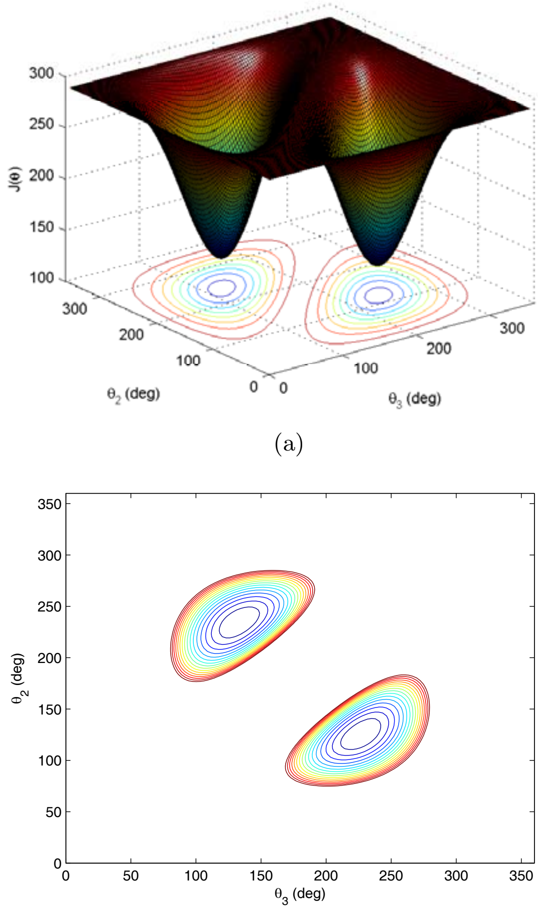

Figure 2: (a) Plot of $J(\boldsymbol{\theta})$, and (b) contour plot of MSE for TDOA localization with $\sigma_1^2 = 0.5$, $\sigma_2^2 = 0.1$ and $\sigma_3^2 = 0.2$. $J(\boldsymbol{\theta})$ is minimized at $\theta_1 = 0$, $\theta_2 = 120°$ and $\theta_3 = 240°$, and MSE is minimized at $\theta_1 = 0$, $\theta_2 = 126°$ and $\theta_3 = 226°$.

where

$$\tilde{\boldsymbol{H}} = \sum_{i=1}^{N} \begin{bmatrix} \sin\theta_i \\ -\cos\theta_i \end{bmatrix} [\sin\theta_i \quad -\cos\theta_i]\,. \tag{32}$$

Unless $\theta_1 = \cdots = \theta_N$, in which case $|\boldsymbol{\Phi}| = 0$ (i.e., $|\boldsymbol{\Phi}|$ is minimized) and the estimation problem has infinite variance, $\tilde{\boldsymbol{H}}$ is positive definite. Then the following quadratic form in (31)

$$\left\| \sum_{i=1}^{N} \boldsymbol{u}_i \right\|_{\tilde{\boldsymbol{H}}}^2 = 0$$

if and only if

$$\sum_{i=1}^{N} \boldsymbol{u}_i = \boldsymbol{0}.$$

Thus $J(\boldsymbol{\theta}) = 0$ if and only if

$$\sum_{i=1}^{N} \boldsymbol{u}_i = \sum_{i=1}^{N} \boldsymbol{v}_i = \boldsymbol{0}. \tag{33}$$

This condition is equivalent to (30). □

We note that for unequal noise variances the minimum of $J(\boldsymbol{\theta})$ cannot always be guaranteed to be zero. We call optimal geometries that minimize $J(\boldsymbol{\theta})$ to zero *maximally optimal* solutions. Maximally optimal solutions may exist for unequal noise variances depending on the number of sensors and noise variances. Consider the previous TDOA localization scenario this time with equal noise variances $\sigma^2 = 0.1$. As shown in Fig. 3, in the case of identical noise variances equiangular sensor separation gives the optimal sensor placement in terms of both uncertainty and MSE minimization.

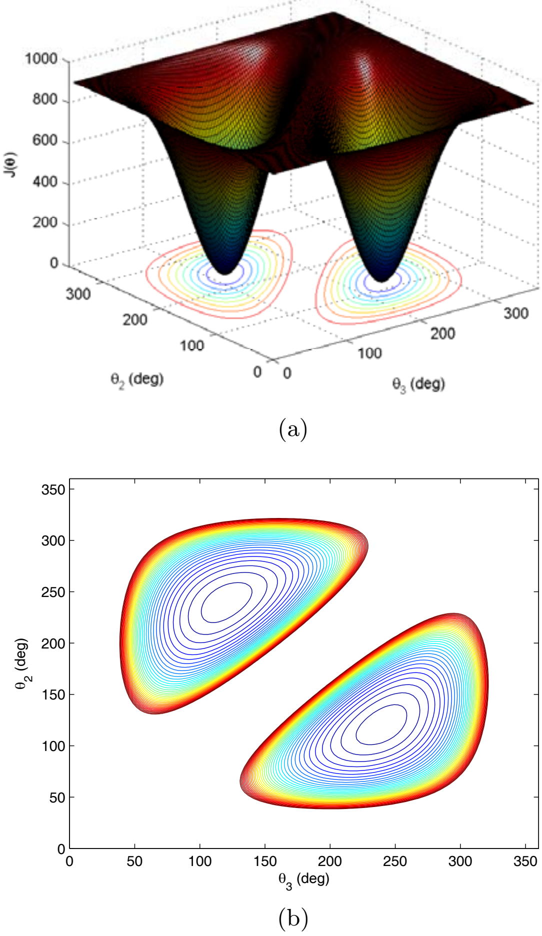

Figure 3: (a) Plot of $J(\boldsymbol{\theta})$, and (b) contour plot of MSE for TDOA localization with $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 0.1$. Both $J(\boldsymbol{\theta})$ and MSE are minimized at $\theta_1 = 0$, $\theta_2 = 120°$ and $\theta_3 = 240°$.

5. UNCERTAINTY-MINIMIZING TDOA GEOMETRIES

5.1 Equal Noise Variances

For equal noise variances maximally optimal sensor geometries that achieve $J(\boldsymbol{\theta}) = 0$ can be found in a straightforward way. Suppose that $\boldsymbol{u}_1 = [1, 0]^T$, i.e., $\theta_1 = 0$, with no loss of generality. Referring to Theorem 2 and (33) we see that a necessary condition for optimal sensor placement is that the directional unit vectors emanating from sensors add to zero. This is readily achieved if all directional unit vectors have equiangular separation. It is straightforward to show that for equally spaced θ_i (i.e., equiangular separation between sensors), the $\boldsymbol{v}_i$ also add to zero (see Corollary 3 in [3]).

What remains to be seen is whether equiangular sensor separation is the unique maximally optimal geometry. It can be shown through geometric arguments that for $N = 3$, 4 and 5 the optimal geometry is uniquely given by equiangular sensor separation. For $N \geq 6$ the optimal geometry is not unique and equiangular sensor separation is a special case of infinitely many optimal geometries. We will illustrate this for $N = 6$. Partition the sensors into two sets of cardinality 3, $\mathcal{I}_1$ and $\mathcal{I}_2$, where $\mathcal{I}_1 \cup \mathcal{I}_2 = \{1, 2, 3, 4, 5, 6\}$. The optimality conditions in (33) are satisfied by

$$\sum_{i \in \mathcal{I}_1} \boldsymbol{u}_i = \boldsymbol{0}, \quad \sum_{j \in \mathcal{I}_2} \boldsymbol{u}_j = \boldsymbol{0}$$
$$\sum_{i \in \mathcal{I}_1} \boldsymbol{v}_i = \boldsymbol{0}, \quad \sum_{j \in \mathcal{I}_2} \boldsymbol{v}_j = \boldsymbol{0}.$$

The two sensor partitions satisfying the above optimality condition would have equiangular sensor separation. The individual partitions can be rotated arbitrarily without affecting the optimality condition; thus creating infinitely many optimal configurations as shown in Fig. 4.

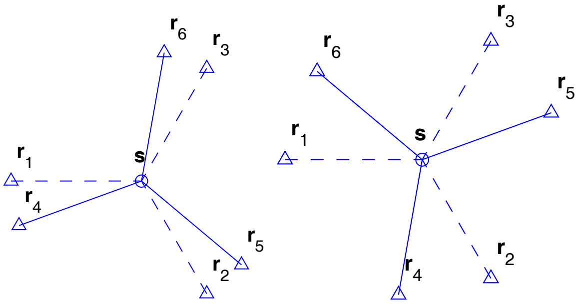

Figure 4: Two optimal sensor placements for $N = 6$. As long as sensor partitions $\mathcal{I}_1 = \{1, 2, 3\}$ and $\mathcal{I}_2 = \{4, 5, 6\}$ have equiangular spacing, the partitions can be rotated freely as shown above without affecting optimality.

5.2 Unequal Noise Variances

Theorem 1 does not lead to a simple geometric interpretation. The main reason for this is the non-trivial nature of the minimization problem in (24) for the general case of unequal noise variances. For three-sensor TDOA localization we have seen that equiangular sensor separation is optimal in terms of minimizing uncertainty. However, this result does not readily carry over to localization scenarios with $N \geq 4$. We have solved (24) numerically for $N = 3$, 4 and 5 with $\theta_1 = 0$, and $\sigma_1^2 = 0.1$, $\sigma_2^2 = 0.13$, $\sigma_3^2 = 0.16$, $\sigma_4^2 = 0.19$ and $\sigma_5^2 = 0.22$. Fig. 5 shows the resulting optimal configurations. We observe that equiangular sensor separation is not in general an optimal configuration. For $N = 3$ and $N = 4$ the minimum values of $J(\boldsymbol{\theta})$ are 31.07 and 9.87, respectively. On the other hand, the optimal geometry for $N = 5$ achieves $J(\boldsymbol{\theta}) = 0$; that is, it is a maximally optimal geometry. For unequal noise variances the existence of maximally optimal geometries depends on the distribution of individual noise variances. If σ_5^2 is changed to 0.4 in the above example, the resulting optimal geometry does not achieve $\min_{\boldsymbol{\theta}} J(\boldsymbol{\theta}) = 0$.

6. CONCLUSION

In this paper we have used estimation uncertainty minimization as the optimization criterion for TDOA sensor placement. If the sensors have equal noise variances, we have shown that optimal TDOA sensor placement corresponds to equiangular sensor separation for $N \leq 5$ regardless of individual sensor-emitter distances. For $N > 5$, optimal sensor placement is given by sensor partitions of appropriate size each with equiangular separation, again independent of individual sensor-emitter distances. Even though equiangular separation is still optimal in this case, it is one of infinitely many optimal solutions. If the sensors have unequal noise variances, the optimal TDOA geometry is no longer given by equiangular sensor separation in general. Finding the optimal geometry requires numerical minimization. The optimal geometries are still independent of individual sensor-emitter distances.

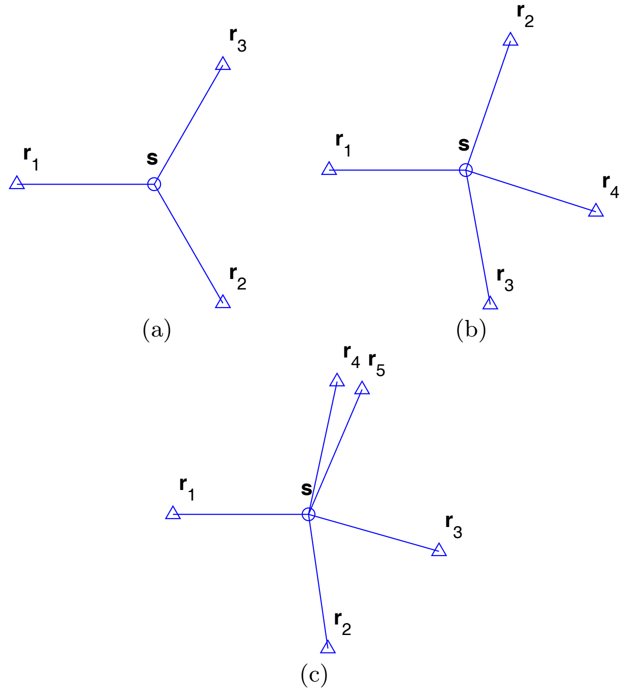

Figure 5: Optimal sensor geometries for (a) $N = 3$, (b) $N = 4$, and (c) $N = 5$. The noise variances are $\sigma_1^2 = 0.1$, $\sigma_2^2 = 0.13$, $\sigma_3^2 = 0.16$, $\sigma_4^2 = 0.19$ and $\sigma_5^2 = 0.22$. The configuration in (c) is a maximally optimal solution with $J(\boldsymbol{\theta}) = 0$.

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