Generalized method of moments for blind near efficient carrier phase acquisition

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***Abstract*— In this paper, a blind carrier phase estimator based on a suitably weighted phase histogram of the received signal samples is phrased as a particular case of the Generalized Moments Method (GMM). More in general, in this contribution, we point out that the estimation of a shift parameter by means of the GMM can be realized using a fast, DFT based, computationally efficient, coarse-to-fine estimation procedure. Furthermore, we develop the statistical analysis needed to extend the estimator to the optimally weighed GMM estimator, *i.e.* the weighted estimator achieving the minimum estimate variance. The theoretical analysis of the optimally weighted estimator performances is carried out in close form. Finally, the theoretical weighted estimator performances are compared with those of the unweighed estimator, of a state-of-art estimator and of the Cramèr-Rao lower bound. The performance improvement due to the optimal weighting is clearly appreciated, since, at medium to high SNR, it approaches the Cramèr-Rao lower bound on all constellations.**

I. Introduction

The Generalized Method of Moments (GMM) has been first introduced by [1] and has been applied in many contexts. An useful review is found in [2], [3].

Here, we recast a blind method for phase acquisition, recently appeared in [4], in the GMM framework, also providing its extension in the sense of the minimum variance (*best*) GMM phase estimator. In [4], the phase offset is evaluated as the cyclic phase shift observed on the weighted phase histogram of the fourth power of the received signal samples. For large sample size, the phase histogram tends to a constellation dependent function, named Constellation Phase Signature (CPS), constituted by a set of pulses whose widths depend on the SNR, and whose locations depend on the signal constellation and on the unknown phase offset. Since the unknown phase offset cyclically shifts the CPS, it can be estimated by evaluating the phase shift between the measured CPS and its expected value corresponding to zero phase offset. We show that this estimation procedure corresponds to the application of the *unweighted* GMM; furthermore, we perform all the statistical analysis needed to derive the *best* GMM estimator, *i.e.* the optimally *weighted* estimator that minimizes the estimation variance. Performance comparison shows that the a significant accuracy gain is guaranteed by the best GMM estimator at medium-high signal-to-noise power ratio values, where the Cramèr-Rao lower bound (CRB) is attained.

The paper is organized as follows: Sect.II depicts the signal model, recalling the definition of the CPS and the phase estimation criterion adopted in [4]. Then, Sect.III describes the GMM algorithm defining the Generalized Moments in terms of the signal CPS; we also derive the GMM estimator in its optimal weighted and unweighted form. Asymptotical performance of the GMM phase estimator are assessed in Sect.IV, in Sect.V we comment the performance comparison with the CRB, and Sect.VI concludes the paper.

II. Signal Model and Background

Let us consider a digital transmission system where the information is carried on by M-ary QAM symbols drawn from a constellation $\mathcal{A}=\{S_0,..,S_{M-1}\}$. At the receiver side, a complex low-pass version of the received signal is extracted by means of front-end processing. Let $X[n]$ be the samples of the complex low-pass received signal extracted at symbol rate. We assume the following analytical model of the N signal samples $X[n], n=0,\cdots N-1$:

$$X[n] = G\, e^{j\theta}\, S[n] + W[n] \tag{1}$$

where $S[n]$ is the n-th transmitted symbol, supposed to have unit power, G is the unknown overall gain, θ is the unknown phase-offset, and $W[n]$ is a realization of a circularly complex Gaussian stationary noise process, statistically independent of $S[n]$, with variance $\sigma_W^2 \stackrel{\text{def}}{=} \mathrm{E}\left\{|W[n]|^2\right\}$. The signal-to-noise ratio (SNR) is thus $\mathrm{SNR} \stackrel{\text{def}}{=} G^2/\sigma_W^2$.

Let us now consider the following nonlinear function of the received signal samples $X[n]$:

$$Y[n] = |X[n]|^P \cdot e^{j4\cdot \arg X[n]} \tag{2}$$

This nonlinearity has the remarkable properties to fold measurements whose phases differ by $\pi/2$. The statistical properties of $Y[n]$ can be usefully exploited to derive estimators of the phase-offset θ. In [4], it is considered the so-called Magnitude weighted Tomographic Projection (MWTP) of the probability density function of $Y[n]$, defined as follows:

$$g_\Phi^{(\mathcal{A},\theta,P)}(\varphi) \stackrel{\text{def}}{=} \int_0^{+\infty} r\cdot p_{R,\Phi}(r,\varphi;\theta)\, dr \tag{3}$$

where we expressed the transformed samples $Y[n]$ in polar coordinates *i.e.* $Y = re^{j\varphi}$, and we denoted its probability density function as $p_{R,\Phi}(r,\varphi;\theta)$.

As widely discussed in [4], the MWTP is a periodic function that depends on the phase-offset θ in the form of a cyclical shift of 4θ, *i.e.* $g_\Phi^{(\mathcal{A},\theta,P)}(\varphi) = g_\Phi^{(\mathcal{A},0,P)}(\varphi - 4\theta)$.

Since $g_\Phi^{(\mathcal{A},\theta,P)}(\varphi)$ substantially behaves like an ordinary pdf, the MWTP can be estimated by subdividing the phase interval $[0,2\pi)$ in K intervals of amplitude $2\pi/K$:

$$I_K^{(k)} \stackrel{\text{def}}{=} \left[k\cdot\frac{2\pi}{K}, (k+1)\cdot\frac{2\pi}{K}\right), \quad \text{for } k=0,\ldots,K-1 \tag{4}$$

and estimating the area of $g_\Phi^{(\mathcal{A},\theta,P)}(\varphi)$ in the k-th phase interval:

$$f^{(\mathcal{A},\theta,P)}(\psi_k) \stackrel{\text{def}}{=} \int_{2\pi k/K}^{2\pi(k+1)/K} g_\Phi^{(\mathcal{A},\theta,P)}(\varphi)\, d\varphi \tag{5}$$

being $\psi_k \stackrel{\text{def}}{=} 2\pi k/K$. The function $f^{(\mathcal{A},\theta,P)}(\psi_k)$, first introduced in [4], is constellation dependent, and it is named *Constellation Phase*

Signature (CPS). Also the CPS, is cyclically shifted of 4θ under a phase-offset θ, and, for a large enough K, it results:

$$f^{(\mathcal{A},\theta,P)}(\psi_k) \simeq \frac{2\pi}{K} g_{\Phi}^{(\mathcal{A},\theta,P)}(\psi_k) \quad (6)$$

The analytical form of the MWTP is quite involved and has been evaluated in [4]; here it is omitted for the sake of conciseness.

Therefore, in [4] the phase-offset estimation problem is rephrased as a (cyclic) shift estimation problem, between the analytically evaluated zero phase offset CPS $f^{(\mathcal{A},0,P)}(\psi_k)$ and the sample estimate of the CPS $\hat{f}^{(\mathcal{A},\theta,P)}(\psi_k)$, this latter calculated as the sample average of $|Y[n]|$ in the k-th phase interval $I_K^{(k)}, k = 0 \ldots K-1$:

$$\hat{f}^{(\mathcal{A},\theta,P)}(\psi_k) = \frac{K}{2\pi} \cdot \frac{1}{N} \sum_{n=0}^{N-1} |Y[n]| \cdot d_K^{(k)}(Y[n]) \quad (7)$$

where $d_K^{(k)}(Y)$ is the function that indicates if $\arg Y \in I_L^{(k)}$:

$$d_K^{(k)}(Y) \stackrel{\text{def}}{=} \begin{cases} 1 & \arg Y \in I_L^{(k)} \\ 0 & \text{otherwise} \end{cases}$$

The cyclic shift estimation is then realized resorting to a cross-correlation based procedure, efficiently evaluated using the properties of the Discrete Fourier Transform (DFT).

In the following Section we will describe how this CPS based phase estimation procedure can be phrased in the framework of the GMM.

III. The Generalized Method of Moments

Let us define the following generalized moment:

$$\mathbf{e}(\xi) \stackrel{\text{def}}{=} \hat{\mathbf{f}} - \mathbf{f}(\xi) \quad (8)$$

being $\hat{\mathbf{f}} \stackrel{\text{def}}{=} \left[\hat{f}^{(\mathcal{A},\theta,P)}(\psi_0), \ldots, \hat{f}^{(\mathcal{A},\theta,P)}(\psi_{K-1})\right]^{\mathrm{T}}$ the observation vector collecting the K values of the sample CPS, and $\mathbf{f}(\xi) \stackrel{\text{def}}{=} \left[f^{(\mathcal{A},\xi,P)}(\psi_0), \ldots, f^{(\mathcal{A},\xi,P)}(\psi_{K-1})\right]^{\mathrm{T}}$ the reference vector collecting the same values of the ideal CPS for a generic phase offset ξ.

Since the CPS estimator defined in (7) is unbiased [4], properly taking into account the four quadrant phase folding in (2), it results:

$$E\{\mathbf{e}(\xi)\} = \mathbf{0} \quad \textit{iff} \quad \xi = 4\theta \quad (9)$$

Then, according to the GMM [1], [2], a Consistent and Asymptotically Normal (CAN) estimate of θ is found by minimizing the generalized moment (8) according to a suitable weighted norm criterion. Namely, considering the elliptic norm:

$$\mathcal{Q}(\xi; \mathbf{W}(\theta)) = (\hat{\mathbf{f}} - \mathbf{f}(\xi))^{\mathrm{T}} \mathbf{W}(\theta) (\hat{\mathbf{f}} - \mathbf{f}(\xi))$$

the GMM estimate of θ is obtained as follows:

$$\hat{\theta}^{(\mathbf{W})} = \frac{1}{4} \arg\min_{\xi} \mathcal{Q}(\xi; \mathbf{W}(\theta)) \quad (10)$$

Obviously, the final estimation accuracy is affected by the weighting matrix $\mathbf{W}(\theta)$, possibly depending on θ.

A. Unweighted GMM Estimation

In the *unweighted* case, *i.e.* $\mathbf{W}(\theta) = \mathbf{I}$, the GMM cost function reduces to the classical Euclidean norm:

$$\mathcal{Q}(\xi; \mathbf{I}) = (\hat{\mathbf{f}} - \mathbf{f}(\xi))^{\mathrm{T}} (\hat{\mathbf{f}} - \mathbf{f}(\xi)) = \hat{\mathbf{f}}^{\mathrm{T}} \hat{\mathbf{f}} + \mathbf{f}(\xi)^{\mathrm{T}} \mathbf{f}(\xi) - 2\, \hat{\mathbf{f}}^{\mathrm{T}} \mathbf{f}(\xi) \quad (11)$$

Let us consider a K large enough to neglect the aliasing (possibly) present in the ideal CPS sampling that forms the vector $\mathbf{f}(\xi)$. Then, since ξ is a cyclic location parameter for the ideal CPS, *i.e.* $f^{(\mathcal{A},\xi,P)}(\psi_k) = f^{(\mathcal{A},0,P)}(\psi_k - \xi)$, we can write:

$$\mathbf{f}(\xi) = \mathbf{S}(\xi) \cdot \mathbf{f}(0)$$

where $\mathbf{S}(\xi)$ is the *periodic sinc* interpolation matrix, with entries:

$$||\mathbf{S}(\xi)||_{k,m} = \frac{1}{K} \frac{\sin((K\xi - 2\pi(k-m))/2)}{\sin((K\xi - 2\pi(k-m))/2K)}$$

The matrix $\mathbf{S}(\xi)$ is orthogonal, *i.e.* $\mathbf{S}(\xi)^{\mathrm{T}} \mathbf{S}(\xi) = \mathbf{I}$, and thus the quadratic norm of the vector $\mathbf{f}(\xi)$ does not depend on ξ:

$$\mathbf{f}(\xi)^{\mathrm{T}} \cdot \mathbf{f}(\xi) = \mathbf{f}(0)^{\mathrm{T}} \cdot \mathbf{S}(\xi)^{\mathrm{T}} \mathbf{S}(\xi) \cdot \mathbf{f}(0) = \mathbf{f}(0)^{\mathrm{T}} \cdot \mathbf{f}(0)$$

Hence, the minimal value of $\mathcal{Q}(\xi; \mathbf{I})$ is found by maximizing the scalar product $\hat{\mathbf{f}}^{\mathrm{T}} \cdot \mathbf{f}(\xi)$ with respect to ξ:

$$\hat{\theta}^{(\mathbf{I})} = \frac{1}{4} \arg\min_{\xi} \mathcal{Q}(\xi; \mathbf{I}) = \frac{1}{4} \arg\max_{\xi}\ \hat{\mathbf{f}}^{\mathrm{T}} \cdot \mathbf{S}(\xi) \cdot \mathbf{f}(0) \quad (12)$$

We recognize that (12) exactly is the cost function employed in [4], here rather (re-)obtained in the framework of the GMM estimation procedure. It is worth noting that maximization of (12) can be conducted even though the sample CPS is estimated apart an amplitude scale factor, *i.e.* without accomplishing a preliminary gain control stage, and the phase offset estimation has to be properly considered *gain-control-free*.

Moreover, as indicated in [4], a fast, DFT based, computational procedure obtains $\hat{\theta}^{(\mathbf{I})}$ in a two-stage, coarse-to-fine, estimation steps. In fact, it results:

$$\mathbf{S}(2\pi k/K) = \mathbf{D}^k$$

being $\mathbf{D} = (\mathbf{D}^{\mathrm{T}})^{-1}$ the following orthogonal unit cyclic shift matrix:

$$\mathbf{D} \stackrel{\text{def}}{=} \begin{bmatrix} 0 & \cdots & \cdots & 0 & 1 \\ 1 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$$

Thus, the index k_c of the coarse estimation with resolution $2\pi/K$, namely $\hat{\theta}_c^{(\mathbf{I})} = 2\pi k_c/K$, is obtained from (12) rewritten as follows:[1]

$$k_c = \arg\min_k \mathcal{Q}\left(\frac{2\pi k}{K}; \mathbf{I}\right) = \arg\max_k\ \hat{\mathbf{f}}^{\mathrm{T}} \cdot \mathbf{D}^k \cdot \mathbf{f}(0) \quad (13)$$

Due to the cyclic shift property of the operator $\mathbf{D}$, in the rightmost hand side of (13) it appears the cyclic cross-correlation between the sequences collected in the vectors $\hat{\mathbf{f}}$ and $\mathbf{f}(0)$; thus, the maximization over k can be conducted by properly using the DFT of both the sequences. Specifically, the DFT of the sample sequence $\hat{f}^{(\mathcal{A},\theta,P)}(\psi_k)$, for $k = 1, \ldots, K$, is computed and multiplied by the pre-calculated, complex conjugated, DFT of the zero phase-offset reference CPS $f^{(\mathcal{A},0,P)}(\psi_k)$, so as to obtain their cross-correlation after inverse DFT, as also indicated in [4]; then, according to (13), the index k_c is read as the *lag* that locates the cross-correlation maximum. The overall computational complexity is significantly reduced by choosing the value of K according to selected FFT algorithms.

B. Minimum Variance GMM Estimation

Let us now consider the *best* GMM estimator, *i.e.* the GMM estimator that minimizes $\mathrm{Var}\left\{\hat{\theta}^{(\mathbf{W})}\right\}$ with respect to $\mathbf{W}$, which uses the following optimal weight matrix:

$$\mathbf{W}_0(\theta) = \arg\min_{\mathbf{W}} \mathrm{Var}\left\{\hat{\theta}^{(\mathbf{W})}\right\}$$

[1] It is worth noting that (13) hold for any integer k, even in presence of aliasing in the ideal CPS sampling.

It is well know [1], [2] that the optimal weight matrix turns out to be $\mathbf{W}_0(\theta) = \mathbf{\Omega}(\theta)^{-1}/N$, being $\mathbf{\Omega}(\theta) \stackrel{\text{def}}{=} \mathrm{E}\{(\hat{\mathbf{f}} - \mathbf{f}(\theta))(\hat{\mathbf{f}} - \mathbf{f}(\theta))^{\mathrm{T}}\}$ the covariance matrix of the measurements, with entries:

$$||\mathbf{\Omega}(\theta)||_{i,j} \stackrel{\text{def}}{=} \mathrm{Cov}\left\{\hat{f}^{(\mathcal{A},\theta,P)}(\psi_i), \hat{f}^{(\mathcal{A},\theta,P)}(\psi_j)\right\} \tag{14}$$

Since the evaluation of the inverse measurements covariance matrix $\mathbf{\Omega}(\theta)^{-1}$ requires the knowledge of the true parameter θ, a coarse-to-fine approach can be envisaged for the minimization of $\mathcal{Q}(\xi; \mathbf{W}_0(\theta))$. Namely, once a coarse GMM estimate $\hat{\theta}_c \approx \theta$ is found,[2] the optimal GMM cost function $\mathcal{Q}(\xi; \mathbf{W}_0(\theta))$ can be approximated as follows:[3]

$$\mathcal{Q}(\xi; \mathbf{W}_0(\theta)) \approx \mathcal{Q}(\xi; \mathbf{W}_0(\hat{\theta}_c)) = (\hat{\mathbf{f}} - \mathbf{f}(\xi))^{\mathrm{T}}\mathbf{W}_0(\hat{\theta}_c)(\hat{\mathbf{f}} - \mathbf{f}(\xi))$$

so to obtain a fine estimate as follows:

$$\hat{\theta}_f = \frac{1}{4} \arg\min_{\xi} \mathcal{Q}(\xi; \mathbf{W}_0(\hat{\theta}_c))$$

Since the term $\hat{\mathbf{f}}^{\mathrm{T}} \cdot \mathbf{W}_0(\hat{\theta}_c) \cdot \hat{\mathbf{f}}$ does not depend on ξ, we can also consider the following modified cost function:[4]

$$\mathcal{J}(\xi; \mathbf{W}_0(\theta)) = \hat{\mathbf{f}}^{\mathrm{T}} \cdot \mathbf{W}_0(\hat{\theta}_c) \cdot \mathbf{f}(\xi) - \frac{1}{2} \cdot \mathbf{f}(\xi)^{\mathrm{T}} \cdot \mathbf{W}_0(\hat{\theta}_c) \cdot \mathbf{f}(\xi) \tag{15}$$

and

$$\hat{\theta}_f = \frac{1}{4} \arg\max_{\xi} \mathcal{J}(\xi; \mathbf{W}_0(\hat{\theta}_c))$$

IV. Performance Analysis

It is well known that the asymptotic estimation variance obtained using the best GMM is [1], [2]:

$$N \cdot \mathrm{Var}\left\{\hat{\theta}_f\right\} = \left(\mathbf{g}(\theta)^{\mathrm{T}} \cdot \mathbf{W}_0(\theta) \cdot \mathbf{g}(\theta)\right)^{-1} \tag{16}$$

where

$$\mathbf{g}(\xi) = \left[\frac{\partial f^{(\mathcal{A},\xi,P)}(\psi_0)}{\partial \xi}, \ldots, \frac{\partial f^{(\mathcal{A},\xi,P)}(\psi_K)}{\partial \xi}\right]^{\mathrm{T}}$$

is the gradient vector collecting the first derivatives of the ideal CPS.

Since the analytical expression of this latter is quite involved, see Appendix A in [4], and in order to avoid the analytical differentiation, we conduct the search for the maximum in (15) by means of a suitable interpolation after a few values of $\mathcal{J}(\xi; \mathbf{W}_0(\hat{\theta}_c))$ have been evaluated around the coarse estimate $\hat{\theta}_c$. Here, following [4], [5] we resort to a parabolic approximation for the GMM cost function $\mathcal{J}(\xi; \mathbf{W}_0(\hat{\theta}_c))$ around its maximum, thus obtaining the following expression for the fine estimate $\hat{\theta}_f$:

$$\begin{aligned} \hat{\theta}_f = \hat{\theta}_c - \frac{\Delta\theta}{8} \\ \cdot \frac{\mathcal{J}(\hat{\theta}_c + \Delta\theta; \mathbf{W}_0(\hat{\theta}_c)) - \mathcal{J}(\hat{\theta}_c - \Delta\theta; \mathbf{W}_0(\hat{\theta}_c))}{\mathcal{J}(\hat{\theta}_c + \Delta\theta; \mathbf{W}_0(\hat{\theta}_c)) - 2\mathcal{J}(\hat{\theta}_c; \mathbf{W}_0(\hat{\theta}_c)) + \mathcal{J}(\hat{\theta}_c - \Delta\theta; \mathbf{W}_0(\hat{\theta}_c))} \end{aligned} \tag{17}$$

[2]As previously described, a coarse GMM estimate of θ can found by minimizing (11), *i.e.* from (13) and $\hat{\theta}_c = 2\pi k_c/K$.

[3]For $\hat{\theta}_c = 2\pi k_c/K$, and since θ is a cyclic location parameter, the covariance matrix can be factorized as follows:

$$\mathbf{\Omega}(\hat{\theta}_c) = \mathbf{D}^{4k_c}\mathbf{\Omega}(0)(\mathbf{D}^{\mathrm{T}})^{4k_c}$$

and its inverse $\mathbf{\Omega}(\hat{\theta}_c)^{-1}$ is simply evaluated:

$$\mathbf{\Omega}(\hat{\theta}_c)^{-1} = \mathbf{D}^{4k_c}\mathbf{\Omega}(0)^{-1}(\mathbf{D}^{\mathrm{T}})^{4k_c}$$

This circumstance can be exploited to pre-calculate and store the inverse matrix $\mathbf{\Omega}(0)^{-1}$, and to obtain the matrix $\mathbf{\Omega}(\hat{\theta}_c)^{-1}$ by columns and rows cyclic shifting.

[4]In principle, (15) requires a preliminary gain estimation stage, which in turn can be substituted by a proper normalization of the CPS vectors, for instance:

$$\hat{\mathbf{f}} \implies \frac{\hat{\mathbf{f}}}{\sqrt{\hat{\mathbf{f}}^{\mathrm{T}} \cdot \hat{\mathbf{f}}}} \quad ; \quad \mathbf{f}(\xi) \implies \frac{\mathbf{f}(\xi)}{\sqrt{\mathbf{f}(0)^{\mathrm{T}} \cdot \mathbf{f}(0)}}$$

The estimation form (17) is analytically tractable in a relatively simple fashion, and it allows to evaluate the asymptotical performance of the optimal GMM based estimation.

Following the guidelines in [5], [4] the performance of the estimator (17) can be expressed as a function of the variances and covariances of $\mathcal{J}(\xi; \mathbf{W}_0(\theta))$; specifically, the variance of $\hat{\theta}_f$ is analytically evaluated as a function of the mean, variances and covariances of $\mathcal{J}(\hat{\theta}_c; \mathbf{W}_0(\theta))$, $\mathcal{J}(\hat{\theta}_c + \Delta\theta; \mathbf{W}_0(\theta))$ and $\mathcal{J}(\hat{\theta}_c - \Delta\theta; \mathbf{W}_0(\theta))$.

Let us set:

$$\begin{aligned} x &= \mathcal{J}(\hat{\theta}_c + \Delta\theta; \mathbf{W}_0(\theta)), & X &= \mathrm{E}\{x\}, \\ y &= \mathcal{J}(\hat{\theta}_c - \Delta\theta; \mathbf{W}_0(\theta)), & Y &= \mathrm{E}\{y\}, \\ z &= \mathcal{J}(\hat{\theta}_c; \mathbf{W}_0(\theta)), & Z &= \mathrm{E}\{z\} \end{aligned}$$

$$c = X - Y, \quad d = X - 2Z + Y.$$

Then, within a first-order approximation of (17), the variance of $\hat{\theta}_f$ is given by:

$$\begin{aligned} \mathrm{Var}\{\hat{\theta}_f\} = \frac{\Delta\theta^2}{64} \Bigg[\left(\frac{d-c}{d^2}\right)^2 \mathrm{Var}\{x\} \\ + \left(\frac{d+c}{d^2}\right)^2 \mathrm{Var}\{y\} + \left(\frac{2c}{d^2}\right)^2 \mathrm{Var}\{z\} \\ - 2\left(\frac{d^2-c^2}{d^4}\right) \mathrm{Cov}\{x,y\} + 2\left(\frac{2dc+2c^2}{d^4}\right) \mathrm{Cov}\{z,y\} \\ + 2\left(\frac{2dc-2c^2}{d^4}\right) \mathrm{Cov}\{x,z\} \Bigg] \end{aligned} \tag{18}$$

The mean values X, Y, Z and the variance and covariances of x, y, z have been analytically evaluated; the details of the derivation are reported in Appendix I.

V. Performance Comparison

Using (18) the accuracy of the *best* GMM estimator has been evaluated. In Figs.1-3 we have reported the results of the analysis, expressed in terms of the normalized standard deviation of the estimation error $\sqrt{N}\cdot\mathrm{StdDev}\{\hat{\theta}_f\}$ versus the SNR,[5] both for square-constellations (16-64-256 QAM) and cross-constellations (32-128-512 QAM). For simplicity, we have limited the analysis to the case $P = 1$.

For the sake of comparison, we have also reported the results pertaining to the unweighted GMM estimator employing the cost function in (11). Moreover, for comparison purposes we have also considered the state-of-the-art estimator [6] (WA03), as well as the CRB [7].

From Figs.1-3 we observe that, for all the herein considered constellations, at medium to high SNR the optimal weighted estimator approaches the CRB, with a clearly appreciated accuracy improvement.

[5]For each SNR value, a suitably L must be chosen such that approximation (6) is verified. In Figs.1-3, L varies from 512 (dense 64-QAM, 128-QAM, 256-QAM, 512-QAM constellations) to 1024 (sparse 16-QAM and 32-QAM constellations).

VI. Conclusion

In this paper, we have phrased a blind method for phase acquisition based on the Constellation Phase Signature (CPS) as a particular case of the GMM, showing that the GMM estimation of a shift parameter can be realized by means of a coarse-to-fine estimation approach using a fast, DFT based, computationally efficient procedure. Then, we have derived the best (minimum variance) CPS GMM estimator, and we have carried out the theoretical performance analysis. Comparison of the best estimator performance with those of the unweighted one and with the CRB shows that at medium to high SNR the optimal weighting definitely improves the estimator performance, and the optimal weighted GMM estimator approaches the CRB.

Appendix I. Statistical Analysis

As far as the first order moments are concerned, since $\mathrm{E}\{\hat{\mathbf{f}}\} = \mathbf{f}(\theta)$ we have:

$$\mathrm{E}\left\{\mathcal{J}(\xi;\mathbf{W}_0(\theta))\right\} = \left(\mathbf{f}(\theta) - \frac{1}{2}\cdot\mathbf{f}(\xi)\right)^{\mathrm{T}}\cdot\mathbf{W}_0(\theta)\cdot\mathbf{f}(\xi)$$

The variances-covariances are evaluated as follows:

$$\begin{aligned} N\cdot\mathrm{Cov}&\left\{\mathcal{J}(\xi_1;\mathbf{W}_0(\theta)),\mathcal{J}(\xi_2;\mathbf{W}_0(\theta))\right\}\\ &= N\cdot\mathbf{f}(\xi_1)^{\mathrm{T}}\cdot\mathbf{W}_0(\theta)\cdot\mathrm{Cov}\left\{\hat{\mathbf{f}},\hat{\mathbf{f}}^{\mathrm{T}}\right\}\cdot\mathbf{W}_0(\theta)\cdot\mathbf{f}(\xi_2)\\ &= N\cdot\mathbf{f}(\xi_1)^{\mathrm{T}}\cdot\mathbf{W}_0(\theta)\cdot\mathbf{\Omega}(\theta)\cdot\mathbf{W}_0(\theta)\cdot\mathbf{f}(\xi_2)\\ &= \mathbf{f}(\xi_1)^{\mathrm{T}}\cdot\mathbf{W}_0(\theta)\cdot\mathbf{f}(\xi_2)\end{aligned} \tag{I.1}$$

Finally, we report the first and second order moments of $\hat{f}^{(\mathcal{A},\theta,P)}(\psi_i)$ as evaluated in [4]:

$$\mathrm{E}\left\{\hat{f}^{(\mathcal{A},\theta,P)}(\psi_k)\right\} = f^{(\mathcal{A},\theta,P)}(\psi_k) \tag{I.2}$$

$$\begin{aligned} N\cdot\mathrm{Cov}\left\{\hat{f}^{(\mathcal{A},\theta,P)}(\psi_k),\hat{f}^{(\mathcal{A},\theta,P)}(\psi_l)\right\} &= f^{(\mathcal{A},\theta,2P)}(\psi_k)\,\delta_{k,l}\\ &- f^{(\mathcal{A},\theta,P)}(\psi_k)\cdot f^{(\mathcal{A},\theta,P)}(\psi_l)\end{aligned} \tag{I.3}$$

where $\delta_{k,l}$ is the Kronecker delta.

Appendix II. On The Optimal Weight Matrix $\mathbf{W}_0(\theta)$

The computation of the optimal weight matrix $\mathbf{W}_0(\theta) = \mathbf{\Omega}(\theta)^{-1}/N$ requires the inversion of the covariance matrix of the measurements $\mathbf{\Omega}(\theta) \overset{\mathrm{def}}{=} \mathrm{E}\{(\hat{\mathbf{f}} - \mathbf{f}(\theta))(\hat{\mathbf{f}} - \mathbf{f}(\theta))^{\mathrm{T}}\}$, which, according to (I.3), can be expressed as follows:

$$N\cdot\mathbf{\Omega}(\theta) = \mathbf{K}^2(\theta) - \mathbf{f}(\theta)\cdot\mathbf{f}(\theta)^{\mathrm{T}}$$

where

$$\mathbf{K}^2(\theta) = \mathrm{diag}\left\{f^{(\mathcal{A},\theta,2P)}(\psi_1),\ldots,f^{(\mathcal{A},\theta,2P)}(\psi_K)\right\}$$

Using the Woobury's identity [8], we obtain:

$$\begin{aligned}\mathbf{W}_0(\theta) &= \mathbf{K}(\theta)^{-2} + \frac{\mathbf{K}(\theta)^{-2}\cdot\mathbf{f}(\theta)\cdot\mathbf{f}(\theta)^{\mathrm{T}}\cdot\mathbf{K}(\theta)^{-2}}{1-\mathbf{f}(\theta)^{\mathrm{T}}\cdot\mathbf{K}(\theta)^{-2}\cdot\mathbf{f}(\theta)}\\ &= \mathbf{K}(\theta)^{-1}\left(\mathbf{I} + \frac{\mathbf{K}(\theta)^{-1}\cdot\mathbf{f}(\theta)\cdot\mathbf{f}(\theta)^{\mathrm{T}}\cdot\mathbf{K}(\theta)^{-1}}{1-\mathbf{f}(\theta)^{\mathrm{T}}\cdot\mathbf{K}(\theta)^{-2}\cdot\mathbf{f}(\theta)}\right)\mathbf{K}(\theta)^{-1}\end{aligned} \tag{II.4}$$

Hence, the elements of the matrix $\mathbf{W}_0(\theta)$ take the following form:

$$\begin{aligned}||\mathbf{W}_0(\theta)||_{k,l} &= \frac{\delta_{kl}}{f^{(\mathcal{A},\theta,2P)}(\psi_k)}\\ &+ \frac{1}{1-\sum_{m=1}^{K}\frac{\left(f^{(\mathcal{A},\theta,P)}(\psi_m)\right)^2}{f^{(\mathcal{A},\theta,2P)}(\psi_m)}}\cdot\frac{f^{(\mathcal{A},\theta,P)}(\psi_k)}{f^{(\mathcal{A},\theta,2P)}(\psi_k)}\cdot\frac{f^{(\mathcal{A},\theta,P)}(\psi_l)}{f^{(\mathcal{A},\theta,2P)}(\psi_l)}\end{aligned}$$

It is worth noting that the computations needed to evaluate (I.1) are significantly reduced when the optimal weight matrix $\mathbf{W}_0(\theta)$ is expressed in the form (II.4).

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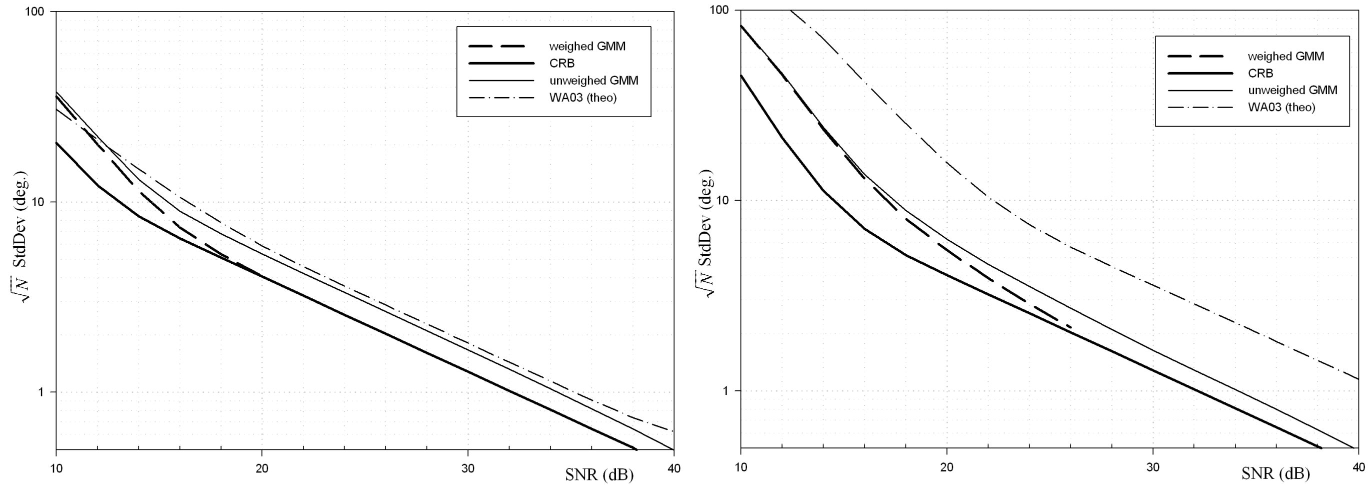

Fig. 1. *Normalized standard deviation of the phase estimation error $\sqrt{N}{\cdot}StdDev\{\hat{\theta}_f\}$ vs. SNR for 16-QAM and 32-QAM constellations: best GMM estimator (dashed line), unweighted GMM estimator (solid gray), estimator of [6] (dot-dashed line) and CRB [7] (solid line).*

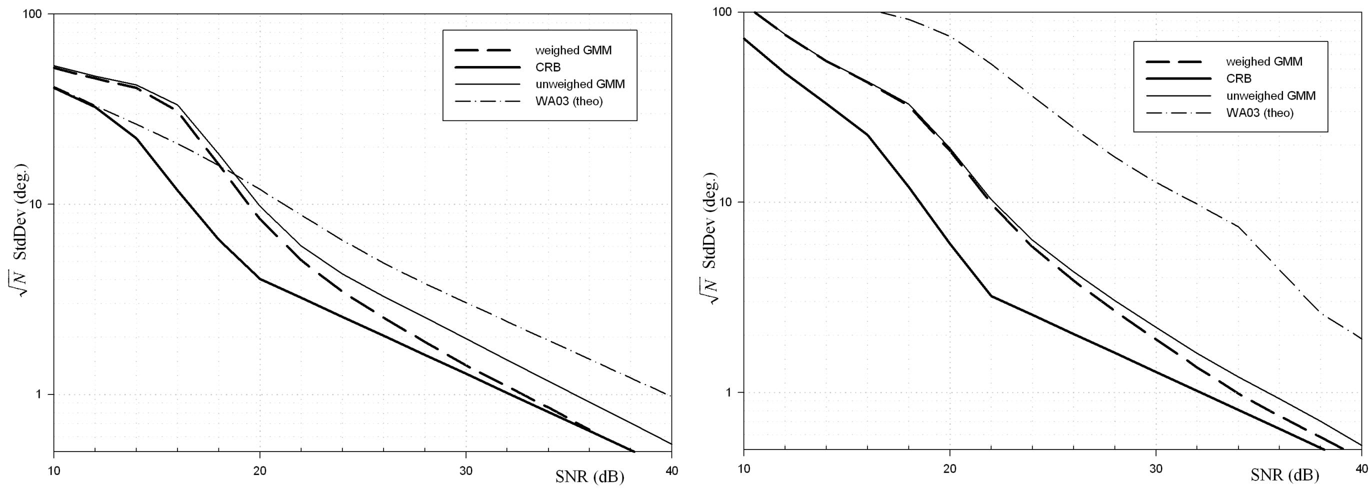

Fig. 2. *Normalized standard deviation of the phase estimation error $\sqrt{N}{\cdot}StdDev\{\hat{\theta}_f\}$ vs. SNR for 64-QAM and 128-QAM constellations: best GMM estimator (dashed line), unweighted GMM estimator (solid gray), estimator of [6] (dot-dashed line) and CRB [7] (solid line).*

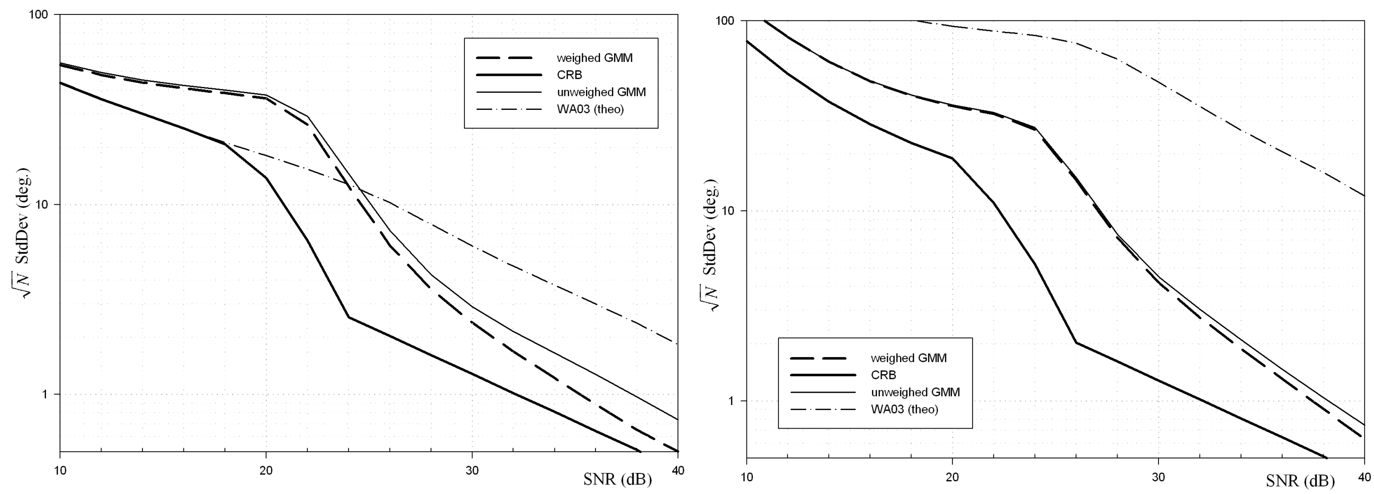

Fig. 3. *Normalized standard deviation of the phase estimation error $\sqrt{N}{\cdot}StdDev\{\hat{\theta}_f\}$ vs. SNR for 256-QAM and 512-QAM constellations: best GMM estimator (dashed line), unweighted GMM estimator (solid gray), estimator of [6] (dot-dashed line) and CRB [7] (solid line).*