DIVERSITY ORDER FOR THE AMPLIFY-AND-FORWARD MULTIPLE-RELAY CHANNEL WITH RANDOMIZED DISTRIBUTED SPACE-TIME CODING

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ABSTRACT

Cooperative transmission is a convenient means of generating spatial diversity without the need of collocating multiple antennas in a single terminal. In this paper, the asymptotic diversity properties of a cooperative transmission system are investigated. The analysis is carried out for an amplify-and-forward multi-relay protocol implementing a distributed randomized space-time block code. According to this approach, each relay multiplies the received signal by its specific spreading matrix, which is selected as having random independent and identically distributed entries. It is shown that the asymptotic diversity order of this system can vary from 2 to the total number of relays plus one. Furthermore, the achieved diversity order critically depends on the quotient between the number of rows and columns of the spreading matrices, which need to be sufficiently tall in order to guarantee full diversity of the system.

1. INTRODUCTION

Cooperative transmission has recently emerged as an alternative means of providing spatial diversity without the need of physically collocating multiple antennas in a single terminal. The basic idea behind this type of transmission is the fact that multiple terminals in the system can emulate the operation of a virtual antenna array by conveniently replicating the signal transmitted by a source node [1–3]. One can basically differentiate between two different families of transmission algorithms in this type of systems, depending on whether relays have access or not to the channel state information between themselves and the destination. If channel state information between relays and destination is available at the relays, then transmission strategies usually try to emulate beamforming [4, 5]. In practice, however, relays only have channel state information associated with the signal that is received (which corresponds to the channel between the source and the relay), and in these circumstances it is more convenient to emulate the operation of a distributed space-time code [6, 7].

In this paper, we focus on this last scenario for cooperative transmission, whereby relays virtually synthesize a distributed space-time block code. We will assume that the relays operate under a half-duplex mode, so that transmission occurs in two time-slots (see further Figure 1). During the first time-slot, the source broadcasts a codeword of K symbols during K time instants, and this transmission is received by both relays and destination nodes. During the second time slot, relays transform these K received samples into a set of N new ones, which basically correspond to a specific column of the corresponding space-time block code. In order to construct these N new samples, each relay can try to decode the transmitted codeword (decode-and-forward) or simply transform the received samples without performing any decoding (amplify-and-forward). In this paper, we will only focus on amplify-and-forward operation, which is basically simpler in terms of signal processing at the relays. Finally, the destination node decodes the original codeword by using the $K+N$ samples received in the two time slots. Note that the cooperative transmission system is basically using $K+N$ samples to transmit a codeword of K symbols, so that there is an inherent loss in degrees of freedom that are used to provide diversity. We will denote by α the ratio between the number of symbols transmitted during the first time slot (K) and the number of symbols transmitted during the second time slot (N), namely $\alpha = K/N$. We will see in this paper that this parameter is very important when describing the asymptotic properties of the cooperative system.

Now, one of the main practical problems associated with the implementation of distributed space time codes is the fact that they are inherently rigid, in the sense that the whole system needs to be designed for each specific number of relays. This means that every time a relay drops in or out of the system, the whole transmission strategy (i.e. the space-time code matrix) needs to be redesigned, and somehow communicated to the relays. In this paper, we consider a different and much more flexible approach, which will be referred to as *randomized distributed space time block coding* (RD-STBC). This cooperative transmission technique basically implements a distributed version of the well-known linear dispersion codes [8, Chapter 10] in which the dispersion matrices are randomly and independently chosen for each relay.

More specifically, each relay in the system is associated with an $N \times K$ matrix with independent and identically distributed entries that is known at the destination (these matrices will be referred to as "spreading matrices" for reasons that will become clear in short). The spreading matrices, which are different for each relay, are fixed at the beginning of the transmission and do not change over time. After the K samples transmitted during the first time slot have been received, each relay multiplies the column vector of K samples by the corresponding $N \times K$ matrix, thus generating a column vector containing the N samples to be transmitted during the second time slot. Note that this is much simpler than a conventional distributed space-time code, because relays do not need to change their transmission strategies when new nodes drop in or out of the system, and also because the signal processing at the relays reduces to a mere matrix-vector multiplication.

In [9] we investigated the performance of this cooperative technique in terms of spectral efficiency. Since the actual performance depends on the choice of spreading matrices, we analyzed the performance in the asymptotic domain assuming that both N and K are large but comparable in magnitude. Using random matrix theory techniques, it was shown that the spectral efficiency converges to a fixed limit, which interestingly is an extraordinarily good approximation of the average behavior of the finite reality. Hence, in

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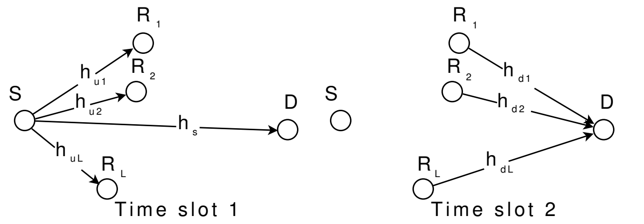

Figure 1: Transmission scheme: the information is broadcasted by the source in slot 1 and then forwarded by the relays in slot 2.

practice we can use these asymptotic expressions even when N,K are finite (in practice, no difference between simulated and asymptotic spectral efficiency was observed even when N,K are chosen as low as 4). The analysis carried out in [9] assumes that the channel coefficients are frequency non-selective and remain constant during the transmission of a codeword (two time-slots). In this paper, we investigate the effect of introducing quasi-static fading at all the channels. This means that all the channels in the system are fixed during the transmission of a codeword, but vary randomly from one codeword to the next. More specifically, we will analyze the effect of fading in terms of outage probability (defined as the probability that the system does not achieve a certain spectral efficiency) when the signal to noise ratio of the system is large. We will show that the diversity order of the system can scale from 2 to $L+1$, L the total number of relays, depending on the value of $\alpha = K/N$.

2. THE SYSTEM MODEL

In this section, the system will be presented in detail and a description of the mathematical model of the signals received by the relays and the destination will be provided. As usual, italic, bold lower-case and bold upper-case letters denote, respectively, scalars, vectors and matrices. The superscripts ^{T} and ^{H} stand for transpose and Hermitian transpose, respectively. Given an integer number M, $\mathbf{I}_M$ is the $M\times M$ identity matrix. $\otimes$ represents the Kronecker matrix product, $\mathbb{E}[\cdot]$ the expectation operator and $\mathbb{1}\{\cdot\}$ the indicator function.

2.1 Time-slot 1

The system under consideration is depicted in Figure 1. Communications are divided into two time-slots, devoted to source and relay transmissions, respectively. During the first phase, the l-th relay, $l = 1 \dots L$, receives

$$\mathbf{r}_l = h_{ul}\mathbf{s} + \mathbf{n}_{ul} \tag{1}$$

where:

- $\mathbf{s} = [s_1, s_2 \dots s_K]^T$ is the vector containing the K symbols transmitted by the source S. The symbols are assumed to be random, complex, circular, independent and identically distributed (i.i.d.) with zero mean and variance $\mathbb{E}[|s_k|^2] = P_s$;
- h_{ul} is the uplink channel coefficient for relay l (that is, the channel between the source and the relay). We will consider this and all the other channels in the system as affected by fading which is flat over the bandwidth of interest and slow time-varying, meaning that the channel coefficients can be modeled as constant during the whole time slot. We will also assume that the destination have perfect and global channel state information;
- the noise $\mathbf{n}_{ul} \sim \mathscr{CN}(\mathbf{0}, \sigma_u^2\mathbf{I}_K)$ is modeled as an i.i.d. circularly symmetric Gaussian random vector with zero mean and covariance matrix $\sigma_u^2\mathbf{I}_K$.

In time-slot 1, also the destination receives the source symbols, namely:

$$\mathbf{d}_1 = h_s\mathbf{s} + \mathbf{n}_1,$$

where h_s is the direct link channel coefficient and $\mathbf{n}_1 \sim \mathscr{CN}(\mathbf{0}, \sigma_d^2\mathbf{I}_K)$ the vector of i.i.d. noise samples.

2.2 Time-slot 2

At the end of the first phase, relay l linearly transforms the K received symbols (1) by multiplication by the complex gain g_l and the signature $N\times K$ matrix $\mathbf{C}_l$. The N-symbol strings $g_l\mathbf{C}_l\mathbf{r}_l$ will be simultaneously transmitted by the relays in time-slot 2. Thus, the signal received by the destination in this second phase is $\mathbf{d}_2 = \sum_{l=1}^{L} g_l h_{dl}\mathbf{C}_l\mathbf{r}_l + \mathbf{n}_2$, where the h_{dl}'s are the downlink (relay–destination) channel coefficient and $\mathbf{n}_2 \sim \mathscr{CN}(\mathbf{0}, \sigma_d^2\mathbf{I}_N)$ the collected additive white Gaussian noise. Recalling the contribution from the first time-slot, the source message has to be estimated from

$$\mathbf{d} = \begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \end{bmatrix} = \begin{bmatrix} h_s\mathbf{I}_K \\ \widetilde{\mathbf{C}}\widetilde{\mathbf{\Psi}}\widetilde{\mathbf{H}}_u \end{bmatrix}\mathbf{s} + \begin{bmatrix} \mathbf{n}_1 \\ \widetilde{\mathbf{C}}\widetilde{\mathbf{\Psi}}\mathbf{n}_u + \mathbf{n}_2 \end{bmatrix}; \tag{2}$$

where:

- $\widetilde{\mathbf{C}} = [\mathbf{C}_1, \mathbf{C}_2, \dots, \mathbf{C}_L]$ collects all the signature matrices;
- $\widetilde{\mathbf{\Psi}} = \mathbf{\Psi}\otimes\mathbf{I}_K$. $\mathbf{\Psi} = \mathrm{diag}\{g_1h_{d1}, g_2h_{d2}, \dots, g_Lh_{dL}\}$ is a diagonal matrix where the l-th entry g_lh_{dl} is the equivalent downlink channel coefficient for relay l;
- $\widetilde{\mathbf{H}}_u = \mathbf{h}_u \otimes \mathbf{I}_K$, with $\mathbf{h}_u = [h_{u1}, h_{u2}, \dots, h_{uL}]^T$;
- $\mathbf{n}_u = [\mathbf{n}_{u1}^T, \mathbf{n}_{u2}^T, \dots, \mathbf{n}_{uL}^T]^T$.

Note that, in spite of assuming all the vector noises temporally white and independent of one another, the equivalent noise in the second time-slot (i.e. $\widetilde{\mathbf{C}}\widetilde{\mathbf{\Psi}}\mathbf{n}_u + \mathbf{n}_2$) is colored with covariance matrix $\sigma_u^2\widetilde{\mathbf{C}}\widetilde{\mathbf{\Psi}}\widetilde{\mathbf{\Psi}}^H\widetilde{\mathbf{C}}^H + \sigma_d^2\mathbf{I}_N$.

2.3 Spectral efficiency

Under the hypothesis of perfect and global channel state information at the destination, the source symbols are estimated by means of the optimal *maximum likelihood* (ML) receiver. The resulting spectral efficiency is clearly a function of the signature matrices $\{\mathbf{C}_l\}_{l=1}^L$. Assume that each element of these $N\times K$ matrices is generated as an i.i.d. random variable with zero mean and variance $1/N$: note that randomness concerns only the generation of the matrices, which are then known at the receiver. It is shown in [9] that the dependence on the matrix statistics disappears in the asymptotic regime, i.e. for K and N growing without bound but with constant ratio α. More specifically, for $K = \alpha N \to +\infty$, the spectral efficiency tends almost surely to a deterministic quantity I. It turns out that, as mentioned before, when averaging over different code realizations in the finite reality, the mean behavior of the system is very well approximated by the asymptotic spectral efficiency I. For this reason, the following diversity analysis is carried out in the asymptotic regime.

To begin with, we fix the relay gain to $|g_l|^2 = P_s/(\alpha(P_s|h_{ul}|^2 + \sigma_u^2))$. In this way, the relay is constrained to transmit at the maximum power P_s. Under this assumption, the asymptotic spectral efficiency (nat/s/Hz) in [9] particularizes to

$$I = \frac{1}{1+\alpha}\left[\alpha\ln\left(1+\rho|h_s|^2\right) + \ln\frac{\phi_2}{\phi_1} + \right.$$
$$\left. + \frac{\sigma_d^2}{\sigma_u^2}(\phi_1-\phi_2) + \alpha\sum_{l=1}^{L}\ln\frac{1+\lambda_l\phi_1}{1+\frac{z\rho|h_{dl}|^2}{\alpha(1+z\rho|h_{ul}|^2)}\phi_2}\right], \tag{3}$$

where:

- $\rho = P_s/\sigma_d^2$ is the reference *signal-to-noise ratio* (SNR) and $z = \sigma_d^2/\sigma_u^2$;
- ϕ_1 is the unique positive solution to

$$\phi_1 = \left(\frac{\sigma_d^2}{\sigma_u^2} + \alpha\sum_{l=1}^{L}\frac{\lambda_l}{1+\lambda_l\phi_1}\right)^{-1};$$

- $\{\lambda_l\}_{l=1}^L$ are the solutions to

$$\frac{1}{1+\rho|h_s|^2}\sum_{l=1}^{L}\frac{z^2\rho^2|h_{ul}h_{dl}|^2}{\alpha(1+z\rho|h_{ul}|^2)\lambda - z\rho|h_{dl}|^2} = 1;$$

- ϕ_2 is the unique positive solution to

$$\phi_2 = \left(\frac{\sigma_d^2}{\sigma_u^2} + \alpha\sum_{l=1}^{L}\frac{z\rho|h_{dl}|^2}{\alpha(1+z\rho|h_{ul}|^2)+z\rho|h_{dl}|^2\phi_2}\right)^{-1}.$$

Hereafter, expression (3) is taken as a starting point to analyze the outage probability of the system. Even if the reported results still hold under more general assumptions, for the sake of simplicity we will only consider here the Rayleigh fading case, i.e. $|h_s|^2$, $\{|h_{ul}|^2\}$ and $\{|h_{dl}|^2\}$ are independent exponential random variables with variances ζ_s, ζ_u and ζ_d, respectively. For each channel gain $|h_x|^2$, we will denote its probability density function with $f_{|h_x|^2}(\cdot)$.

3. OUTAGE PROBABILITY

The outage probability $P_{out}(R)$ is defined as the probability that a communications system cannot support a target transmission rate R, namely

$$P_{out}(R) = \Pr[I < R].$$

Recall that the spectral efficiency I is a random quantity since it is a function of the random variables that model the channels.

In most cases, it is not possible to give a close-form expression of the outage probability, due to the involved structure of the spectral efficiency, e.g. (3). Nevertheless, it is usually sufficient to characterize $P_{out}(R)$ in the high-SNR regime and derive the diversity order d and the outage gain κ of the system. The relationship between these two parameters is given by

$$\kappa = \lim_{\rho\to+\infty}\rho^d P_{out}(R). \quad (4)$$

For the system under consideration, the diversity order and the outage gain can be shown to depend on the ratio $\alpha = K/N$ as follows:

Proposition. *For the L-relay system described in Section 2, with spectral efficiency as in (3), the diversity order is*

$$d = \begin{cases} L+1 & \text{for } \alpha < \dfrac{1}{L-1}, \\ M+1 & \text{for } \dfrac{1}{M} \le \alpha < \dfrac{1}{M-1}, \end{cases}$$

with $1 \le M < L$. The outage gain κ can be hence computed according to its definition (4).

Proof. The exhaustive proof is quite large and complex, so it is omitted due to space constraints. For a general idea, the interested reader can refer to [10] or [11]: the relays are divided into four different and disjoint sets, according to whether their uplink, downlink or both channels can support the target rate. Then, for all the possible choices of the four relay sets, one has to investigate which is the lowest number of badly faded channels (roughly speaking, with gain in the order of $1/\rho$) that results in the outage event. The worst case corresponds to the diversity order of the system. Section 4 analyzes the single-relay case, which is much more easy to handle than the general case. □

We will now comment on the diversity order and the resulting outage gain for the different values of α.

3.1 Case $\alpha < 1/(L-1)$

When $\alpha = K/N < 1/(L-1)$, it can be shown that the contributions to the outage probability only come from the partitions where each relay has either one, but not both, of the channels (uplink or downlink) in outage. The interpretation is the following. When using orthogonal spreading sequences, whose length would be at least $N = LK$, the system is equivalent to a transmission over $L+1$ parallel channels (same as time-division, see, e.g., [6]), counting also the direct link. With random, non-orthogonal signatures with $N > K(L-1)$, the ML receiver still sees $L+1$ parallel channels. Thus, the system falls in outage only when all the relays cannot support the required rate, because either of their links cannot. The probability that heavy fading affects both links for some relays and only one link for all the others is negligible.

This reasoning yields to the following conclusion: if $\alpha < 1/(L-1)$, the system achieves diversity $L+1$ (relays + direct link) and the outage gain is given by

$$\lim_{\rho\to+\infty}\rho^{L+1}P_{out}(R) = \sum_{k=0}^{L}\binom{L}{k}\zeta_s\zeta_u^k\zeta_d^{L-k}z^{-L}\cdot$$
$$\cdot\int_{\mathbb{R}_+^{L+1}}\mathbb{1}\left\{f_k\left(a,\mathbf{b}_1^k,\mathbf{c}_{k+1}^L\right) < (1+\alpha)R\right\}\mathrm{d}a\,\mathrm{d}\mathbf{b}_1^k\,\mathrm{d}\mathbf{c}_{k+1}^L,$$

where we used the notation

$$\mathrm{d}\mathbf{y}_i^j \equiv \mathrm{d}y_i\,\mathrm{d}y_{i+1}\cdots\mathrm{d}y_j, \quad i \le j,$$

as well as the definition

$$f_k\left(a,\mathbf{b}_1^k,\mathbf{c}_{k+1}^L\right) =$$
$$= \alpha\ln\left[1+a+\sum_{l=1}^{k}b_l+\frac{1}{\alpha}\left(\sum_{l=k+1}^{L}c_l\right)\bar{\phi}_1\right]+$$
$$-(1-\alpha k)\left[1-\ln(1-\alpha k)\right]-(1-\alpha k)\ln\left(z\bar{\phi}_1\right)+z\bar{\phi}_1,$$

where

$$\bar{\lambda} = \frac{1}{\alpha}\frac{\sum_{l=k+1}^{L}c_l}{1+a+\sum_{l=1}^{k}b_l}$$

$$\bar{\phi}_1 = \begin{cases} \frac{[z-(1-\alpha(k+1))\bar{\lambda}]}{2z\bar{\lambda}}\left[-1+\sqrt{1+\frac{4z\bar{\lambda}(1-\alpha k)}{[z-(1-\alpha(k+1))\bar{\lambda}]^2}}\right] & k < L \\ \dfrac{1-\alpha L}{z} & k = L. \end{cases}$$

Note that the spectral efficiency (3) tends, for $\rho\to+\infty$, to $f_k\left(a,\mathbf{b}_1^k,\mathbf{c}_{k+1}^L\right)/(1+\alpha)$ whenever k relays experience heavy fading in the uplink channel and the other $L-k$ do in the downlink one. Roughly speaking, the assumption of badly faded channel implies $b_l = z\rho|h_{ul}|^2 < +\infty$ and $c_l = z\rho|h_{dl}|^2 < +\infty$.

3.2 Case $1/M \le \alpha < 1/(M-1)$

If $1/M \le \alpha < 1/(M-1)$, with $1 \le M < L$, the contributions to the outage probability are brought by the cases with exactly M relays with heavy fading in the uplink, independently of their downlinks. The other $L-M$ relays do not experience bad fading in either channel. This means that the source–relay link is the dominant one in terms of outage:

1. if the source symbols are received by all relays with a high SNR, then it is always possible to convey information to the destination (except, of course, the case where all the downlinks and the direct link are corrupted. This case, however, has very low probability and should be accounted for only when $\alpha < 1/(L-1)$, see the previous section);

2. conversely, as α increases, there is a reduction in the minimum number of badly faded uplink channels which is sufficient to generate the outage event. For instance, when the direct link is corrupted and $\alpha \geq 1$, it is enough that one single relay, out of L, receives a low-SNR signal to fail data transmission.

The intuition for the second point is that the non-orthogonal, random coding employed in the presented scheme correlates the contributions of the relays, as can be noticed in (3): the quantities ϕ_1, ϕ_2 and all λ_l's depend on the totality of the links (direct, source–relay, relay–destination) of the system.

Resuming the analysis of the results in the above proposition, whenever $1/M \leq \alpha < 1/(M-1)$, with $1 \leq M < L$, the system achieves diversity of order $M+1$, and the corresponding outage gain is given by

$$\lim_{\rho\to+\infty} \rho^{M+1} P_{out}(R) = \binom{L}{M} \zeta_s \zeta_u^M z^{-M} \cdot$$
$$\cdot \int_{\mathbb{R}_+^{L+M+1}} \mathbb{1}\left\{ g(a, \mathbf{b}_1^M, \mathbf{y}_1^L) < (1+\alpha)R \right\} \mathrm{d}a \, \mathrm{d}\mathbf{b}_1^M f_{|h_d|^2}(\mathbf{y}_1^L) \, \mathrm{d}\mathbf{y}_1^L,$$

where we used the notation

$$f_{|h_d|^2}(\mathbf{y}_1^L) \equiv f_{|h_d|^2}(y_1) f_{|h_d|^2}(y_2) \cdots f_{|h_d|^2}(y_L),$$

as well as the definition

$$g(a, \mathbf{b}, \mathbf{y}_1^L) = \alpha \ln(1+a) + \ln \frac{\theta_2}{\theta_1} + \alpha \sum_{l=L-M}^{L} \ln(1 + \nu_l \theta_1)$$
$$- \alpha \sum_{k=1}^{M} \ln \left[1 + z \frac{y_k}{\alpha(1+b_k)} \theta_2 \right].$$

The quantities θ_1 and θ_2 are the positive solutions to the equations

$$\alpha(M+1) - 1 = \alpha \sum_{l=L-M}^{L} \frac{1}{1 + \nu_l \theta_1}$$
$$\alpha M - 1 = \alpha \sum_{k=1}^{M} \frac{1}{1 + \frac{z y_k}{\alpha(1+b_k)} \theta_2}$$

with $\{\nu_l\}_{l=L-M}^{L}$ being the solutions to the following equation in ν

$$\sum_{k=1}^{M} \frac{b_k y_k}{(1+b_k)\nu - \frac{z y_k}{\alpha}} + \frac{1}{\nu} \sum_{k=M+1}^{L} y_k = \frac{\alpha(1+a)}{z}.$$

As before, the hypothesis of M and only M relays with badly faded uplink channels implies $b_l = z\rho |h_{ul}|^2 < +\infty$ and $(1+\alpha)I \to g(a, \mathbf{b}, \mathbf{y}_1^M)$ for $\rho \to +\infty$.

4. THE SINGLE-RELAY CASE

The single-relay case is much simpler than the general one, mainly because ϕ_1 and ϕ_2 can be explicitly computed as the positive root of a second-order polynomial and the close-form expression for λ is

$$\lambda = \left(\frac{z\rho |h_u|^2}{1 + \rho |h_s|^2} + 1 \right) \frac{z\rho |h_d|^2}{\alpha(1 + z\rho |h_u|^2)}.$$

According to the result for the general case, given in the above proposition, systems with a single relay always achieve diversity order $d = 2$, for any value of α. The proof of $\lim_{\rho\to+\infty} \rho^2 P_{out}(R) = \kappa$, where the outage gain κ is finite and strictly positive, can be sketched as follows. First, we define the four events

- $\mathscr{E}_1$ the system is in outage ($I < R$) and $|h_u|^2 = O(\rho^{-1})$, $|h_d|^2 > O(\rho^{-1})$ for $\rho \to +\infty$;
- $\mathscr{E}_2$ the system is in outage ($I < R$) and $|h_u|^2 > O(\rho^{-1})$, $|h_d|^2 = O(\rho^{-1})$ for $\rho \to +\infty$;
- $\mathscr{E}_3$ the system is in outage ($I < R$) and $|h_u|^2 > O(\rho^{-1})$, $|h_d|^2 > O(\rho^{-1})$ for $\rho \to +\infty$;
- $\mathscr{E}_4$ the system is in outage ($I < R$) and $|h_u|^2 = O(\rho^{-1})$, $|h_d|^2 = O(\rho^{-1})$ for $\rho \to +\infty$.

Note that the four events are disjoint and, thus, $P_{out}(R) = \sum_{i=1}^{4} \Pr[\mathscr{E}_i]$. Then, we study the probability of each event, namely

$$\lim_{\rho\to+\infty} \rho^2 \Pr[\mathscr{E}_i] =$$
$$= \lim_{\rho\to+\infty} \rho^2 \int_{\mathbb{R}_+^3} \mathbb{1}\{\mathscr{E}_i\} f_{|h_s|^2}(x) f_{|h_u|^2}(w) f_{|h_d|^2}(y) \, \mathrm{d}x \, \mathrm{d}w \, \mathrm{d}y. \quad (5)$$

4.1 Event $\mathscr{E}_1$

With the change of variables $a = \rho x$ and $b = z\rho w$, one has

$$\lambda = \left(\frac{b}{1+a} + 1 \right) \frac{z\rho y}{\alpha(1+b)} \to +\infty$$

when $\rho \to +\infty$. As a result, it is straightforward to show that

$$\phi_1, \phi_2 \to \begin{cases} 0 & \text{for } \alpha > 1; \\ \dfrac{1-\alpha}{z} & \text{for } \alpha \leq 1 \end{cases}$$

and that, for $\alpha > 1$,

$$\lambda \phi_1, \frac{z\rho y}{\alpha(1+b)} \phi_2 \to \frac{1}{\alpha - 1}.$$

Knowing these facts, the outage event $\mathscr{E}_1$ can be shown to be equivalent to

$$\mathscr{E}_1' : \alpha \ln(1+a) + \min\{1, \alpha\} \ln \left(\frac{b}{1+a} + 1 \right) < (1+\alpha)R$$

in the limit for $\rho \to +\infty$. Thus, $\mathbb{1}\{\mathscr{E}_1\}$ represents a finite volume and (5) can be computed by means of the *Lebesgue's dominated convergence theorem* (LDCT):

$$\lim_{\rho\to+\infty} \rho^2 \Pr[\mathscr{E}_1] =$$
$$= \frac{1}{z} \lim_{\rho\to+\infty} \int_{\mathbb{R}_+^3} \mathbb{1}\{\mathscr{E}_1\} f_{|h_s|^2}\left(\frac{a}{\rho}\right) f_{|h_u|^2}\left(\frac{b}{z\rho}\right) f_{|h_d|^2}(y) \, \mathrm{d}a \, \mathrm{d}b \, \mathrm{d}y =$$
$$= \frac{\zeta_s \zeta_u}{z} \int_{\mathbb{R}_+^2} \mathbb{1}\{\mathscr{E}_1'\} \, \mathrm{d}a \, \mathrm{d}b.$$

4.2 Event $\mathscr{E}_2$

With the change of variables $a = \rho x$ and $c = z\rho y$ one has, for $\rho \to +\infty$:

$$\lambda \to \frac{c}{\alpha(1+a)},$$
$$\phi_1 \to \bar{\phi}_1 < +\infty,$$
$$\phi_2 \to \frac{1}{z},$$
$$\lambda \bar{\phi}_1 = \frac{1 - z\bar{\phi}_1}{\alpha - 1 + z\bar{\phi}_1}.$$

Note that the last equation implies $z\bar{\phi}_1 > 1 - \alpha$.

The limit of the spectral efficiency is hence

$$\lim_{\rho\to+\infty} (1+\alpha)I = \alpha \ln(1+a) + \alpha \ln \alpha +$$
$$+ z\bar{\phi}_1 - \ln(z\bar{\phi}_1) - 1 - \alpha \ln(z\bar{\phi}_1 + \alpha - 1).$$

Note that, from the definition of ϕ_1:

$$\frac{1}{z\bar{\phi}_1} = 1 + \frac{\alpha c}{\alpha z(1+a) + z\bar{\phi}_1 c}.$$

Resolving the last equation with respect to c, the outage event $\mathscr{E}_2$ can be shown to be equivalent to

$$\mathscr{E}_2' : c < \frac{\alpha z(1+a)}{\gamma\left(\frac{\alpha}{1-\gamma} - 1\right)}.$$

The quantity γ, with $\max\{0, 1-\alpha\} < \gamma < 1$, can be computed as

$$\gamma = f_0^{-1}((1+\alpha)R - \alpha \ln(1+a)),$$

where $f_0^{-1}(\cdot)$ is the inverse of

$$f_0(t) = t - \ln t - \alpha \ln(t + \alpha - 1) + \alpha \ln \alpha - 1,$$

monotonically decreasing in t for $\max\{0, 1-\alpha\} < t < 1$.

Again, $\mathbb{1}\{\mathscr{E}_2\}$ represents a finite volume and the limit in (5) is finite and can be computed with the LDCT:

$$\lim_{\rho \to +\infty} \rho^2 \Pr[\mathscr{E}_2] = \frac{\zeta_s \zeta_d}{z} \int_{\mathbb{R}_+^2} \mathbb{1}\{\mathscr{E}_2'\}\, \mathrm{d}a\, \mathrm{d}c.$$

4.3 Event $\mathscr{E}_3$

The contribution of this event is negligible. Indeed, with the change of variables $a = \rho x$, $b = z\rho w$ and $c = z\rho y$, the event $\mathscr{E}_3$ has a finite volume and

$$\lim_{\rho \to +\infty} \rho^2 \Pr[\mathscr{E}_3] = \frac{\zeta_s \zeta_u \zeta_d}{z^2} \lim_{\rho \to +\infty} \frac{1}{\rho} \int_{\mathbb{R}_+^3} \mathbb{1}\{\mathscr{E}_3\}\, \mathrm{d}a\, \mathrm{d}b\, \mathrm{d}c = 0.$$

4.4 Event $\mathscr{E}_4$

This case does not bring any contribution since, with both the relay channels "not small", the spectral efficiency grows without bound, i.e. $\mathbb{1}\{\mathscr{E}_4\} \to 0$, for $\rho \to +\infty$.

Summarizing, we just have to account for the contributions of $\mathscr{E}_1$ and $\mathscr{E}_2$, which both tend to zero as ρ^{-2} for large ρ.

Figure 2 reports two examples of outage-probability curves and the relative large-SNR approximations (the outage gains have been computed numerically): as predicted by the theory, the diversity order (the slope for large SNR values) is independent of the choice of the coding ratio α and equal to 2. However, α can be tuned to minimize the outage gain and, thus, the outage probability.

5. CONCLUSIONS

Following the spectral-efficiency analysis in [9], this paper studies the large-SNR approximation of the outage probability for a communications system with L amplify-and-forward, half-duplex relays which implement a randomized, distributed, space-time block code. Defining α as the ratio between the number of source symbols K and that of relay symbols N, the achieved diversity order is shown to be maximum, i.e. $d = L+1$, only for $\alpha < 1/(L-1)$, whereas it is $d = M+1$, with $1 \le M < L$, whenever $1/M \le \alpha < 1/(M-1)$. We can observe two main aspects regarding the large-SNR regime:

1. the spreading matrices $\mathbf{C}_l$ have to be tall ($N > (L-1)K$) to minimize the outage probability. Note, however, that this implies a sensible waste of degrees of freedom, since the source is transmitting K symbols every $K+N$ channel accesses;
2. relaying is always superior to the direct link, which only achieves a unitary diversity order.

Further work will concentrate on studying the dependence of the outage gain κ on the coding rate α in order to minimize the outage probability for any given diversity order.

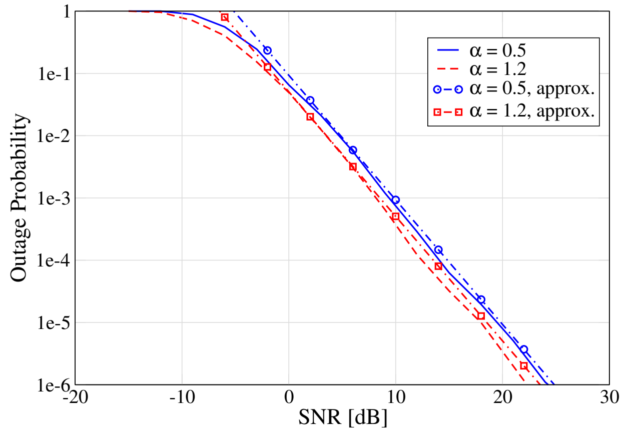

Figure 2: Outage probability vs. SNR for the single-relay case and different values of α. Channel and noise variances are set to 1 and the target rate is $R = 0.1$ nat/s/Hz.

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