OPTIMAL SPECTRUM BALANCING IN MULTI-USER SIMO xDSL NETWORKS

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ABSTRACT

When multiple xDSL users coexist in the same network, crosstalk can become a major performance limiting factor, e.g. in so-called near-far scenarios. By employing multiple receiver signals, i.e. by operating in a SIMO (Single Input Multiple Output) rather than the standard SISO (Single Input Single Output) mode, each user can estimate and compensate the crosstalk more efficiently, thereby increasing its performance. In this paper, an algorithm is presented for the optimal allocation of transmit power in these multi-user SIMO networks. Secondly, since transmitters are usually limited to integer bit loadings, an optimal bit allocation algorithm is presented which has the added advantage of being computationally more efficient than the power allocation algorithm. The focus is on multi-tone xDSL systems, where the use of multiple tones allows the transmit spectra to be easily shaped and where near-far scenarios frequently occur when some of the users are serviced from remote terminals. Simulation results show that in these cases an improvement of the data rate of 10% is possible by using existing twisted pairs in SIMO configurations, compared to the standard SISO (Single Input Single Output) configuration.

1. INTRODUCTION

To remain competitive with other emerging broadband access technologies such as in cable and wireless networks, xDSL operators must continue to improve their technologies for data transmission over the existing telephone network. To maximize the capacity of the twisted pair lines, these should be kept as short as possible so as to minimize the effect of attenuation. Therefore, xDSL networks are gradually extended by deploying high data rate connections from remote terminals (RT's) close to the end-users. Lines deployed from an RT can share the same binder as lines deployed from the central office (CO) for which a lower data rate is acceptable. This, however, creates a so-called near-far problem. At the point where the RT deployed lines enter the binder, the signals on the CO deployed lines have already traveled some distance and are attenuated. Strong transmit signals on the RT lines then cause crosstalk interference on the CO lines that can sometimes completely overpower the desired signal. This far-end crosstalk (FEXT) is a major performance limiting factor.

In typical xDSL networks, the last section of the twisted pairs is laid out in a loop, going from the cabinet to the end of the street and then returning to the cabinet. When users are inserted into such a twisted pair loop, they are actually connected twice to the xDSL network. Only one of the two resulting twisted pair connections is used to transmit data, preferably the shortest connection so as to maximize the achievable data rate. Many of these connections share the same binder, resulting in a multi-user SISO (Single Input Single Output) transmission system. An example is shown in figure 1(a), where a near-far scenario creates considerable crosstalk on the CO deployed line. In such a scenario, the RT deployed user has to apply some power backoff in order to protect the CO deployed user, thereby limiting its data rate.

Figure 1(b) shows a multi-user SIMO extension, where an extra signal is used from the twisted pair that does not carry a transmitted signal. The extra signal can be used to estimate the crosstalk that is present on the twisted pair carrying the xDSL signal. This allows each user to clean up the received xDSL signal, thereby creating a higher data rate. Since CO deployed lines can then to some extent mitigate the crosstalk they receive, RT deployed users have to apply less power backoff and can thus transmit at higher data rates.

In this paper, an algorithm is presented for the optimal allocation of transmit power in these multi-user SIMO networks. Secondly, since transmitters are usually limited to integer bit loadings, an optimal bit allocation algorithm is presented which has the added advantage of being computationally more efficient than the power allocation algorithm.

The paper is organized as follows: section 2 introduces the system model that is used and section 3 introduces a method for bit loading in multi-user SIMO networks. Sections 4 and 5 then present a method for optimal power and optimal bit allocation in multi-user SIMO networks. Section 6 presents some simulation results and section 7 concludes the paper. In the Appendix, proofs are presented for a number of theorems.

2. SYSTEM MODEL

Most current DSL systems use Discrete Multi-Tone (DMT) modulation. The available frequency band is divided in a number of parallel subchannels or tones. When we assume that all users in the network are synchronized, each tone can be treated independently from other tones, and so the transmit power and the number of bits can be assigned individually for each tone. This gives a large flexibility in optimally shaping the transmit spectra.

Synchronized transmission for a binder of N users can be modelled on each tone k by

$$\mathbf{y}_k = \mathbf{H}_k \mathbf{x}_k + \mathbf{z}_k \qquad k = 1 \dots K. \tag{1}$$

The vector $\mathbf{x}_k = [x_k^1, x_k^2, \dots, x_k^N]^T$ contains the transmitted signals on tone k for all N users. $\mathbf{H}_k$ is an $N \times N$ block matrix where each block element $[\mathbf{H}_k]_{i,j} = \mathbf{h}_k^{i,j} = [\mathbf{h}_k^{i,j}(1), \dots, \mathbf{h}_k^{i,j}(I)]^T$ with I the number of receivers of user i, is a vector containing the channel coefficient of the transmitter of user j to each of the receivers of user i. $[\mathbf{z}_k]_i = \mathbf{z}_k^i$ is the block vector of additive noise on tone k, containing thermal noise, alien crosstalk, RFI (radio frequency interference),..., where each block element $\mathbf{z}_k^i = [\mathbf{z}_k^i(1), \dots, \mathbf{z}_k^i(I)]^T$ contains the additive noise on the receivers of user i. The block vector $\mathbf{y}_k$ contains the received symbols where each block element $[\mathbf{y}_k]_i = [\mathbf{y}_k^i(1), \dots, \mathbf{y}_k^i(I)]^T$ is a vector with the received signals at each of the receivers of user i.

We denote the transmit power as $s_k^n \triangleq \Delta_f E\{|x_k^n|^2\}$, the positive definite noise covariance matrix as $\mathbf{N}_k^n \triangleq \Delta_f E\{\mathbf{z}_k^n \mathbf{z}_k^{nH}\}$. The vector containing the transmit power of user n on all tones is $\mathbf{s}^n \triangleq [s_1^n, s_2^n, \dots, s_K^n]^T$. The DMT symbol rate is denoted as f_s, the tone spacing as Δ_f.

It is assumed that each user treats interference from other users as noise. When the number of interfering users is large, the interference is well approximated by a Gaussian distribution. Under this assumption and under optimal receiver processing, the achievable bit rate of user n on tone k, given the transmit spectra

$\mathbf{s}_k \triangleq [s_k^1, s_k^2, \ldots, s_k^N]^T$ of all users in the system, is

$$b_k^n = \log_2 \det \left(\mathbf{I} + \frac{1}{\Gamma} \mathbf{h}_k^{n,n} s_k^n \mathbf{h}_k^{n,nH} \left(\mathbf{N}_k^n + \sum_{j \neq n} \mathbf{h}_k^{n,j} s_k^j \mathbf{h}_k^{n,jH} \right)^{-1} \right), \tag{2}$$

where Γ denotes the SNR-gap to capacity, which is a function of the desired BER, the coding gain and noise margin. The data rate and total power for user n are

$$R^n = f_s \sum_k b_k^n \qquad \text{and} \qquad P^n = \sum_k s_k^n \tag{3}$$

respectively.

3. MULTI-USER BIT LOADING

Formula (2) provides a relation between the transmit powers and bit rates. Given the transmit powers $s_k^i, i = 1 \ldots N$ of all the users it can be used to calculate the achievable bit rates. In practice however, it will be more interesting to be able to calculate the required transmit powers for all the users, given the bit rates $b_k^i, i = 1 \ldots N$. In this section a procedure is given to calculate these transmit powers.

Using the property $\det(\mathbf{I} + \mathbf{x}\mathbf{y}^H) = 1 + \mathbf{y}^H \mathbf{x}$, (2) can be transformed into:

$$b_k^n = \log_2 \left(1 + \frac{1}{\Gamma} s_k^n \mathbf{h}_k^{n,nH} \left(\mathbf{N}_k^n + \sum_{j \neq n} \mathbf{h}_k^{n,j} s_k^j \mathbf{h}_k^{n,jH} \right)^{-1} \mathbf{h}_k^{n,n} \right). \tag{4}$$

Given the bit rates b_k^n, this leads a nonlinear system of equations that can be solved for the transmit powers $s_k^i, i = 1 \ldots N$, with for each user an equation of the form:

$$\Gamma \left(2^{b_k^n} - 1 \right) = s_k^n \mathbf{h}_k^{n,nH} \left(\mathbf{N}_k^n + \sum_{j \neq n} \mathbf{h}_k^{n,j} s_k^j \mathbf{h}_k^{n,jH} \right)^{-1} \mathbf{h}_k^{n,n}. \tag{5}$$

An iterative procedure is now proposed where each user calculates an update of its required transmit power based on the transmit powers $s_k^j(t)$ of the previous iteration:

$$s_k^n(t+1) = \left[\Gamma \left(2^{b_k^n} - 1 \right) \left(\mathbf{h}_k^{n,nH} \left(\mathbf{N}_k^n + \textstyle\sum_{j \neq n} \mathbf{h}_k^{n,j} s_k^j(t) \mathbf{h}_k^{n,jH} \right)^{-1} \mathbf{h}_k^{n,n} \right)^{-1} \right]^+ \tag{6}$$

where $[x]^+ = max(x, 0)$. Starting from a specific initialization, this iterative procedure will provide a monotonically decreasing sequence of transmit powers, converging to a unique solution, as will be detailed next.

Theorem 1 (Initialization)**.** *The SISO power loading, where each user only uses one of its receiver signals, provides an upper bound for the solution of (5).*

Intuitively, since optimal receiver processing can always choose to ignore all but one of its receiver signals, the SIMO case can never require more transmit power than the SISO case to transmit the same number of bits. In the SISO case, (5) reduces to a linear system of equations that can be easily solved [1]. Reducing the SIMO case to a SISO case by arbitrarily selecting one of the receiver signals for each of the users thus leads to a bit loading problem that will result in a power loading that is an upper bound for the power loading in the SIMO case. Using this upper bound in formula (6) results in updated power levels $s_k^n(t+1)$ that can never increase. A formal proof of this theorem is given in Appendix A1.

Theorem 2 (Iteration)**.** *Update formula (6) exhibits monotonic behaviour: when the transmit powers $s_k^j(t), j \neq n$ are decreased, the updated transmit power $s_k^n(t+1)$ cannot increase. Vice versa, when the transmit powers $s_k^j(t), j \neq n$ are increased, the updated transmit power $s_k^n(t+1)$ cannot decrease.*

Intuitively, when some of the other users decrease their transmit power, the crosstalk on the user under consideration decreases and thus this user can decrease its transmit power and still achieve the same bit rate. A formal proof of this theorem is given in Appendix A2. Iterating formula (6), when initialized with the SISO solution, thus results in a decreasing series of transmit powers that leads to a stationary point of the nonlinear system of equations (5). Convergence is guaranteed since the transmit powers cannot become negative.

Theorem 3 (Uniqueness)**.** *The nonlinear system of equations (5) has a unique solution.*

A formal proof of this theorem is given in Appendix A3. As a consequence, each bit loading corresponds to a unique power loading and vice versa. A summary of the resulting procedure to calculate the power loading corresponding to a given bit loading is outlined in Algorithm 1.

Algorithm 1 Multi-user bit loading

```
t = 1, s_k^n(t) = SISO solution
while s_k^n, n = 1...N not converged do
  t = t + 1
  for all users n = 1...N do
    calculate s_k^n(t) with eq (6), based on s_k^j(t-1), j ≠ n
  end for
end while
```

4. OPTIMAL SPECTRUM BALANCING

The Optimal Spectrum Balancing (OSB) algorithm [1] tackles the spectrum management problem by formulating spectrum management as an optimization problem. The objective is to maximize the data rate of the whole binder, subject to a number of constraints.

First, there is a *total power constraint* $P^{n,tot}$ for each user $n = 1 \ldots N$, indicating that the user's total power should not exceed the maximum allowed total transmit power. On top of this constraint there is a *spectral mask constraint* $s_k^{n,mask}$ for each tone to guarantee electromagnetic compatibility with other systems. Secondly, there is a *rate constraint* $R^{n,target}$ for each user. The rate constraint indicates a minimum target data rate required by the user.

Mathematically, the optimization problem is expressed as a maximization of the sum of the data rates of the users R^n, subject to the power and rate constraints [1]:

$$\begin{array}{lll} \text{maximize}_{\mathbf{s}^1 \ldots \mathbf{s}^N} \sum_{n=1}^N R^n & & \\ \text{subject to} \quad P^n \leq P^{n,tot} & n = 1 \ldots N & \\ \qquad 0 \leq s_k^n \leq s_k^{n,mask} & n = 1 \ldots N, k = 1 \ldots K \\ \qquad R^n \geq R^{n,target} & n = 1 \ldots N & \end{array} \tag{7}$$

It is observed that (7) is a non-convex problem. Finding the global optimum requires an exhaustive search over all possible combinations of transmit spectra. Because the objective function is coupled over the users and some of the constraints couple the problem over the tones, this results in an exponential complexity in both the number of users N and the number of tones K.

In (7) the optimization is carried out over the transmit power levels of all the users. This procedure is also referred to as 'power loading'. Alternatively, the optimization can be carried out over the number of bits transmitted by each user and is then referred to as 'bit loading'.

OSB uses the dual decomposition technique to make the complexity linear in the number of tones K. The constraints coupled

$$obj_k(\mathbf{s}_k) = \sum_{n=1}^{N} \omega_n \log_2 \det \left(\mathbf{I} + \frac{1}{\Gamma} \mathbf{h}_k^{n,n} s_k^n \mathbf{h}_k^{n,nH} \left(\mathbf{N}_k^n + \sum_{j \neq n} \mathbf{h}_k^{n,j} s_k^j \mathbf{h}_k^{n,jH} \right)^{-1} \right) - \sum_{n=1}^{N} \lambda_n s_k^n \quad (10)$$

over the tones are moved into the objective function by using Lagrange multipliers $\omega = [\omega_1 \dots \omega_N]^T$ and $\lambda = [\lambda_1 \dots \lambda_N]^T$:

$$\mathbf{s}^{opt} \text{ or } \mathbf{b}^{opt} = \underset{\mathbf{s} \text{ or } \mathbf{b}}{\operatorname{argmax}} \sum_{n=1}^{N} \omega_n R^n + \sum_{n=1}^{N} \lambda_n \left(P^{n,tot} - \sum_{k=1}^{K} s_k^n\right) \quad (8)$$

$$\text{subject to} \quad \begin{array}{ll} 0 \le s_k^n \le s_k^{n,mask} & n = 1 \dots N \\ \lambda_n \ge 0, \omega_n \ge 0 & n = 1 \dots N \end{array}$$

In the first term of the objective function, the ω's weigh the rate sum over the users. Some users can be given priority over other users such that by allocating the proper weights, the rate constraints can be satisfied. Similarly, λ's represent costs for power. A larger λ_n results in less power allocated to the n-th user. Again, allocating proper costs for power results in enforcing the total power constraints. Finding the Lagrange multipliers that enforce the constraints is a convex problem and can be solved by using a subgradient type of search method [2].

For fixed Lagrange multipliers ω and λ, (8) is reduced to an optimization of a sum over tones, which can be performed by optimizing each tone individually:

$$\text{for } k = 1 \dots K : \mathbf{s}_k^{opt} \text{ or } \mathbf{b}_k^{opt} = \underset{\mathbf{s}_k \text{ or } \mathbf{b}_k}{\operatorname{argmax}} \sum_{n=1}^{N} \omega_n R_k^n - \sum_{n=1}^{N} \lambda_n s_k^n \quad (9)$$

$$\text{subject to} \quad \begin{array}{ll} 0 \le s_k^n \le s_k^{n,mask} & n = 1 \dots N \\ \lambda_n \ge 0, \omega_n \ge 0 & n = 1 \dots N \end{array}$$

Due to this decoupling of the spectrum management problem over the tones, the complexity of solving the problem becomes linear in the number of tones K instead of exponential. This is a significant reduction since in xDSL typically a large number of tones is used.

The per-tone optimization problem in (9) is still a non-convex problem. This problem is discussed in section 5.

5. EXHAUSTIVE SEARCH

One way to solve the per-tone optimization problem is to exhaustively search over all possible loadings and choose the loading that maximizes (9) for a specific tone. There are two possible approaches, namely an exhaustive search over the power loadings and an exhaustive search over the bit loadings.

5.1 Power loading

To determine the optimal power loading, an exhaustive search over all possible power loadings is performed. Applying (2) to (9) results in objective function (10) on tone k. This objective function is then evaluated for all possible combinations of transmit powers for the users. Each user can select a transmit power $s_k^n \in \mathscr{S}$, where $\mathscr{S}$ represents a discretized set of transmit powers chosen over the domain $[0 \dots s_k^{n,mask}]$. With a set $\mathscr{S}$ of cardinality S, this exhaustive search procedure requires S^N evaluations of the objective function to find the optimal transmit powers for tone k, that is, the transmit powers that maximize the objective function.

In practice, the transmit power of xDSL modems can be configured with an accuracy of 0.1 dBm/Hz [3][4][5]. A typical set $\mathscr{S}$ then has a cardinality S of more than 500.

5.2 Bit loading

To determine the optimal bit loading, an exhaustive search over all possible bit allocations is performed. The objective function on tone k is now

$$obj_k(\mathbf{b}_k) = \sum_{n=1}^{N} \omega_n f_s b_k^n - \sum_{n=1}^{N} \lambda_n s_k^n(\mathbf{b}_k), \quad (11)$$

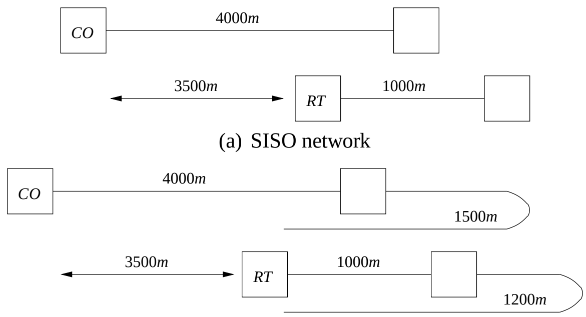
where the transmit power $s_k^n(\mathbf{b}_k)$ corresponding to a bit allocation $\mathbf{b}_k$ has to be determined using Algorithm 1 from section 3. This objective function is then evaluated for all possible combinations of bit loadings for the users. Each user can select a bit loading $b_k^n \in \mathscr{B}$, where $\mathscr{B}$ represents the set of allowed bit loadings. With a set $\mathscr{B}$ of cardinality B, this exhaustive search procedure requires B^N evaluations of the objective function to find the optimal bit loading for tone k, that is, the bit loading that maximizes the objective function.

In practice, xDSL modems can load up to 15 bits on a tone [3][4][5]. The cardinality B of the search domain for bit loading is thus significantly smaller than the cardinality S of the search domain for power loading. Performing exhaustive bit loading is therefore significantly faster than exhaustive power loading, even with a more complex objective function.

6. SIMULATION RESULTS

In this section, the performance of a 2-user SISO and SIMO system is compared. The scenario that is considered is shown in figure 1: one user is serviced by a 4000m CO line, the other by a 1000m RT line. In the SIMO case, both users have an extra receiver on the second twisted pair of their local loop. These twisted pairs are deployed from the remote terminal and are respectively 1500m and 1200m long.

Downstream ADSL2+ transmission is considered over the shortest pair in the local loop. A line diameter of 0.5mm (24 AWG) is used and the maximum total transmit power is 20.4 dBm. The SNR gap Γ is set to 12.9 dB. The tone spacing Δf = 4.3125 kHz and the DMT symbol rate f_s = 4 kHz.



(b) SIMO network

Figure 1: SISO (a) and SIMO (b) scenario

Figure 2 shows the rate region for both the SISO and SIMO case. Looking at the operating point where the CO deployed user transmits at a data rate of 3 Mbps, the RT deployed user can transmit at a rate of 27.5 Mbps in the SISO case, whereas in the SIMO case this is 30 Mbps, i.e. an increase of 9%. Vice versa, if the data rate of the RT deployed user is kept at 30 Mbps, the CO deployed user can transmit at 2 Mbps in the SISO case and 2.9 Mbps in the SIMO case, i.e. an increase of 45%.

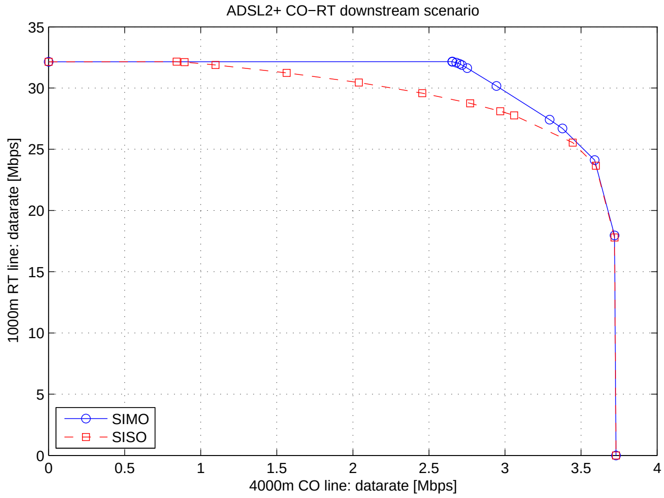


Figure 2: Rate regions for the scenarios in figure 1

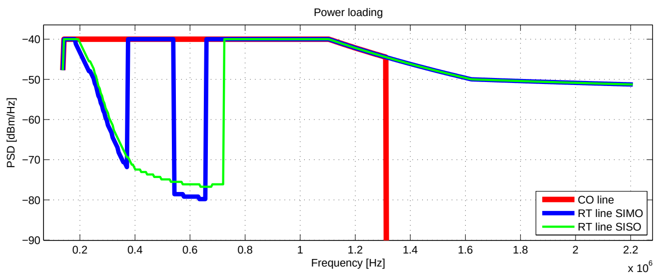


Figure 3: Transmit PSD's; CO line @ 3 Mbps.

Figure 3 shows a comparison of the transmit PSD's when the CO deployed user is transmitting at 3 Mbps. These PSD's were obtained using the power loading method described in section 5 with a set $\mathscr{S}$ of cardinality 100, equally spaced in dBm/Hz between -100 dBm/Hz and the spectral mask. To allow the CO deployed user to transmit at this data rate, the RT deployed user has to apply considerable power backoff in the SISO case. It can only allocate 65% of its available power budget in order to sufficiently protect the CO deployed user. In the SIMO case, the RT deployed user can allocate 85% of its available power budget. In the range from 0.4 MHz to 0.55 MHz, the RT deployed user can now transmit at the spectral mask since the CO deployed user can use its multiple receiver signals to reduce the effect of the crosstalk.

Figure 4 shows the result when the bit loading procedure of section 5 is applied, where in the exhaustive search integer bit loadings from 0 to 24 are evaluated. For both the SISO and SIMO case, the CO deployed user is transmitting a 2.9 Mbps. In the SISO case, the RT deployed user can achieve a data rate of 28.3 Mbps while using 50% of the available power budget. In the SIMO case, the RT deployed user can use 80% of its available power budget, leading to an increased data rate of 31.1 Mbps. Averaged over the exhaustive search, Algorithm 1 required only 2 iterations to converge when calculating the required transmit powers for a given bit loading.

7. CONCLUSION

In this paper, an algorithm for the optimal allocation of power and bits in multi-user SIMO xDSL networks has been presented. Optimal power allocation can be performed based on the well known capacity formula for SIMO systems. For optimal multi-user bit allocation, an algorithm was presented to calculate the required transmit powers for a given bit loading. The resulting optimal bit loading procedure is not only computationally more efficient than optimal power loading, it is also more relevant in practice since xDSL transmitters are usually restricted to integer bit loadings.

Finally, the simulation results showed that the use of SIMO receivers in the xDSL network can lead to significant performance improvements in cases where crosstalk would otherwise be a limiting factor.

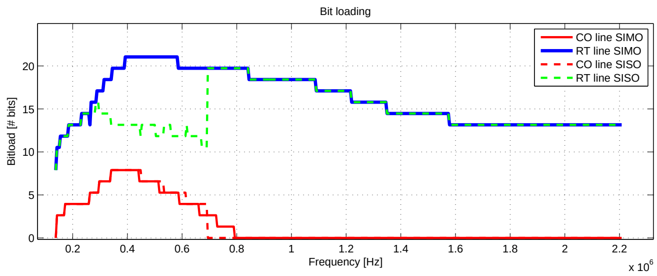


Figure 4: Bit loading; CO line @ 2.9 Mbps.

APPENDIX

A1. Proof theorem 1

Theorem (Initialization). *The SISO power loading, where each user only uses one of its receiver signals, provides an upper bound for the solution of (5).*

Proof. In the SISO case, update formula (6) reduces to a scalar equation of the form

$$s^n(t+1) = \left[\Gamma\left(2^{b^n}-1\right)\left(\underbrace{\alpha^*(a)^{-1}\alpha}_{f_{\mathrm{SISO}}}\right)^{-1}\right]^+, \qquad (12)$$

with α the channel transfer coefficient and a the total noise power, including crosstalk from other users. In the SIMO case with 2 receivers (without loss of generality) the equation is of the form

$$s^n(t+1) = \left[\Gamma\left(2^{b^n}-1\right)\left(\underbrace{[\alpha^*\beta^*]\begin{bmatrix} a & b \\ b^* & c \end{bmatrix}^{-1}\begin{bmatrix} \alpha \\ \beta \end{bmatrix}}_{f_{\mathrm{SIMO}}}\right)^{-1}\right]^+ \qquad (13)$$

In this case there is a second channel transfer coefficient β and the total noise is characterized by a positive definite Hermitian noise covariance matrix, where a is the total noise power at the first receiver and equal to the noise power in the SISO case if the transmit powers of all the users are kept the same, c is the total noise power at the second receiver and b is the correlation between the noise signals. x^* denotes the complex conjugate of x.

For the SISO case, a stationary point can be easily calculated by solving a linear system of equations. By showing that for this stationary point $f_{\mathrm{SIMO}} \geq f_{\mathrm{SISO}}$, the updated transmit powers for the SIMO case will not increase:

$$\begin{aligned} f_{\mathrm{SIMO}} &\geq f_{\mathrm{SISO}} \\ \alpha^2 \tfrac{c}{ac-|b|^2} - 2\alpha\beta \tfrac{|b|}{ac-|b|^2} + \beta^2 \tfrac{a}{ac-|b|^2} &\geq \tfrac{\alpha^2}{a} \\ (\alpha|b| - \beta a)^2 &\geq 0 \end{aligned} \qquad (14)$$

□

$$s^n(t+1) = \left[\Gamma\left(2^{b^n}-1\right)\left(\underbrace{\mathbf{h}^{n,nH}\mathbf{N}'^{n-1}\mathbf{h}^{n,n}}_{\text{original}} - \left(\frac{\Delta s^i(t)}{\underbrace{\mathbf{h}^{n,i^H}\mathbf{N}'^{n-1}\mathbf{h}^{n,i}}_{>0}\Delta s^i(t)+1}\right)\underbrace{\mathbf{h}^{n,nH}\mathbf{N}'^{n-1}\mathbf{h}^{n,i}(\mathbf{h}^{n,nH}\mathbf{N}'^{n-1}\mathbf{h}^{n,i})^H}_{>0}\right)^{-1}\right]^+ \quad (16)$$

A2. Proof theorem 2

Theorem (Iteration)**.** *Update formula (6) exhibits monotonic behaviour: when the transmit powers $s^j(t), j \neq n$ are decreased, the updated transmit power $s^n(t+1)$ cannot increase. Vice versa, when the transmit powers $s^j(t), j \neq n$ are increased, the updated transmit power $s^n(t+1)$ cannot decrease.*

Proof. Consider the case where one of the original transmit powers $s^i(t)$ is changed with an amount $\Delta s^i(t)$. Update formula (6) can then be written as $s^n(t+1) =$

$$\left[\Gamma\left(2^{b^n}-1\right)\left(\mathbf{h}^{n,nH}\left(\mathbf{N}'^n + \mathbf{h}^{n,i}\Delta s^i(t)\mathbf{h}^{n,i^H}\right)^{-1}\mathbf{h}^{n,n}\right)^{-1}\right]^+, \quad (15)$$

with $\mathbf{N}'^n = \mathbf{N}^n + \sum_{j\neq n}\mathbf{h}^{n,j}s^j(t)\mathbf{h}^{n,j^H}$ the original total noise. Using the matrix inversion lemma this becomes the scalar equation (16). It follows that if $\Delta s^i(t) > 0$ (transmit power increases), the transmit power $s^n(t+1)$ also increases.

The term $\mathbf{h}^{n,i^H}\mathbf{N}'^{n-1}\mathbf{h}^{n,i}$ can be further decomposed as

$$\mathbf{h}^{n,i^H}\mathbf{N}'^{n-1}\mathbf{h}^{n,i} = \mathbf{h}^{n,i^H}\left(\mathbf{N}''^n + \mathbf{h}^{n,i}s^i(t)\mathbf{h}^{n,i^H}\right)^{-1}\mathbf{h}^{n,i}. \quad (17)$$

Using the matrix inversion lemma this becomes scalar:

$$\begin{aligned}
&\mathbf{h}^{n,i^H}\mathbf{N}''^{n-1}\mathbf{h}^{n,i} - \\
&\quad \mathbf{h}^{n,i^H}\mathbf{N}''^{n-1}\mathbf{h}^{n,i}\left(s^i(t)^{-1} + \mathbf{h}^{n,i^H}\mathbf{N}''^{n-1}\mathbf{h}^{n,i}\right)^{-1}\mathbf{h}^{n,i^H}\mathbf{N}''^{n-1}\mathbf{h}^{n,i} \\
&= \frac{\mathbf{h}^{n,i^H}\mathbf{N}''^{n-1}\mathbf{h}^{n,i}}{\mathbf{h}^{n,i^H}\mathbf{N}''^{n-1}\mathbf{h}^{n,i}s^i(t)+1} \\
&\leq \frac{\mathbf{h}^{n,i^H}\mathbf{N}''^{n-1}\mathbf{h}^{n,i}}{\mathbf{h}^{n,i^H}\mathbf{N}''^{n-1}\mathbf{h}^{n,i}s^i(t)} = \frac{1}{s^i(t)} \\
&\leq \frac{1}{|\Delta s^i(t)|}
\end{aligned} \quad (18)$$

where the last inequality follows from the fact that transmit powers are not negative and thus a power decrease cannot be larger than the current power level. Applying this to equation (16) it follows that if $\Delta s^i(t) < 0$ and $|\Delta s^i(t)| < s^i(t)$ (transmit power decreases), the transmit power $s^n(t+1)$ also decreases.

The case where more than one of the original transmit powers $s^j(t)$ is increased can be decomposed into a series of 1-user updates. In each of these updates the transmit power $s^n(t+1)$ increases. When more than one of the original transmit powers $s^j(t)$ is decreased, decomposition leads to a series of decreases for the transmit power $s^n(t+1)$. ☐

A3. Proof theorem 3

Lemma. *To increase s^n by a factor $\alpha > 1$, at least one of the transmit powers s^j of the other users has to be increased by a factor larger than α.*

Proof. Equation (5) can be written as

$$s^n = \Gamma\left(2^{b^n}-1\right)\left(\tilde{\mathbf{h}}^{n,nH}\left(\mathbf{I} + \sum_{j\neq n}\tilde{\mathbf{h}}^{n,j}s^j\tilde{\mathbf{h}}^{n,jH}\right)^{-1}\tilde{\mathbf{h}}^{n,n}\right)^{-1} \quad (19)$$

where $\tilde{\mathbf{h}}^{n,i} = \mathbf{L}^{n-1}\mathbf{h}^{n,i}$ with $\mathbf{N}^n = \mathbf{L}^n\mathbf{L}^{nH}$ is the prewhitened channel. Using the singular value decomposition $\sum_{j\neq n}\tilde{\mathbf{h}}^{n,j}s^j\tilde{\mathbf{h}}^{n,jH} = \mathbf{USU}^H$ this becomes

$$s^n = \Gamma\left(2^{b^n}-1\right)\left((\mathbf{U}^H\tilde{\mathbf{h}}^{n,n})^H(\mathbf{I}+\mathbf{S})^{-1}\mathbf{U}^H\tilde{\mathbf{h}}^{n,n}\right)^{-1} \quad (20)$$

where $\mathbf{S}$ is diagonal with nonnegative elements.

If all transmit powers s^j are multiplied by a factor $\alpha > 1$, $\mathbf{U}$ is unchanged and $\mathbf{S}$ is multiplied by α. Writing this as

$$s^n = \alpha\Gamma\left(2^{b^n}-1\right)\left((\mathbf{U}^H\tilde{\mathbf{h}}^{n,n})^H\left(\frac{\mathbf{I}}{\alpha}+\mathbf{S}\right)^{-1}\mathbf{U}^H\tilde{\mathbf{h}}^{n,n}\right)^{-1} \quad (21)$$

it follows that s^n is increased by a factor smaller than α. Or, one way to increase s^n by a factor α, is to increase all transmit powers s^j by a factor larger than α.

More generally, to increase s^n by a factor α, at least one of the transmit powers s^j of the other users has to be increased by a factor larger than α. Indeed, if it where sufficient to increase the transmit powers s^j by a factor smaller than α, these could then be increased further until they are α times the initial value and then due to the monotonic behaviour described by theorem 2, s^n would be increased by a factor larger than α. This contradicts the observation above that if the transmit powers s^j are increased by a factor α, s^n is increased by a factor smaller than α. ☐

Theorem (Uniqueness)**.** *The nonlinear system of equations (5) has a unique solution.*

Proof. Assume there are two solutions a and b for the nonlinear system of equations (5) with $a = (s_a^1, s_a^2, \ldots, s_a^N)$ and $b = (s_b^1, s_b^2, \ldots, s_b^N)$ and assume that (without loss of generality) the users are ordered such that $s_a^1/s_b^1 > 1$ and $s_a^1/s_b^1 >= s_a^2/s_b^2 >= \ldots >= s_a^N/s_b^N$ meaning that by moving from solution b to solution a the transmit power s^1 increases the most.

This increase of s^1 when moving from solution b to solution a is with some factor α. From the lemma, it follows that at least one of the other transmit powers s^j has to increase with a factor larger than α which contradicts the fact that s^1 increases most. Therefore it is not possible to have 2 different solutions to the nonlinear system of equations (5). ☐

REFERENCES

[1] R. Cendrillon, W. Yu, M. Moonen, J. Verlinden, and T. Bostoen, “Optimal multiuser spectrum management for digital subscriber lines,” *IEEE Transactions on Communications*, vol. 54, no. 5, pp. 922–933, May 2006.

[2] P. Tsiaflakis, J. Vangorp, M. Moonen, J. Verlinden, and K. Van Acker, “An efficient lagrange multiplier search algorithm for optimal spectrum balancing in crosstalk dominated xDSL systems,” in *IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, Toulouse, France, May 2006, pp. IV 101–104.

[3] *Asymmetric digital subscriber line transceivers2 (ADSL2)*, ITU-T Std. G.992.3, 2005.

[4] *Asymmetric Digital Subscriber Line (ADSL) transceivers Extended bandwidth ADSL2 (ADSL2+)*, ITU-T Std. G.992.5, 2005.

[5] *Transmission and Multiplexing (TM); Access transmission systems on metallic access cables; Very high speed Digital Subscriber Line (VDSL); Functional Requirements, ETSI Standard TS 101 270*, ETSI Std. TS 101 270, 2003.