A SHORT REVIEW OF SIGNALS AND SYSTEMS FOR SPATIAL AUDIO

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ABSTRACT

Processing of one-dimensional signals for all kinds of applications is based on the mature theory of signals and systems. On the other hand, processing methods for acoustical signals have been developed in quite diverse application fields like seismic engineering, sonar, audio engineering, and communication acoustics. Similar techniques and algorithms appear under different names and with different notation. This short course tries to unify some of the general concepts of acoustical signal processing. First, it covers some basics from multidimensional signals and systems. By considering the acoustic wave equation as description of a special multidimensional system, its solutions can be represented in various domains regarding time, space, and their associated frequency domains. Spatial descriptions in various coordinate systems and their corresponding Fourier transforms are discussed.

1. INTRODUCTION

The theory of one-dimensional signals and systems provides unifying concepts for entities like convolution, sampling, signal transformations, transfer functions, correlation, power spectral density, stability, controllability, observability, and alike. These entities are defined without connection to certain applications, but they are highly useful for solving practical problems in signal processing applications.

Such a statement applies to multidimensional signals and systems only to a lesser extent. Although equally mature, the theory is less elegant than its one-dimensional counterpart. For example, since multidimensional polynomials cannot be factorized in general, tasks like filter design or stability tests are much more tedious than in the one-dimensional case [8, 15]. Theoretical tools for multidimensional systems are often application specific, e.g. for image processing, beamforming, seismics, control of distributed parameter systems, etc.

Apart from these applications, the emerging field of spatial audio is considered here. In this context, spatial audio means the processing of acoustic signals that are picked up by spatially distributed microphones or that serve as driving signals for spatially distributed loudspeakers. Such arrays of microphones and loudspeakers have usually some kind of regular arrangement (e.g. line, planar, circular, spherical arrays). The number of microphones may be several tens, the number of loudspeakers in large arrays reaches figures of many hundreds.

The acoustic signals picked up by microphones and produced by loudspeakers are a special case of space-time (i.e. multidimensional) signals, since their behaviour is governed by the acoustic wave equation. This fact distinguishes the theoretical foundations for spatial audio from image and video processing, since the content of images or video signals is usually not subject to mathematical constraints.

The theory of space-time signal processing has been considered from various viewpoints. Historically the oldest application dates back to the 1960s in seismic processing [2,6,14] for the identification of reflecting layers. Microphone array applications triggered the interest in directivity and beamforming [9, 11, 18]. For audio reproduction with stereophony and surround sound, any processing and coding is usally channel related. The creation of the spatial auditory impression is left to the art and experience of the sound engineer (Tonmeister). The theory of the spatial component has been neglected until recently [5].

Other spatial reproduction methods are rooted more deeply in mathematical representations of spatial functions. For example, recording and reproduction of spatial sound events with Ambisonics are based on a decomposition into spherical harmonics. The spatial microphone and loudspeaker configurations for recording and reproduction are decoupled by an intermediate signal representation, the so-called Ambisonics signals.

Wave field synthesis, a related reproduction method, attempts to recreate a sound field in an extended listening area. Its acoustical and mathematical foundation is the Kirchhoff-Helmholtz Integral and other integral relations derived from it. Here, the spatial information is represented by the Green's function of the acoustic wave equation.

It appears that the different scientific communities involved in beamforming, acoustic imaging, spatial audio coding, surround sound, Ambisonics, and wave field synthesis have so far not strived for a unifying mathematical theory for their activities. This leaves novices to any of these fields with little assistance in bridging the gap between two seemingly different worlds: The world of signal processing based on the theory of signals and systems with convolution, signal transformations, and correlations as working tools and the world of acoustics based on physical principles like conservation laws and balance equations with integral relations (Gauß, Green) and differential equations as mathematical tools.

This contribution attempts to fill this gap between signal processing and acoustics by reviewing some basic concepts from both fields on the basis of the theory of multidimensional signals and systems. A comprehensive presentation is beyond the scope of this review article, therefore only a few topics can be highlighted. Sec. 2 lists some foundations from one-dimensional signals and systems for reference in the multidimensional case in Sec. 3. Some comments on sampling are added in Sec. 4.

2. ONE-DIMENSIONAL SIGNALS AND SYSTEMS

To show the connections between one- and multidimensional signals and systems, this section briefly recalls some basic facts from the one-dimensional case. For simplicity, time-dependent signals are assumed. The selection of fundamentals given here serves as reference for the corresponding multidimensional counterparts. No derivations and not even references are given, since all standard textbooks on signals and systems cover this material.

2.1 One-Dimensional Signals

The various forms of one-dimensional signals can be classified in continuous-time and discrete-time signals, in periodic and non-periodic signals, and in their corresponding representations in the time and frequency domain. They are connected by their respective Fourier-type transforms and by the operations of sampling and periodization.

Sampling and periodization are described most concisely by a train of Dirac impulses $\delta(t)$ with unit distance

$$\text{III}(t) = \sum_{\kappa=-\infty}^{\infty} \delta(t-\kappa)\ . \tag{1}$$

Sampling a continuous-time signal $u_c(t)$ with $t \in \mathbb{R}$ at the time instants $t = nT$ with the time step size T gives a discrete-time signal (sequence) $u_d(n) = u_c(nT)$ with the index $n \in \mathbb{Z}$. This process is described by a multiplication with an impulse train

$$u_c(t)\,\frac{1}{T}\,\text{III}\left(\frac{t}{T}\right) = \sum_{n=-\infty}^{\infty} u_d(n)\delta(t-nT)\ . \tag{2}$$

On the other hand, nearly arbitrary continuous-time signals $u_c(t)$ may be turned into periodic continuous-time signals $\mathring{u}_c(t)$ with period T_0 by convolution with an impulse train

$$\mathring{u}_c(t) = u_c(t) * \frac{1}{T_0}\,\text{III}\left(\frac{t}{T_0}\right)\ . \tag{3}$$

In a similar fashion, an arbitrary sequence $u_d(n)$ can be turned into a periodic sequence $\mathring{u}_d(n)$. The front face of the cube in Fig. 1 shows these four types of signals and their relations by sampling and periodization.

For each type of signal there is a corresponding transformation from the time into the frequency domain: Fourier transform (FT), Fourier series (FS), Discrete time Fourier transform (DTFT), and Discrete Fourier transform (DFT) as listed in Table 1. The respective inverse transformations are not shown, but the one-sided Laplace transform has been added for comparison. These four frequency domain representations are shown in the back face of the cube.

Here, continuous and discrete signals as well as periodic and non-periodic signals along with their frequency domain counterparts fit nicely in one cube. These relations are more involved for multidimensional signals.

2.2 1D Systems

The most important class of systems for one-dimensional signal processing are linear and time-invariant systems (LTI systems). They allow an easy transition between time and frequency domain. There are two different widely used analysis methods, the steady state analysis with phasors and the dynamic analysis with impulse responses and transfer functions.

FT	$U_c(\omega)$	$=$	$\int_{-\infty}^{\infty} u_c(t)e^{-j\omega t}\,dt$	
FS	$\mathring{U}_c(\mu)$	$=$	$\frac{1}{T_0}\int_{0}^{T_0} \mathring{u}_c(t)e^{-j(2\pi\mu/T_0)t}\,dt$	
DTFT	$U_d(\Omega)$	$=$	$\sum_{n=-\infty}^{\infty} u_d(n)e^{-jn\Omega}$	$(\Omega = \omega T)$
DFT	$\mathring{U}_d(\mu)$	$=$	$\sum_{n=0}^{M} \mathring{u}_d(n)e^{-jn(2\pi\mu/M)}$	
$\mathscr{L}$	$\bar{U}_c(s)$	$=$	$\int_{0}^{\infty} u_c(t)e^{-st}\,dt$	

Table 1: Fourier type and Laplace signal transformations

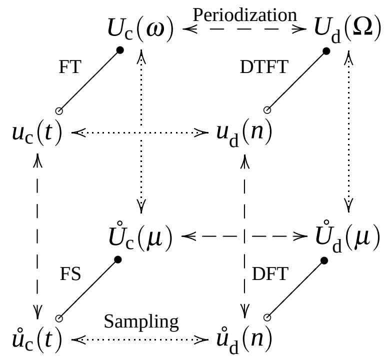

Figure 1: Relations between 1D signal representations.

Phasors are eigenfunctions of one-dimensional LTI-systems and have the form

$$u_{c0}(t) = e^{j\omega_0 t}\ . \tag{4}$$

The response to a phasor is also a phasor with a complex amplitude factor $H(\omega_0)$, the complex frequency response, see Fig. 2.

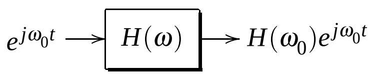

Figure 2: Steady state analysis: Response of an LTI-system to a phasor.

For the dynamic analysis, the one-sided Laplace transform allows the consideration of initial conditions and it provides a well-defined region of convergence. The response to arbitrary input signals $u_c(t)$ can be expressed either in the time domain by a convolution with the impulse response $h_c(t)$ of the LTI-system or in the frequency domain by a multiplication with its transfer function $H_c(s)$. The respective quantities in the time and frequency domain as related by the Laplace transform as shown in Fig. 3. Similar relations hold for discrete-time LTI systems with the z-transform.

The complex transfer function $H_c(s)$ can be obtained from the impulse response $h_c(t)$ or from knowledge about the structure of the system. Depending on the type of system, this knowledge is expressed by an ordinary or partial differential equation or by a combination with other LTI building blocks, e.g. delay elements.

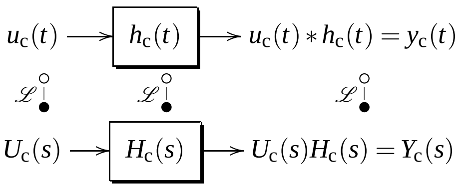

Figure 3: Dynamic analysis: Convolution and transfer function.

This short review of one-dimensional signals and systems has introduced the basic notation and serves as a reference for the multidimensional case.

3. MULTIDIMENSIONAL SIGNALS AND SYSTEMS

Only a subset of the theory of multidimensional signals and systems is considered here. At first, the multidimensional variables are restricted to the time and space coordinates. Secondly, only acoustical quantities are of interest, i.e. all signals have to obey the acoustic wave equation. Until further notice all variables are continuous and non-periodic, i.e. the multidimensional counterpart of the top-left edge of Fig. 1 is considered. The subscript c is omitted for simplicity.

3.1 Multidimensional Signals

The multidimensional signals considered here depend on time and space. When $N = 1,2,3$ is the number of spatial dimensions, then these signals have are $N+1$-dimensional or short $(N+1)$D signals. To show the different nature of time and space variables explicitly, these space-time signals are denoted by $u(t,\mathbf{x})$, where $\mathbf{x}$ is the vector of space variables. For $N = 3$, its elements are $\mathbf{x} = [x,\ y,\ z]^{\mathrm{T}}$.

With respect to the time variable, the Laplace transform $\mathscr{L}$ and the Fourier transform FT_t from Table 1 can be applied to $u(t,\mathbf{x})$. A subscript t has been added to FT_t to distinguish it from the Fourier transform $\mathrm{FT}_{\mathbf{x}}$ with respect to the space variable

$$\tilde{u}(t,\mathbf{k}) = \mathrm{FT}_{\mathbf{x}}\{u(t,\mathbf{x})\} = \int u(t,\mathbf{x})\, e^{-j\mathbf{k}^{\mathrm{T}}\mathbf{x}}\, d\mathbf{x} \qquad (5)$$

with the vector of spatial frequencies (wave numbers) $\mathbf{k} = [k_x,\ k_y,\ k_z]^{\mathrm{T}}$. The tilde denotes functions in the spatial frequency domain.

The Laplace transform $\mathscr{L}$ and Fourier transforms FT_t and $\mathrm{FT}_{\mathbf{x}}$ can be applied in four different orders as shown in Fig. 4. This figure expands the top-left edge in Fig. 1 to space-time signals.

3.2 Multidimensional Systems

For considering spatial audio in the context of multidimensional signals and systems, only one system is of paramount interest: the mechanism of acoustic wave propagation. Its physical properties are well understood and usually formulated in terms of partial differential equations. Under some

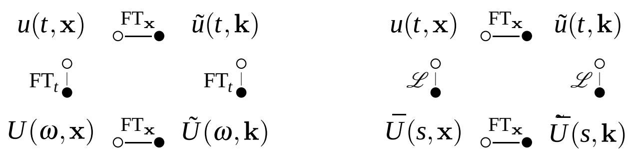

Figure 4: Fourier and Laplace transforms in time and space.

admissible simplifications, it can be described by the so-called acoustical wave equation [3,10,17,18]

$$\frac{\partial^2 u(t,\mathbf{x})}{\partial t^2} - c^2 \Delta u(t,\mathbf{x}) = v(t,\mathbf{x}), \qquad (6)$$

where Δ is the Laplace operator consisting of the second order derivatives calculated along each spatial dimension. As a physical quantity $u(t,\mathbf{x})$ corresponds to sound pressure. The excitation term $v(t,\mathbf{x})$ describes sources of acoustical energy.

3.2.1 Wave Propagation in Source Free Media

At first the propagation of waves in a source free medium is investigated. Setting the source term $v(t,\mathbf{x})$ in (6) to zero yields the corresponding autonomous behavior. To this end, usually an approach is chosen which resembles the steady state analysis shown in Fig. 2. For one-dimensional systems, the phasor from (3) is the kernel of the Fourier transform (s. Table 1). Its counterpart for space-time systems is chosen as the combination of the kernels of FT_t and $\mathrm{FT}_{\mathbf{x}}$ as

$$u_0(t,\mathbf{x}) = e^{j(\omega_0 t + \mathbf{k}_0^{\mathrm{T}}\mathbf{x})}\ . \qquad (7)$$

Due to the periodicity of the complex exponential function in (7), also $u_0(t,\mathbf{x})$ is periodic both in time and space

$$u_0(t,\mathbf{x}) = u_0(t + T_0,\ \mathbf{x} + \lambda_0 \mathbf{n}_0)\ , \qquad (8)$$

$$\text{with}\quad T_0 = \frac{2\pi}{\omega_0},\ \ \lambda_0 = \frac{2\pi}{k_0},\ \ k_0 = |\mathbf{k}_0|,\ \text{ and } \mathbf{n}_0 = \frac{1}{k_0}\mathbf{k}_0\,. \qquad (9)$$

3.2.2 Harmonic Plane Waves

The multidimensional phasor $u_0(t,\mathbf{x})$ solves the wave equation only for certain combinations between the angular frequencies in time and space, ω_0 and k_0. Inserting (7) into (6) gives

$$\left[k_0^2 - \left(\frac{\omega_0}{c}\right)^2\right] u_0(t,\mathbf{x}) = 0 \quad \text{and thus} \quad \omega_0 = \pm c k_0\ . \qquad (10)$$

The acoustic wave equation enforces this close tie between ω_0 and k_0. It is also called the *dispersion relation*.

A closer look at $u_0(t,\mathbf{x})$ under the restriction (10) shows that

$$u_0(t,\mathbf{x}) = \text{const} \quad \text{for} \quad \mathbf{n}_0^{\mathrm{T}}\mathbf{x} \pm ct = \text{const}\,. \qquad (11)$$

For each of the two signs, (11) describes a plane in space which propagates with the speed c in the direction of the normal vector $\mathbf{n}_0$. Therefore $u_0(t,\mathbf{x})$ is called a *plane wave* or due to its complex exponential nature also a *harmonic plane wave*. Physics-oriented texts sometimes emphasize the analogy to monofrequent light by the designation *monochromatic plane wave*.

3.2.3 General Plane Waves

Harmonic plane waves are very special solutions of the wave equation. Mainly they serve as building blocks for more general solutions. These are obtained by summing harmonic plane waves with different frequency dependent weighting factors $G(\omega_0)$. This process is conveniently described by an inverse Fourier transform as

$$u(t,\mathbf{x}) = \frac{1}{2\pi}\int G(\omega_0) u_0(t,\mathbf{x})\, d\omega_0 = g(t + \frac{1}{c}\mathbf{n}_0^{\mathrm{T}}\mathbf{x}) \quad (12)$$

where $g(t) = \mathrm{FT}_t^{-1}\{G(\omega)\}$. Due to the form of its argument, $u(t,\mathbf{x})$ is also a plane wave. Its wave form is no longer harmonic; instead it is an arbitray function of time $g(t)$. This wave form may be observed at a fixed point in space, e.g. by recording with a microphone.

The Fourier transform of the plane wave $u(t,\mathbf{x})$ with respect to time is given by

$$U(\omega,\mathbf{x}) = G(\omega) e^{j\frac{\omega}{c}\mathbf{n}_0^{\mathrm{T}}\mathbf{x}} \,. \quad (13)$$

The delay term indicates the time $t_0 = \frac{1}{c}\mathbf{n}_0^{\mathrm{T}}\mathbf{x}$ until the wave front reaches the point $\mathbf{x}$.

Spatial Fourier transformation of (13) gives (compare Fig. 4)

$$\tilde{U}(\omega,\mathbf{k}) = \mathrm{FT}_{\mathbf{x}}\{U(\omega,\mathbf{x})\} = G(\omega)\delta\left(\mathbf{k} - \frac{\omega}{c}\mathbf{n}_0\right) \quad (14)$$

since $\mathrm{FT}_{\mathbf{x}}\{e^{j\mathbf{k}_0^{\mathrm{T}}\mathbf{x}}\} = \delta\left(\mathbf{k} - \mathbf{k}_0\right)$. Here, δ denotes a multdimensional Dirac function; for a thorough introduction see [4].

The Dirac function in (14) restricts the values of the space-time spectrum $\tilde{U}(\omega,\mathbf{k})$ to those values of ω for which the argument of the Dirac function is zero. This statement is again the dispersion relation (10), now valid for all space-time frequencies simultaneously. Fig. 5 illustrates the behaviour of $\tilde{U}(\omega,\mathbf{k})$ for one dimension in space (i.e. (1+1)D), i.e. $\mathbf{x} = x$ and $\mathbf{k} = k$. In this case, the possible directions of the normal vector $\mathbf{n}_0$ correspond to the scalar values ± 1. Therefore the spectrum $\tilde{U}(\omega,k)$ takes the values $G(\omega_0)$ only for $\omega_0 = \pm k_0 c$ (s. (10)) and is zero elsewhere.

Note that it is important to distinguish between the frequency variables ω and k, which span the whole (1+1)D frequency plane, and the restricted values $\omega_0 = \pm k_0 c$ for which solutions of the wave equation exist.

3.2.4 Response to Source Functions

When source functions are present ($v(t,\mathbf{x}) \neq 0$) then an equivalent to the dynamic analysis from Sec. 2.2 is necessary. The excitation function $v(t,\mathbf{x})$ acts as an input and $u(t,\mathbf{x})$ is the output signal. But also initial values and – if present – boundary conditions can act as inputs.

As a simple (1+1)D example consider a problem without source term but with the initial values $u(0,x) = v(x)$ and $\dot{u}(0,x) = 0$. Starting from the wave equation in the form (dots and primes denote time and space derivatives, respectively)

$$\ddot{u}(t,x) - c^2 u''(t,x) = 0 \,, \quad (15)$$

Laplace transformation for the time variable gives

$$s^2\bar{U}(s,x) - c^2\bar{U}''(s,x) = s v(x) \quad (16)$$

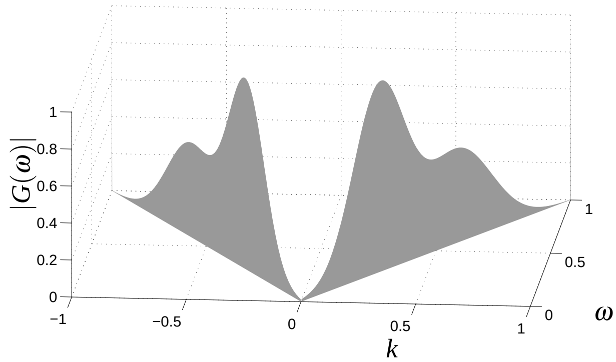

Figure 5: Frequency-domain representation in the k-ω-plane. The k- and ω axes are labeled in multiples of π. The waves travelling in both directions are assumed to be the same.

and spatial Fourier transformation results in an algebraic relation

$$s^2\tilde{\bar{U}}(s,k) + (ck)^2\tilde{\bar{U}}(s,k) = s\tilde{v}(k) \,. \quad (17)$$

It can be solved for the the transform of the solution $\tilde{\bar{U}}(s,k)$

$$\tilde{\bar{U}}(s,k) = \tilde{\bar{G}}(s,k)\tilde{v}(k) \quad (18)$$

with the two-dimensional transfer function

$$\tilde{\bar{G}}(s,k) = \frac{s}{s^2 + (ck)^2} \,. \quad (19)$$

This transfer function description of the response to an initial value in the space-time frequency domain is shown in the lower right corner of Fig. 6. The remaining entries show the corresponding representations in the other domains from Fig. 4.

$$\begin{array}{ccc} u(t,x) = g(t,x) \overset{x}{*} v(x) & \overset{\mathrm{FT}_x}{\circ\!\!-\!\!\bullet} & \tilde{u}(t,k) = \tilde{g}(t,k)\tilde{v}(k) \\ \mathscr{L}\ \circ\!\!\mid\!\!\bullet & & \circ\!\!\mid\!\!\bullet\ \mathscr{L} \\ \bar{U}(s,x) = \bar{G}(s,x) \overset{x}{*} v(x) & \overset{\mathrm{FT}_x}{\circ\!\!-\!\!\bullet} & \tilde{\bar{U}}(s,k) = \tilde{\bar{G}}(s,k)\,\tilde{v}(k) \end{array}$$

Figure 6: Input-output relations with convolution and transfer functions. Spatial convolution is denoted by $\overset{x}{*}$.

$$\begin{array}{ccc} g(t,x) = \frac{1}{2}\delta(x-ct) + \frac{1}{2}\delta(x+ct) & \overset{\mathrm{FT}_x}{\circ\!\!-\!\!\bullet} & \tilde{g}(t,k) = \cos(ckt) \\ \mathscr{L}\ \circ\!\!\mid\!\!\bullet & & \circ\!\!\mid\!\!\bullet\ \mathscr{L} \\ \bar{G}(s,x) = \frac{1}{2c}e^{-s|x|/c} & \overset{\mathrm{FT}_x}{\circ\!\!-\!\!\bullet} & \tilde{\bar{G}}(s,k) = \dfrac{s}{s^2+(ck)^2} \end{array}$$

Figure 7: Multidimensional transfer functions and Green's function. $g(t,x)$ and $\tilde{g}(t,k)$ are valid for $t \geq 0$.

The various forms of the transfer function $\tilde{\bar{G}}(s,k)$ after inverse Fourier and/or Laplace transformation are shown in Fig. 7. For an intepretation consider $g(t,x)$ in the space-time domain. It is the *Green's function* of this initial-value problem. Performing the spatial convolution with $g(t,x)$ in the top left corner of Fig. 6 gives the so-called *d'Alembert solution* in terms of forward and backward travelling waves starting at $t = 0$

$$u(t,x) = \frac{1}{2}v(x-ct) + \frac{1}{2}v(x-ct), \quad t \geq 0 . \qquad (20)$$

This simple example shows the relation between multidimensional transfer functions and the Green's functions approach for describing physical effects.

So far, a Cartesian spatial coordinate system has been adopted. Using polar or spherical coordinates leads to signals which are non-periodic in the radial direction and periodic in the angular direction(s). Then the periodic coordinates can be expanded into Fourier series (see Fig. 1) to obtain modal representations by *circular harmonics* or *spherical harmonics*. See [12] for two-dimensional case.

4. SAMPLING OF SOUND FIELDS

Up to now, a continuous representation of the acoustic wave fields was considered. For practical systems employing digital signal processing appropriate sampling of the sound field is required. Both temporal and spatial sampling has to be considered. We will focus on spatial sampling, since time domain sampling is covered by the traditional theory of one-dimensional signals (see Sec. 2.1). The sampling of multidimensional sound fields is not straightforward since the nature of acoustic fields and the underlying coordinate system has to be considered. However, this also opens up the potential for sophisticated solutions.

A first approach to sample a field would be to place microphones in an equidistant Cartesian grid in the area of interest. If evanescent contributions are neglected, two sampling positions per wave length are suitable for accurate representation. However, sound fields can be sampled more efficiently. The Rayleigh integrals or in the general case Kirchhoff-Helmholtz integrals [17] state that the sound field to be captured can be characterized by measurements taken on the boundary of the area of interest. This principle is frequently used in linear, planar and cylindrical and spherical microphone arrays. However, the underlying coordinate system implies major differences in the sampling strategy and the resulting sampling artifacts. For linear and planar microphone arrays refer e.g. to [1, 7], for cylindrical arrays to [16] and for spherical arrays to [13].

5. CONCLUSIONS

This paper gave a short review of multidimensional systems theory as applied to signals that are solutions of the wave equation. It has been shown that many concepts from acoustics (dispersion relation, plane waves, d'Alembert solution, Green's functions) follow directly by rigorous application of well-known signal transformations, once the acoustic wave equation is accepted as the description of a multidimensional system. The combination of time and space, continuous and discrete signal representations, various spatial coordinate systems, and their respective signal transformations generates a wealth of possibilites that could only be touched upon.

Different application fields have adopted their favourite representations. Spatial signal processing with line arrays of microphones usually relies on plane wave representations. Spatial recording and reproduction with Ambisonics is based on the decomposition into spherical harmonics, while wave field synthesis uses Green's functions. The link to geometrical acoustics (e.g. mirror image sources, ray and beam tracing) is provided by the decomposition into plane waves, since the normal direction may be interpreted as an acoustical ray.

REFERENCES

[1] T. Ajdler and M. Vetterli. The plenacoustic function and its sampling. *IEEE Transactions on Signal Processing*, 54(10):3790–3804, 2006.

[2] A. Berkhout. *Applied Seismic Wave Theory*. Elsevier, Amsterdam, 1987.

[3] D. Blackstock. *Foundamentals of Physical Acoustics*. John Wiley & Sons, New York, 2000.

[4] R. Bracewell. *Fourier analysis and imaging*. Kluwer Academic/Plenum Publishers, New York, 2003.

[5] J. Breebaart and C. Faller. *Spatial Audio Processing: MPEG Surround and Other Applications*. John Wiley & Sons, Chichester, 2008.

[6] K. Cattermole. *Determinate Theory of Signals and Waves*. Pentech, London, 1985.

[7] J. Coleman. Ping-pong sample times on a linear array halve the Nyquist rate. In *Proc. Int. Conf. on Speech, Acoustics, and Signal Processing (ICASSP)*, pages iv–925–928. IEEE, 2004.

[8] D. Dudgeon and R. Mersereau. *Multidimensional Digital Signal Processing*. Prentice-Hall, Englewood Cliffs, 1978.

[9] D. Johnson and D. Dudgeon. *Array Signal Processing: Concepts and Techniques*. Prentice Hall, Englewood Cliffs, 1993.

[10] A. Pierce. *Acoustics: An Introduction to Its Physical Priciples and Applications*. Acoustical Society of America, Woodbury, 1991.

[11] S. Pillai. *Array Signal Processing*. Springer-Verlag, New York, 1989.

[12] R. Rabenstein, P. Steffen, and S. Spors. Representation of two-dimensional wave fields by multidimensional signals. *Signal Processing*, 86:1341–1351, 2006.

[13] B. Rafaely, B. Weiss, and E. Bachmat. Spatial aliasing in spherical microphone arrays. *IEEE Transactions on Signal Processing*, 55(3):1003–1010, 2007.

[14] E. Robinson, T. Durrani, and L. Peardon. *Geophysical Signal Processing*. Prentice-Hall, Englewood Cliffs, 1986.

[15] J. Shanks, S. Treitel, and J. Justice. Stability and synthesis of two-dimensional recursive filters. *Audio and Electroacoustics, IEEE Transactions on*, 20(2):115–128, Jun 1972.

[16] H. Teutsch. *Modal Array Signal Processing*. Springer, Berlin, 2007.

[17] E. Williams. *Fourier Acoustics: Sound Radiation and Nearfield Acoustical Holography*. Academic Press, London, 1999.

[18] L. Ziomek. *Fundamentals of Acoustic Field Theory and Space Time Signal Processing*. CRC Press, Boca Raton, 1995.