# STATISTICAL EFFICIENCY STUDY OF BLIND IDENTIFICATION/EQUALIZATION OF TWO-PATH CHANNELS WITH THE CONSTANT-MODULUS-CRITERION

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## ABSTRACT

The statistical efficiency of a batch-processing constant-modulus blind equalizer for estimating a) the complex-valued tap weight within a two-path channel model and b) the equalized signal is investigated. Expanding the constant-modulus cost-function in a multidimensional Taylor series up to third order we derive closed-form expressions for the first-order bias and variance of the path weight and the equalized symbols as a function of the variance of the Gaussian distributed noise, the block length, and the actual channel parameters. We study random as well as deterministic symbol sequences. In the first case we compute the average of the bias and variance over zero-mean random (real-valued) signals of binary pulse amplitude modulation (PAM), and (complex-valued) signals of phase shift keying (PSK) modulation. We compare our analytical results with Monte-Carlo simulations and find good agreement for small to medium noise variance.

## 1. INTRODUCTION

Radio communication can be severely distorted due to multipath propagation. Then, an equalizer either exploiting training sequences or operating in a blind way is needed to remove the intersymbol interference. The constant-modulus-algorithm (CMA) is by far the most known and studied method for blind channel equalization. It was first introduced for blind equalization of quadrature amplitude modulation (QAM) signals in [1] and of PAM and FM signals in [2]. A review including a large list of publications about the constant modulus criterion for blind equalization can be found in [3]. Theoretical analysis of the CMA mainly deal with a study of the convergence behavior, see e.g. [4, 5, 6], or of the error surface [3]. Results concerning an upper bound for the mean-squared-error (MSE) can be found in [7] and the steady-state MSE for the equalization of noise-free non-constant-modulus signals with CMA variants have been published in [8].

In this contribution, we present results of a statistical analysis, both theoretical and numerical, of a batch-processing constant-modulus blind equalizer using a parametric channel model. Our theoretical calculations are based on a multidimensional Taylor series of the constant-modulus cost-function in a power series in terms of the additive noise and the deviations of the estimated parameters from the true ones. In order to compute the bias, we have to extend the Taylor series up to third order, as within a second-order expansion the parameter deviations depend linearly on the additive noise which is assumed to be zero-mean. We present expressions for the first-order bias and variance of the path parameters as well as an expression for the variance of the equalized symbols and compare them with the results of Monte-Carlo simulations. We find that although several approximations are involved in our theoretical results deviations from the numerical simulations are rather insignificant.

The channel model that we employed in our analysis is often used as a simple model for a HF-communication channel. It follows the considerations leading to the Watterson model [9] which underlies the ITU recommendation [10] for testing HF modems: Although the HF ionospheric channels are non-stationary both in frequency and time they can be considered as nearly stationary for band-limited signals and sufficiently short times. Furthermore, in most cases the HF channel is of specular nature, and the representative channel parameter combinations of the ITU recommendations include only two fading paths without frequency shifts following a complex Gaussian random process. As for quiet and moderate conditions, which have more than 90 % probability of occurrence, the frequency spread of the random process is rather small (up to 1 Hz) we assume the tap-gain function of each path to be constant for each block of received data. In this contribution we consider two cases: a channel with purely real-valued tap-gains distorting a PAM signal, and the more general case of a channel with complex-valued tap gains distorting a complex-valued PSK signal.

The paper is organized as follows: we start with the definition of the constant-modulus cost-function. After listing general expressions for the bias and variance of the path parameters and the equalized signal we present the main results for the CM cost-function and compare with MC-simulations.

## 2. CONSTANT-MODULUS COST-FUNCTION

The algorithm which is analyzed in this work is a variant of the well known CMA. One modification consists in employing a parametric zero-forcing filter and the second difference is that the channel parameters are then estimated batch-wise from the global minimum of a cost-function using the constant-modulus-criterion. The impulse response of the channel is modeled as

$$h(t) = \delta(t) + \lambda\,\delta(t-\tau) \tag{1}$$

with complex-valued path attenuation $\lambda = \alpha e^{j\phi}$ and delay $\tau$. (As the channel can only be identified up to an unknown overall factor, we set the amplitude of the first path equal to one.)

The received lowpass signal $x(t)$ is then the convolution of the complex envelope

$$y(t) = \sum_{i=-\infty}^{\infty} s_i g(t-iT)\,, \tag{2}$$

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with the channel $h(t)$. Here, $g(t)$ is the combined transmitter and receiver filter which fulfills the first Nyquist condition, and $s_i$ are the (complex-valued) information-bearing symbols with magnitude $|s_i| = 1$. Over-sampling the received signal by a factor of $M$ times the symbol rate $T$, we denote the data in one batch $\mathbf{x} = (x_1, \ldots, x_{\bar{N}M})^T \in \mathbb{C}^{\bar{N}M}$ where $\bar{N}$ is the number of symbols in the batch and $(\cdot)^T$ denotes transposition. The delay is $\tau = LT/M$ with $L$ being an integer. In order to equalize the received signal a zero-forcing filter $\mathbf{w}$ of length $WL+1$, with $W$ being an integer, is applied to the received data. The filter is parameterized by the path weight $\lambda$ and the delay. The estimation of the delay will not be the issue of the statistical analysis presented here. The delay estimation can be carried out before, independently from the estimation of $\lambda$, and does typically not rely on the constant-modulus criterion. Furthermore, provided that the delay does not change in time, the estimation of the delay is rather robust for a sufficiently long observation period. We proceed on the assumption that the estimated delay equals the true one. In the following, we will treat the case that the second path is the less dominant one with weight $|\lambda| < 1$, so that the filter acts on the past data $\mathbf{x}_{nM} = (x_{nM-WL}, \ldots, x_{nM})^T$. (The other case where the first path is weaker than the following one can be treated analogously. Then $|\lambda| > 1$ and the transversal filter acts on the 'future' data.) Explicitly, the zero-forcing equalized signal is given by:

$$z_n = \mathbf{x}_{nM}^T \mathbf{w} = \sum_{k=0}^{W} x_{nM-kL} (-\lambda)^k . \tag{3}$$

We note that in our Monte-Carlo simulations for a given value of $\lambda$, the filter order $W$ is chosen such that $|\lambda|^W < 10^{-10}$. On the other hand, in our theoretical analysis we assume an infinite filter length $W \to \infty$ as otherwise the estimate for the path parameter would have an additional bias.

It is obvious, that from a batch of length of $\bar{N}$ symbols we can get out only a smaller number of $N = \bar{N} - \lceil WL/M \rceil + 1$ equalized symbols. ($\lceil x \rceil$ denotes the smallest possible integer larger than or equal to $x$.)

The cost-function in which $N$ equalized symbols are considered depends on the received data, the path parameters, the particular transmitted symbol sequence $\mathbf{s} = \{s_i\}_{i=-\infty}^{\infty}$, and an unknown, overall scaling factor $\gamma \in \mathbb{R}$ for adjusting the magnitude of the equalized signal:

$$c(\mathbf{x}; \alpha, \phi, \gamma | \mathbf{s}) = \sum_{n=1}^{N} (|\gamma z_n|^2 - 1)^2 . \tag{4}$$

## 3. GENERAL EXPRESSIONS FOR FIRST-ORDER BIAS AND VARIANCE

The path parameters are found by minimizing the CM cost-function $c(\mathbf{x}; \rho | \mathbf{s})$ with respect to $\rho = (\alpha, \phi, \gamma)^T$. (In case of complex-valued receive data, we define the real-valued receive vector $\mathbf{x} = (x_{1-WL}^{(r)}, x_{1-WL}^{(i)}, \ldots, x_{NM}^{(r)}, x_{NM}^{(i)})^T$ and consider the real-valued cost-function as a function of real variables only.) In case of no noise, the minimum of the cost function is at the position of the true parameter $\hat{\rho}$. With additional noise on the received signal $\hat{\mathbf{x}}$, i.e. $\hat{\mathbf{x}}$ changes to $\mathbf{x} = \hat{\mathbf{x}} + \delta\mathbf{x}$, the position of the minimum will change correspondingly from $\hat{\rho}$ to $\rho = \hat{\rho} + \delta\rho$.

An approximation to the position of the new minimum can be found by using a Taylor expansion of the cost function around $(\hat{\mathbf{x}}, \hat{\rho})$. A general expression for the multi-dimensional Taylor series can be found for example in [11]:

$$f(\mathbf{y}) = \sum_{m=0}^{\infty} \frac{1}{m!} \left[ (\delta \mathbf{y}^T \nabla_{\mathbf{y}})^m f \right]_{\mathbf{y}=\hat{\mathbf{y}}} . \tag{5}$$

Here, we have to partition the variables into one part containing the received signal and another part containing the channel parameter, $\mathbf{y} = (\mathbf{x}^T, \rho^T)^T$. The Taylor series up to third order then involves partial derivatives of second order which we collect in the following matrices

$$\mathbf{D}_{ij}^{(xx)} = \left.\frac{\partial^2 c}{\partial x_i \partial x_j}\right|_{\hat{\mathbf{x}},\hat{\rho}} \quad \mathbf{D}_{ik}^{(x\rho)} = \left.\frac{\partial^2 c}{\partial x_i \partial \rho_k}\right|_{\hat{\mathbf{x}},\hat{\rho}} \quad \mathbf{D}_{kl}^{(\rho\rho)} = \left.\frac{\partial^2 c}{\partial \rho_k \partial \rho_l}\right|_{\hat{\mathbf{x}},\hat{\rho}} \tag{6}$$

and partial derivatives of third order which are comprised in the tensors

$$\mathscr{D}_{ikl}^{(x\rho\rho)} = \left.\frac{\partial^3 c}{\partial x_i \partial \rho_k \partial \rho_l}\right|_{\hat{\mathbf{x}},\hat{\rho}} , \quad \mathscr{D}_{ijk}^{(xx\rho)} = \left.\frac{\partial^3 c}{\partial x_i \partial x_j \partial \rho_k}\right|_{\hat{\mathbf{x}},\hat{\rho}} . \tag{7}$$

Furthermore, we define the multiplication of a tensor $\mathscr{D}$ with a vector $\mathbf{z}$ with respect to the first dimension of the tensor to be the resulting matrix $(\mathscr{D} *_1 \mathbf{z})$ with elements

$$(\mathscr{D} *_1 \mathbf{z})_{jk} \equiv \mathscr{D}_{ijk} z_i , \tag{8}$$

where we used Einstein's summation convention, i.e. if an index occurs twice in a term, summation over the index is implied. Then, the third-order Taylor expansion reads

$$\begin{aligned} c(\mathbf{x}; \rho | \mathbf{s}) \approx\, & c(\hat{\mathbf{x}}; \hat{\rho} | \mathbf{s}) + \left[ \nabla_{\mathbf{x}}^T c \; \nabla_{\rho}^T c \right] \begin{bmatrix} \delta\mathbf{x} \\ \delta\rho \end{bmatrix} \\ & + \frac{1}{2} \left[ \delta\mathbf{x}^T \delta\rho^T \right] \begin{bmatrix} \mathbf{D}^{(xx)} & \mathbf{D}^{(x\rho)} \\ \mathbf{D}^{(\rho x)} & \mathbf{D}^{(\rho\rho)} \end{bmatrix} \begin{bmatrix} \delta\mathbf{x} \\ \delta\rho \end{bmatrix} \\ & + \frac{1}{6} (\delta\mathbf{x}^T \nabla_{\mathbf{x}})^3 c + \frac{1}{2} \delta\mathbf{x}^T (\mathscr{D}^{(xx\rho)} *_1 \delta\mathbf{x}) \delta\rho \\ & + \frac{1}{2} \delta\rho^T (\mathscr{D}^{(x\rho\rho)} *_1 \delta\mathbf{x}) \delta\rho + \frac{1}{6} (\delta\rho^T \nabla_{\rho})^3 c . \end{aligned} \tag{9}$$

As the cost-function contains $P = MN + WL$ values of the received signal, each perturbed by an amount of $\delta\mathbf{x}$, the estimation error $\delta\rho$ of a small number of parameters, here just three, is typically much smaller than any one of the components of $\delta\mathbf{x}$. Moreover, the second last term consists of $P$ parts, and the third last term even of $P^2$ parts. Then, provided that the third order partial derivative with respect to $\rho$ is not significantly larger than the other partial derivatives, we will skip the last term in Eq. 9 which is cubic in $\delta\rho$.

From the necessary condition for the minimum $\nabla_{\delta\rho} c = 0$ we get

$$\begin{aligned} \delta\rho = & -\left( \mathbf{D}^{(\rho\rho)} + (\mathscr{D}^{(x\rho\rho)} *_1 \delta\mathbf{x}) \right)^{-1} \\ & \cdot (\mathbf{D}^{(\rho x)} \delta\mathbf{x} + \frac{1}{2} (\mathscr{D}^{(xx\rho)} *_1 \delta\mathbf{x})^T \delta\mathbf{x}) . \end{aligned} \tag{10}$$

The bias$(\delta\rho) = \mathrm{E}\left[\delta\rho\right]$ is obtained by taking the expectation operation, using for $\delta\mathbf{x}$

$$\mathrm{E}\left[ \delta\mathbf{x} \delta\mathbf{x}^T \right] = \begin{cases} \sigma_n^2 \mathbf{I} & \text{for real-valued receive data} \\ \frac{1}{2} \sigma_n^2 \mathbf{I} & \text{for complex-valued receive data} \end{cases}$$

$$\mathrm{E}[\delta\mathbf{x}] = 0 \ , \quad \mathrm{E}[\delta x_i \delta x_j \delta x_k] = 0 \,, \tag{11}$$

and expanding the denominator of Eq. 10 for small $\delta\mathbf{x}$ (provided that none of the eigenvalues of the matrix $(\mathbf{D}^{(\rho\rho)})^{-1}(\mathscr{D}^{(x\rho\rho)} *_1 \delta\mathbf{x})$ equals one),

$$\begin{aligned}\left(\mathbf{D}^{(\rho\rho)} + (\mathscr{D}^{(x\rho\rho)} *_1 \delta\mathbf{x})\right)^{-1} \approx \\ (\mathbf{D}^{(\rho\rho)})^{-1}\left(1 - (\mathbf{D}^{(\rho\rho)})^{-1}(\mathscr{D}^{(x\rho\rho)} *_1 \delta\mathbf{x}) + \ \ldots \ \right).\end{aligned} \tag{12}$$

Finally, defining the vectors with elements $\Delta_k^{(1)} = \mathscr{D}_{jmk}^{(x\rho\rho)}\mathbf{D}_{mj}^{(\rho x)}$ and $\Delta_k^{(2)} = \mathscr{D}_{iik}^{(xx\rho)}$, we get for the leading order term of the bias

$$\begin{aligned}\mathrm{bias}(\rho) &= \mathrm{E}[\delta\rho] \\ &= \sigma_n^2(\mathbf{D}^{(\rho\rho)})^{-1}\left((\mathbf{D}^{(\rho\rho)})^{-1}\Delta^{(1)} - \frac{1}{2}\Delta^{(2)}\right).\end{aligned} \tag{13}$$

If we consider only the term of order $\mathscr{O}(\delta\mathbf{x})$ in Eqs. (10,12) which would have been obtained from a second-order Taylor series which, of course, would lead to a zero bias, the leading order variance is identical to the leading order term of the MSE. The first-order variance is then contained in:

$$\begin{aligned}\mathrm{var}(\rho) &= \mathrm{E}\left[\delta\rho\delta\rho^T\right] \\ &= \sigma_n^2(\mathbf{D}^{(\rho\rho)})^{-1}\mathbf{D}^{(\rho x)}\mathbf{D}^{(x\rho)}(\mathbf{D}^{(\rho\rho)})^{-1}.\end{aligned} \tag{14}$$

An expression for the variance of the equalized signal can be derived by expanding Eq. (3) in a Taylor series in $\mathbf{x}, \lambda$ (which is trivial for $\mathbf{x}$ as $z_n$ depends linearly on $\mathbf{x}$). Considering only the leading order in $\delta\mathbf{x}, \delta\lambda$ we obtain for the error of the equalized signal $z_n$

$$\begin{aligned}\delta z_n &= z(\hat{\mathbf{x}}_{nM} + \delta\mathbf{x}_{nM}, \hat{\lambda} + \delta\lambda) - \hat{\mathbf{x}}_{nM}^T\mathbf{w}(\hat{\lambda}) \\ &\approx \delta\mathbf{x}_{nM}^T\mathbf{w} + \hat{\mathbf{x}}_{nM}^T \left.\frac{\partial\mathbf{w}}{\partial\lambda}\right|_{\hat{\lambda}} \delta\lambda \ .\end{aligned} \tag{15}$$

The first-order variance is computed straightforward:

$$\begin{aligned}\mathrm{E}[\delta z_n \delta z_n^*] = \sigma_n^2|\mathbf{w}|^2 + \left|\hat{\mathbf{x}}_{nM}^T\frac{\partial\mathbf{w}}{\partial\lambda}\right|^2 \mathrm{E}[\delta\alpha^2] \\ - 2\sigma_n^2 \,\Re\left\{\hat{\mathbf{x}}_{nM}^T\frac{\partial\mathbf{w}}{\partial\lambda}\left(D^{(\lambda\lambda^*)}\right)^{-1} D^{(\lambda^* x)}\mathbf{w}^*\right\}.\end{aligned} \tag{16}$$

Here, we have used the second-order Taylor series of the cost function in terms of the complex signal $\mathbf{x}$ and the complex path parameter $\lambda$, ignoring the scaling factor $\gamma$ in the first instance.

The bias and variance can be computed for a particular symbol sequence $\mathbf{s}$. On the other hand, we are interested in the average over all possible symbol sequences with a given probability distribution $w(\mathbf{s})$. Regarding the expectation of the above expressions with respect to the symbol sequence, e.g. $\mathrm{E}_s[\mathrm{E}[\delta\alpha]] = \int d\mathbf{s}\ \mathrm{E}[\delta\alpha]w(\mathbf{s})$ we see that an analytical calculation is not feasible. On the other hand, the expectations of the individual terms in the expressions for the bias and variance can be carried out analytically. Because these individual terms have distributions which are very well localized around their mean for large block length $N$ we approximate/replace the above expectation by proper expressions containing the expectations of the individual terms. More details of this argument will be published elsewhere.

## 4. THEORETICAL RESULTS FOR THE CONSTANT-MODULUS COST-FUNCTION

We studied three different cases with respect to the nature of the channel and symbols. All different cases have the following assumptions in common:

**A0.1** The filter $g(t)$ is a rectangular pulse:

$$g(t) = \begin{cases} T^{-1} & 0 \le t \le T \\ 0 & \text{otherwise}\end{cases} \tag{17}$$

(Note: For path delays larger than the length of the pulse the results below are the same for arbitrary pulse which fulfills the first Nyquist condition.)

**A0.2** The path delay $\tau$ is an integer multiple of the symbol period $T$. (Note: For different path delays the results below change slightly.)

The calculation of the results below can be carried out by computing the formulas like Eqs. (13,14,16) containing the partial derivatives of the cost function. The terms involving multiple sums can be resolved with the help of the assumptions Eqs. (18,24) and formulas for geometric series. Details of the rather long calculations will be published elsewhere.

### 4.1 Random complex-valued symbol sequence transmitted over complex two-path channel

In this case, a n-PSK signal passes the two-path channel Eq. (1) with complex path weight $\lambda = \alpha e^{j\phi}$ and we make the following assumptions:

**A1.1** The symbols $s_n \in \mathbb{C}$ have the expectations

$$\mathrm{E}[s_n] = 0, \mathrm{E}[s_n s_m^*] = \delta_{n,m}, \mathrm{E}[s_n s_m] = 0\,. \tag{18}$$

**A1.2** The noise $\delta x_i$ is a discrete complex-valued Gaussian random process with

$$\mathrm{E}[\delta x_i] = 0 \ , \quad \mathrm{E}\left[\delta x_i \delta x_j^*\right] = \sigma_n^2\delta_{i,j} \quad . \tag{19}$$

Under the above assumptions, the term of leading order in $\sigma_n^2$ for the variance and bias of the CM-estimate for the path parameters of a two-path channel with tap gain $\lambda = \alpha e^{j\phi}$ is asymptotically (for large block length $N$)

$$\mathrm{E}_s\left[\delta\alpha^2\right] = \frac{\sigma_n^2}{N(1-\alpha^2)} + \mathscr{O}(N^{-2}) \tag{20}$$

$$\mathrm{E}_s[\delta\alpha] = -\sigma_n^2\frac{\alpha}{1-\alpha^2} + \mathscr{O}(N^{-1}) \tag{21}$$

$$\mathrm{E}_s\left[\delta\phi^2\right] = \frac{1}{\alpha^2}\mathrm{E}_s\left[\delta\alpha^2\right] , \ \ \mathrm{E}_s[\delta\phi] = 0 \tag{22}$$

and the leading order term for the variance of the equalized symbols in the case $\delta\gamma = 0$ reads

$$\mathrm{E}\left[|\delta z|^2\right] = \frac{\sigma_n^2}{1-\alpha^2} + \mathscr{O}(N^{-1})\,. \tag{23}$$

### 4.2 Random real-valued symbol sequence transmitted over real two-path channel

Here, a binary PAM signal passes the channel Eq. (1) with real path weight $\lambda = \alpha$ and we assume:

**A2.1** The symbols $s_n \in \{+1, -1\}$ have the expectations

$$\mathrm{E}[s_n] = 0 \ , \quad \mathrm{E}[s_n s_m] = \delta_{n,m} \quad . \tag{24}$$

**A2.2** The noise $\delta x_i$ is a discrete real-valued Gaussian random process with

$$\mathrm{E}[\delta x_i] = 0 \ , \quad \mathrm{E}[\delta x_i \delta x_j] = \sigma_n^2 \delta_{i,j} \quad . \tag{25}$$

Under the above assumptions, the term of leading order in $\sigma_n^2$ for the variance and bias of the CM-estimate for the path parameter of a two-path channel with tap gain $\alpha$ is asymptotically (for large block length $N$)

$$\mathrm{E}\left[\delta\alpha^2\right] = \frac{\sigma_n^2}{N}\frac{1+\alpha^2}{1-\alpha^2} + \mathcal{O}(N^{-2}) \tag{26}$$

$$\mathrm{E}[\delta\alpha] \ = -\sigma_n^2 \frac{\alpha}{1-\alpha^2} + \frac{\sigma_n^2}{N}\frac{6\alpha}{(1-\alpha^2)^2} + \mathcal{O}(N^{-2}) \, . \tag{27}$$

### 4.3 Periodic real-valued symbol sequence transmitted over real two-path channel

It is interesting to compute results for a specific symbol sequence. There are some particular symbol sequences which show a relatively large deviation from the average. E.g. for a deterministic periodic binary PAM signal $\{\overline{+1-1}\}$ and the assumption A2.2 we find for the first-order variance and bias

$$\mathrm{E}\left[\delta\alpha^2\right] = \frac{\sigma_n^2}{N}\left(1 - \frac{1}{N}\frac{2\alpha}{1-\alpha^2}\right) + \mathcal{O}(N^{-2}) \tag{28}$$

$$\mathrm{E}[\delta\alpha] = -\frac{\sigma_n^2}{2}\frac{3+5\alpha}{(1+\alpha)^2} + \mathcal{O}(N^{-1}) \, . \tag{29}$$

## 5. NUMERICAL COMPARISONS

We compared the analytical results with Monte-Carlo (MC) simulations. We considered a block length of $N = 100$, i.e. there are $2^N \approx 10^{30}$ different symbol sequences. In order to obtain smooth curves, we had to use a large number of at least 200,000 MC runs, resulting in a computation time of about one day on a PC.

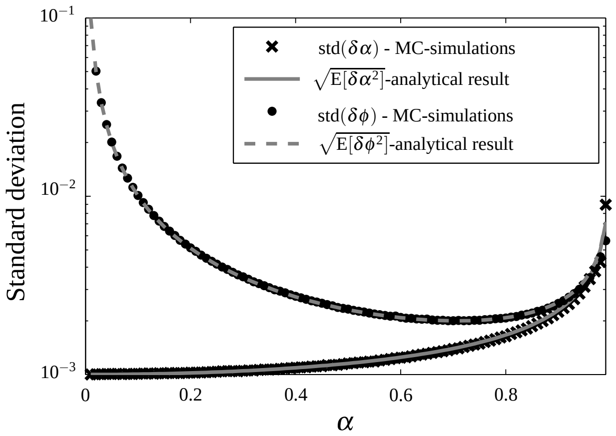


Figure 1: Standard deviation of the path parameters $\alpha, \phi$ for $\sigma_n = 0.01$ and $N = 100$. The transmitted symbols are 4-PSK.

For the case of a complex path weight and 4-PSK symbols Figs. 1 and 2 show the standard deviation and bias for varying path attenuation $\alpha$. In order to avoid confusion, we note that the theoretical variance is based on a second order Taylor expansion which by itself does not lead to a bias.

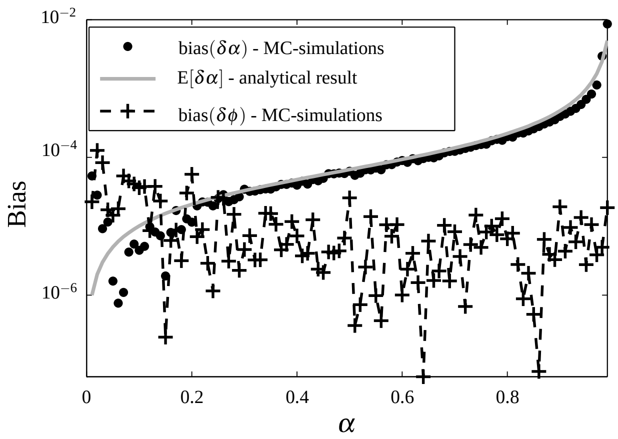


Figure 2: Bias of the path parameters $\alpha, \phi$ for $\sigma_n = 0.01$ and $N = 100$. The transmitted symbols are 4-PSK.

Therefore, we have to compare with the standard deviation of the numerical results, not with the MSE. On the other hand, the equalized symbols do not show a bias, therefore, Fig. 3 displays the MSE of the equalized signals. We find very good overall agreement, only for values of $\alpha$ close to one, the numerical results are problematic due to the filter length approaching infinity.

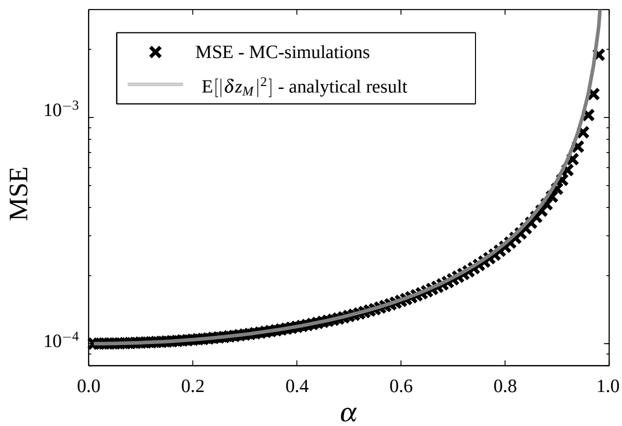


Figure 3: Mean-Squared-Error of the equalized 4-PSK symbols at $\sigma_n = 0.01$ and $N = 100$.

For the case of real path weight and BPSK symbols Figs. 4 and 5 display the bias and standard deviation for varying path attenuation $\alpha$. For the bias we plotted two lines: the first one corresponds to the first term in Eq. 27, which is the leading order term in $N$. The small gap to the numerical result is filled by including both terms. In Fig. 6 the bias and the square root of the variance are plotted for varying noise variance. We find that the results start to differ for a noise variance larger than about $\sigma_n^2 \approx 0.03$, which shows the limitations of the Taylor series expansion. Finally, in Fig. 7 we show the dependence of the results on the block length $N$.

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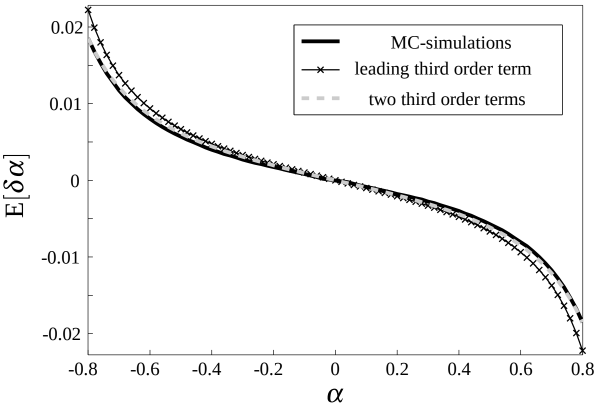


Figure 4: Comparison of analytical results with Monte-Carlo simulations for $\sigma_n = 0.1$ and $N = 100$. Including both third order terms gives reasonable agreement.

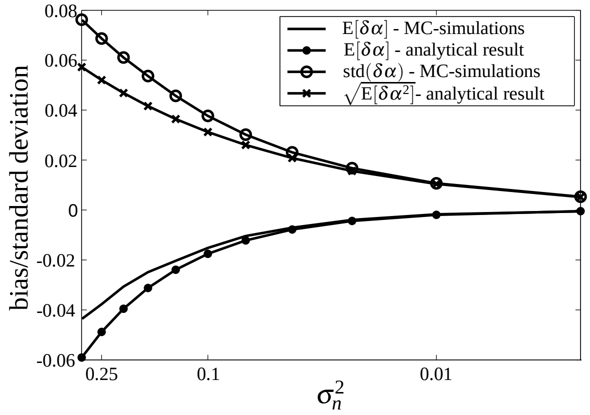


Figure 6: Comparison of analytical and numerical results for bias and standard deviation for $\alpha = 0.2$ and $N = 100$ as a function of the noise variance.

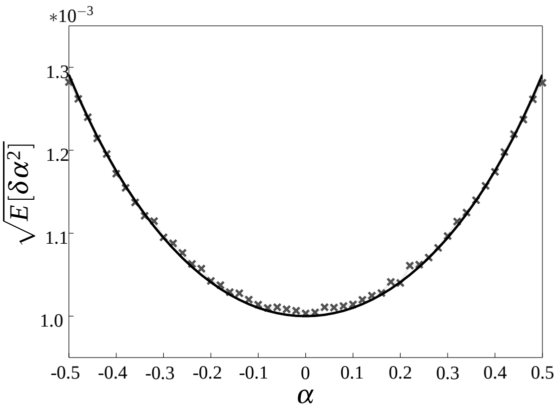


Figure 5: Square root of the variance of the path parameter $\alpha$ for $\sigma_n = 0.01$ and $N = 100$: solid line: analytical result. Crosses: Standard deviation from MC-simulations.

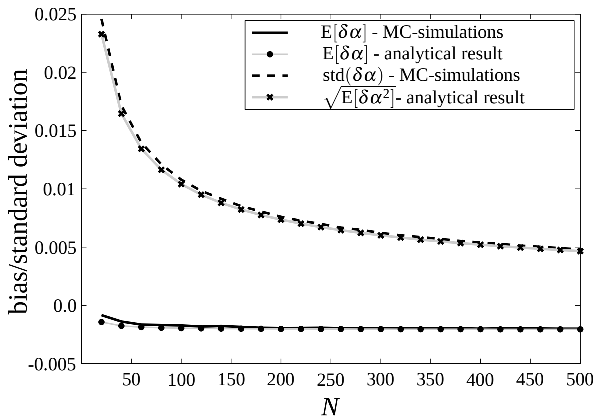


Figure 7: Comparison of analytical and numerical results for bias and standard deviation for $\alpha = 0.2$ and $\sigma_n = 0.1$ as a function of the block length $N$.


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