# A NEW DESIGN METHOD FOR IIR DIAMOND-SHAPED FILTERS 

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#### Abstract

This paper proposes a new design method for two-dimensional diamond-shaped IIR filters. The design starts from a desired 1D prototype filter, to which a frequency transformation is applied, which leads to the $2 D$ filter with the desired shape. The frequency transformation is derived by specifying the filter shape in polar coordinates in the frequency plane. The design method combines the analytical approach with numerical approximations. Starting from a digital prototype filter, we approach two design cases, namely diamond-shaped filters with complex transfer functions, then zero-phase diamond-shaped filters which are particularly useful in image processing due to the absence of phase distortions.


## 1. INTRODUCTION

The field of two-dimensional filters and their design methods has been approached by many researchers [1], [2]. A commonly-used design technique for 2D filters is to start from a specified 1D prototype filter and transform its transfer function using various frequency mappings in order to obtain a 2D filter with a desired frequency response. Some important papers regarding 2D filter design through spectral transformations are [3]-[5]. In [5] the problem of 2D filter stability is studied in detail.
Diamond-shaped filters are currently used as anti-aliasing filters for the conversion between signals sampled on the rectangular sampling grid and the quincunx sampling grid. Various aspects and methods of design for diamond filters were approached in [6]-[8].
In this paper we propose a new analytical design method for dia-mond-shaped filters. This technique can be generally used for designing a class of filters specified by a periodic function expressed in polar coordinates in the frequency plane. This idea was also used in [9], where a class of zero-phase diamond-shaped filters were designed. Zero-phase filters are particularly useful in various image processing applications due to the absence of phase distortions. Here the basic ideas of the method are reconsidered and a more general case is approached
The contour plots of their frequency response, resulted as sections with planes parallel with the frequency plane, can be defined as closed curves which can be described in terms of a variable radius which is a periodic function of the current angle formed with one of the axes. This periodic radius can be developed as a rational periodic function. Then, using a desired 1D prototype filter with factorized transfer function, the 2D diamond filters can be obtained by a 1 D to 2 D frequency transformation. The 2D filter function results directly in a factorized form, which is an advantage in implementation.

This paper is focused mainly on presenting the proposed design method and describes in detail the design steps. Some design examples are also provided. We did not approach here any applications of diamond-shaped filters, which are extensively treated in other works.

## 2. 2D FILTERS DEFINED IN POLAR COORDINATES

### 2.1 Spectral Transformation for the Design of 2D Filters in Polar Coordinates from 1D Prototypes

We will approach here a particular class of 2D filters, namely filters whose frequency response is symmetric about the origin and has at the same time an angular periodicity. For such filters, if we consider any level contour resulted from the intersection of the frequency response with a horizontal plane, the contour has to be a closed curve which can be described in polar coordinates by: $\rho=\rho(\varphi)$ where $\varphi$ is the angle formed by the radius OP with the $\omega_{1}$ - axis, as shown in Fig.1(a) for a four-lobe filter. Therefore $\rho(\varphi)$ is a periodic function of the angle $\varphi$, for $\varphi \in[0,2 \pi]$.
The proposed design method for this class of 2D filters is based on a frequency transformation of the form [9]:

$$
\begin{equation*}
F: \mathbb{R} \rightarrow \mathbb{C}^{2}, \omega^{2} \rightarrow F\left(z_{1}, z_{2}\right) \tag{1}
\end{equation*}
$$

Through this transformation we will obtain low-pass type filters, in the sense that their frequency characteristic contains the origin of the frequency plane, and they are symmetric about the origin, as in fact are most 2D filters currently used in image processing.
The frequency transformation (1) is a mapping from the real frequency axis $\omega$ to the complex plane $\left(z_{1}, z_{2}\right)$ and will be defined through the intermediate frequency mapping:

$$
\begin{equation*}
F_{1}: \mathbb{R} \rightarrow \mathbb{R}^{2}, \omega \rightarrow F_{1}\left(\omega_{1}, \omega_{2}\right)=\sqrt{\omega_{1}^{2}+\omega_{2}^{2}} / \rho\left(\omega_{1}, \omega_{2}\right) \tag{2}
\end{equation*}
$$

Here the function $\rho\left(\omega_{1}, \omega_{2}\right)$ plays the role of a radial compressing function and is initially determined in the angular variable $\varphi$ as $\rho(\varphi)$. In the frequency plane $\left(\omega_{1}, \omega_{2}\right)$ we have:

$$
\begin{equation*}
\cos \varphi=\omega_{1} / \sqrt{\omega_{1}^{2}+\omega_{2}^{2}} \tag{3}
\end{equation*}
$$

where $\varphi$ is the angle formed by the radius with axis $\overline{O \omega_{1}}$.
Generally the function $\rho(\varphi)$ will be determined as a polynomial or a ratio of polynomials in variable $\cos \varphi$. For instance, the four-lobe filter whose contour plot is shown in Fig.1(a) corresponds to:

$$
\begin{equation*}
\rho(\varphi)=a+b \cos 4 \varphi=a+b-8 b \cos ^{2} \varphi+8 b \cos ^{4} \varphi \tag{4}
\end{equation*}
$$

which is plotted in Fig.1(b) in the range $\varphi \in[0,2 \pi]$.
If the radial function $\rho(\varphi)$ can be expressed in the variable $\cos \varphi$, using (3) we obtain by substitution a function $\rho\left(\omega_{1}, \omega_{2}\right)$.
In this paper the notion of template is used, common in the field of cellular neural networks (CNNs), to denominate the coefficient matrices corresponding to the numerator and denominator of a 2D filter transfer function $H\left(z_{1}, z_{2}\right)$. Odd-sized templates (e.g. $3 \times 3$, $5 \times 5$ ) correspond to even order filters and allow for using both negative and positive powers of $z_{1}, z_{2}$.

It can be shown that in this general situation the cosine of the current angle $\varphi$ with initial angle $\varphi_{0}$ can be expressed as:

$$
\begin{equation*}
\cos ^{2}\left(\varphi+\varphi_{0}\right)=\frac{\cos ^{2} \varphi_{0} \cdot \omega_{1}^{2}+\sin ^{2} \varphi_{0} \cdot \omega_{2}^{2}+0.5 \sin 2 \varphi_{0} \cdot \omega_{1} \omega_{2}}{\omega_{1}^{2}+\omega_{2}^{2}} \tag{5}
\end{equation*}
$$

corresponding to the simple expression (3). Replacing $\omega_{1}$ and $\omega_{2}$ by the complex variables $s_{1}=j \omega_{1}$ and $s_{2}=j \omega_{2}, \cos ^{2}\left(\varphi+\varphi_{0}\right)$ can be written in 2D Laplace domain.

### 2.2 Description of Diamond-Shaped Filters in Polar Coordinates in the Frequency Plane

In this section, we determine analytically the mapping which transforms a circle of given radius, in the frequency plane, into a square, having its vertices on the same circle [9]. We refer to the geometrical construction in Fig.2. In the frequency plane ( $\omega_{1}, \omega_{2}$ ) spanned by the axes $O \omega_{1}, O \omega_{2}$, we consider the circle of radius $R$. The default value will be $R=\pi$.
Let us take an arbitrary point $P_{1}$ situated on the first side of the square ( $A_{1} A_{2}$ ), and let $\varphi$ be the angle between the segment $O P_{1}$ and the axis $O \omega_{1} ; \varphi_{0}$ is the angle between $O A_{1}$ and axis $O \omega_{1}$ (initial phase), where $A_{1}$ is the first vertex of the square. In the triangle $P_{1} O A_{1}$ we have the angles:
$\prec O A_{1} P_{1}=\pi / 4 ; \prec P_{1} O A_{1}=\varphi-\varphi_{0} ; \prec O P_{1} A_{1}=3 \pi / 4-\varphi+\varphi_{0}$
Applying the sine theorem in the triangle $P_{1} O A_{1}$, we find the measure of segment $O P_{1}$ as a function of $R$ and $\varphi$ :

$$
\begin{equation*}
O P_{1}=\frac{R \cdot \sin \left(O A_{1} P_{1}\right)}{\sin \left(O P_{1} A_{1}\right)}=\frac{R \sqrt{2} / 2}{\cos \left(\varphi-\varphi_{0}-\pi / 4\right)} \tag{6}
\end{equation*}
$$

Thus we found the measure of $O P_{1}$ as a function of the current angle. However, (6) is valid only for $\varphi$ in the range: $\varphi \in\left[\varphi_{0}+2 n \pi / 4, \varphi_{0}+2(n+1) \pi / 4\right]$.
For a standard diamond-shaped filter we have $\varphi_{0}=0, R=1$ and in the first quadrant of the frequency plane we obtain:

$$
\begin{equation*}
\rho(\varphi)=1 / \sqrt{2} \cos (\varphi-\pi / 4) \tag{7}
\end{equation*}
$$

To express the value $O P_{n}$ for an arbitrary angle $\varphi$, when point $P_{n}$ is located on any side of the square, including the vertices, we find a periodic function $\rho(\varphi)$ of the current angle $\varphi$. This function has the period $\Phi=\pi / 2$ and is plotted in Fig.3(a). A very convenient way to obtain a closed-form periodic approximation of this function is to use a rational approximation. As mentioned earlier, the Chebyshev-Padé approximation usually gives best results. We look for such a rational approximation for the function:

$$
\begin{equation*}
\rho_{1}(\varphi)=1 / \sqrt{2} \cos \varphi \tag{8}
\end{equation*}
$$

over the phase range $\varphi \in[-\pi / 4, \pi / 4]$, in powers of the variable $\cos 4 \varphi$, which is a periodic function with period $\pi / 2$. In this way, the rational function will actually approximate the function $\rho_{1}(\varphi)$ over the entire range $[0,2 \pi]$.
Since the function $\rho(\varphi)$ is not differentiable in the points $\varphi=-\pi,-\pi / 2,0, \pi / 2$ (corresponding to square vertices) as can be noticed in Fig.3(a), we will consider the function $\rho_{1}(\varphi)$ on the range $\varphi \in[-\pi / 4, \pi / 4]$, which is differentiable everywhere within
this interval. We first obtain:

$$
\begin{equation*}
\rho_{1}(\varphi)=1 / \cos \varphi \cong\left(1+0.087481 \cdot \varphi^{2}\right) /\left(1-0.413 \cdot \varphi^{2}\right) \tag{9}
\end{equation*}
$$

At this step we make the change of variable:

$$
\begin{equation*}
x=\cos (4 \varphi) \Leftrightarrow \varphi=0.25 \cdot \arccos x \tag{10}
\end{equation*}
$$

and we get the intermediate function:

$$
\begin{equation*}
\rho_{i}(x)=\frac{1.082679+1.189232 \cdot x+0.202714 \cdot x^{2}}{1+1.202559 \cdot x+0.271879 \cdot x^{2}} \tag{11}
\end{equation*}
$$

Returning to the initial variable $\varphi=0.25 \cdot \arccos x$, by substituting back $x=\cos (4 \varphi)$, we obtain a rational approximation in powers of $\cos (4 \varphi)$. In this expression we must replace $\varphi$ by $\varphi-\pi / 4$, to get the final approximation for the function $\rho(\varphi)$ :

(a)

(b)

Figure 1 - (a) Contour plot of a four-lobe filter; (b) Periodic func-


Figure 2 - Square inscribed in the circle of radius $R$ in the frequency plane, with an initial phase $\varphi_{0}$

(a)

(b)

Figure 3 - (a) Periodic function $\rho(\varphi)$; (b) its periodic approximation $\rho_{1}(\varphi)$

$$
\begin{equation*}
\rho(\varphi)=\frac{1.04234-1.046915 \cdot \cos (4 \varphi)+0.089227 \cdot \cos (8 \varphi)}{1-1.058647 \cdot \cos (4 \varphi)+0.119671 \cdot \cos (8 \varphi)} \tag{12}
\end{equation*}
$$

This function is plotted in Fig.3(b) and is a very accurate approximation of the original function in Fig.3(a). Using trigonometric identities, this can be expressed as a rational expression in $(\cos \varphi)^{2 n}$, with $n=1 \ldots 4$.

$$
\begin{equation*}
\rho(x)=0.7456 \cdot \frac{(x+0.347)(x+0.0156)(x-1.0156)(x-1.347)}{(x+0.2342)(x+0.0136)(x-1.0136)(x-1.2342)} \tag{13}
\end{equation*}
$$

where by $x$ we denoted here $(\cos \varphi)^{2}$. At this point we finally reach an expression of the radial function $\rho(\varphi)$ of the frequency variables $\omega_{1}$ and $\omega_{2}$, i.e. $\rho\left(\omega_{1}, \omega_{2}\right)$.

### 2.3 Design method for 2D diamond-shaped filters based on frequency transformations and numerical approximation

In this section we will propose a more general design method for a diamond shaped filter. It will start from a digital filter prototype, with a transfer function $H(z)$ of a certain order $N$. We discuss the common case when the numerator and denominator of $H(z)$ are polynomials in $z$ of the same degree. Let us consider for instance a transfer function $H(z)$ of even order $N$ factorized into second order functions which may be referred to as biquads. Such a biquad function will have the general form $H_{b}(z)$ :

$$
\begin{equation*}
H_{b}(z)=\frac{b_{2} z^{2}+b_{1} z+b_{0}}{z^{2}+a_{1} z+a_{0}}=\frac{B_{b}(z)}{A_{b}(z)} \tag{14}
\end{equation*}
$$

in which all coefficients have been normalized to the coefficient of $z^{2}$ of the denominator, for simplicity. The frequency response of $H_{b}(z)$ can be put into the simple form:

$$
\begin{equation*}
H_{b}(\omega)=\frac{b_{1}+\left(b_{0}+b_{2}\right) \cos \omega+j\left(b_{2}-b_{0}\right) \sin \omega}{a_{1}+\left(1+a_{0}\right) \cos \omega+j\left(1-a_{0}\right) \sin \omega}=\frac{B_{b}(\omega)}{A_{b}(\omega)} \tag{15}
\end{equation*}
$$

The expression (12), using trigonometric identities, can be written in powers of $(\cos \varphi)^{2}$; then, according to (3) we have:

$$
\begin{equation*}
(\cos \varphi)^{2}=\omega_{1}^{2} /\left(\omega_{1}^{2}+\omega_{2}^{2}\right) \tag{16}
\end{equation*}
$$

and by substitution we obtain an expression of the radial function $\rho(\varphi)$ in the two frequency variables $\omega_{1}$ and $\omega_{2}$, denoted $\rho\left(\omega_{1}, \omega_{2}\right)$. Finally we get an expression of the real frequency transformation of the general form (2). The next step is to find numerically an approximation of the functions:

$$
\begin{align*}
& C\left(\omega_{1}, \omega_{2}\right)=\cos \left(\sqrt{\omega_{1}^{2}+\omega_{2}^{2}} / \rho\left(\omega_{1}, \omega_{2}\right)\right)  \tag{17}\\
& S\left(\omega_{1}, \omega_{2}\right)=\sin \left(\sqrt{\omega_{1}^{2}+\omega_{2}^{2}} / \rho\left(\omega_{1}, \omega_{2}\right)\right) \tag{18}
\end{align*}
$$

We would like to approximate the above functions with a trigonometric series of the general form:

$$
\begin{equation*}
F\left(\omega_{1}, \omega_{2}\right) \cong \sum_{m=-N}^{N} \sum_{n=-N}^{N} a_{m n} \cos \left(m \omega_{1}+n \omega_{2}\right) \tag{19}
\end{equation*}
$$

where $N$ is imposed by the required accuracy of approximation. We will derive this approximation indirectly, using the following change of variables:

$$
\begin{align*}
& \omega_{1}=\arccos x_{1} \Leftrightarrow x_{1}=\cos \omega_{1} \\
& \omega_{2}=\arccos x_{2} \Leftrightarrow x_{2}=\cos \omega_{2} \tag{20}
\end{align*}
$$

Using this change of variables we obtain from $C\left(\omega_{1}, \omega_{2}\right)$ and $S\left(\omega_{1}, \omega_{2}\right)$ the functions $C_{x}\left(x_{1}, x_{2}\right)$ and $S_{x}\left(x_{1}, x_{2}\right)$ which have rather complicated expression. However, using a symbolic calculation software like MAPLE, we can derive immediately the bivariate Taylor series expansion in $x_{1}$ and $x_{2}$, of the general form:

$$
\begin{equation*}
F_{x}\left(x_{1}, x_{2}\right) \cong \sum_{k=-N}^{N} \sum_{l=-N}^{N} b_{k l} \cdot x_{1}^{k} x_{2}^{l} \tag{21}
\end{equation*}
$$

Finally by substituting back in (21) the new variables $x_{1}=\cos \omega_{1}$ and $x_{2}=\cos \omega_{2}$ we return to the former variables and applying again trigonometric identities we obtain at last the desired expansions of the form (19).
For instance with $N=2$ the expansion for $C\left(\omega_{1}, \omega_{2}\right)$ is:

$$
\begin{align*}
& C\left(\omega_{1}, \omega_{2}\right) \cong-0.419822+0.517714 \cdot\left(\cos \omega_{1}+\cos \omega_{2}\right) \\
& +0.177207 \cdot\left(\cos \left(\omega_{1}+\omega_{2}\right)+\cos \left(\omega_{1}-\omega_{2}\right)\right) \\
& -0.054476 \cdot\left(\cos \left(\omega_{1}+2 \omega_{2}\right)+\cos \left(\omega_{1}-2 \omega_{2}\right)\right.  \tag{22}\\
& \left.+\cos \left(2 \omega_{1}+\omega_{2}\right)+\cos \left(2 \omega_{1}-\omega_{2}\right)\right)+0.094109 \cdot\left(\cos 2 \omega_{1}+\cos 2 \omega_{2}\right) \\
& -0.008439 \cdot\left(\cos \left(2 \omega_{1}+2 \omega_{2}\right)+\cos \left(2 \omega_{1}-2 \omega_{2}\right)\right)
\end{align*}
$$

and for $S\left(\omega_{1}, \omega_{2}\right)$ is:

$$
\begin{align*}
& S\left(\omega_{1}, \omega_{2}\right) \cong 0.552617+0.393861 \cdot\left(\cos \omega_{1}+\cos \omega_{2}\right) \\
& -0.233406 \cdot\left(\cos \left(\omega_{1}+\omega_{2}\right)+\cos \left(\omega_{1}-\omega_{2}\right)\right) \\
& -0.041057 \cdot\left(\cos \left(\omega_{1}+2 \omega_{2}\right)+\cos \left(\omega_{1}-2 \omega_{2}\right)\right.  \tag{23}\\
& \left.+\cos \left(2 \omega_{1}+\omega_{2}\right)+\cos \left(2 \omega_{1}-\omega_{2}\right)\right)-0.1238 \cdot\left(\cos 2 \omega_{1}+\cos 2 \omega_{2}\right) \\
& +0.009519 \cdot\left(\cos \left(2 \omega_{1}+2 \omega_{2}\right)+\cos \left(2 \omega_{1}-2 \omega_{2}\right)\right)
\end{align*}
$$

Next if we express each cosine term as a function of the complex variables $Z_{1}=e^{j \omega_{1}}$ and $z_{2}=e^{j \omega_{2}}$ like:

$$
\begin{equation*}
\cos \left(m \omega_{1}+n \omega_{2}\right)=0.5\left(z_{1}^{m} z_{2}^{n}+z_{1}^{-m} z_{2}^{-n}\right) \tag{24}
\end{equation*}
$$

we obtain the functions (17), (18) as $C_{Z}\left(z_{1}, z_{2}\right)$ and $S_{Z}\left(z_{1}, z_{2}\right)$, which are real. Therefore through the real frequency transformation (2) we finally reached the mappings:

$$
\begin{equation*}
\cos \omega \rightarrow C_{Z}\left(z_{1}, z_{2}\right) \quad \sin \omega \rightarrow S_{Z}\left(z_{1}, z_{2}\right) \tag{25}
\end{equation*}
$$

Taking into account the expression (15) of $H_{b}(\omega)$, the 1D biquad function $H_{b}(z)$ given in (14) is mapped into the following 2D function $H_{2 D}\left(z_{1}, z_{2}\right)$ in the variables $z_{1}$ and $z_{2}$ :

$$
\begin{align*}
& H_{2 D}\left(z_{1}, z_{2}\right)=B\left(z_{1}, z_{2}\right) / A\left(z_{1}, z_{2}\right) \\
& \frac{b_{1}+\left(b_{0}+b_{2}\right) \cdot C_{Z}\left(z_{1}, z_{2}\right)+j\left(b_{2}-b_{0}\right) \cdot S_{Z}\left(z_{1}, z_{2}\right)}{a_{1}+\left(1+a_{0}\right) \cdot C_{Z}\left(z_{1}, z_{2}\right)+j\left(1-a_{0}\right) \cdot S_{Z}\left(z_{1}, z_{2}\right)} \tag{26}
\end{align*}
$$

We remark that the obtained 2D filter function has complex coefficients if it is expressed in the 2D $\mathbf{Z}$ transform.
The real functions $C_{Z}\left(z_{1}, z_{2}\right)$ and $S_{Z}\left(z_{1}, z_{2}\right)$ can further be written in matrix form as:

$$
\begin{equation*}
C_{Z}\left(z_{1}, z_{2}\right)=\mathbf{Z}_{1} \times \mathbf{C} \times \mathbf{Z}_{2}^{T} ; \quad S_{Z}\left(z_{1}, z_{2}\right)=\mathbf{Z}_{1} \times \mathbf{S} \times \mathbf{Z}_{2}^{T} \tag{27}
\end{equation*}
$$

where the vectors $\mathbf{Z}_{1}$ and $\mathbf{Z}_{2}$ are given by:

$$
\mathbf{Z}_{1}=\left[\begin{array}{lllll}
z_{1}^{-2} & z_{1}^{-1} & 1 & z_{1} & z_{1}^{2}
\end{array}\right] ; \mathbf{Z}_{2}=\left[\begin{array}{lllll}
z_{2}^{-2} & z_{2}^{-1} & 1 & z_{2} & z_{2}^{2} \tag{28}
\end{array}\right]
$$

and $\mathbf{C}, \mathbf{S}$ are matrices of size $5 \times 5$ which have as elements the coefficients identified from the expressions (22) and (23) of $C\left(\omega_{1}, \omega_{2}\right)$ and $S\left(\omega_{1}, \omega_{2}\right)$. The matrices $\mathbf{C}$ and S result as:

$$
\begin{align*}
& \mathbf{C}=\left[\begin{array}{rrrrr}
0.0471 & -0.0272 & -0.0042 & -0.0272 & 0.0471 \\
-0.0272 & 0.0886 & 0.2588 & 0.0886 & -0.0272 \\
-0.0042 & 0.2588 & -0.4198 & 0.2588 & -0.0042 \\
-0.0272 & 0.0886 & 0.2588 & 0.0886 & -0.0272 \\
0.0471 & -0.0272 & -0.0042 & -0.0272 & 0.0471
\end{array}\right]  \tag{29}\\
& \mathbf{S}=\left[\begin{array}{rrrrr}
0.0047 & -0.0205 & -0.0619 & -0.0205 & 0.0047 \\
-0.0205 & -0.1167 & 0.1969 & -0.1167 & -0.0205 \\
-0.0619 & 0.1969 & 0.5526 & 0.1969 & -0.0619 \\
-0.0205 & -0.1167 & 0.1969 & -0.1167 & -0.0205 \\
0.0047 & -0.0205 & -0.0619 & -0.0205 & 0.0047
\end{array}\right]
\end{align*}
$$

where the elements were limited to 4 decimals. The matrix $\mathbf{S}$ has a similar form. The matrices $\mathbf{C}$ and $\mathbf{S}$ are symmetric horizontally and vertically. Since the element values decrease rapidly towards margins, the size $5 \times 5$ for the templates $\mathbf{C}$ and $\mathbf{S}$ is sufficient to ensure the accuracy of the numerical approximation, and higher order terms can be ignored with a negligible error.
Taking into account relation (26) and (27), we can finally express the complex matrices $\mathbf{B}$ and $\mathbf{A}$ that correspond to the numerator and denominator of $H_{2 D}\left(z_{1}, z_{2}\right)$, i.e. $B\left(z_{1}, z_{2}\right)$ and $A\left(z_{1}, z_{2}\right)$ :

$$
\begin{align*}
& \mathbf{B}=b_{1} \cdot \mathbf{E}+\left(b_{0}+b_{2}\right) \cdot \mathbf{C}+j\left(b_{2}-b_{0}\right) \cdot \mathbf{S}  \tag{31}\\
& \mathbf{A}=a_{1} \cdot \mathbf{E}+\left(1+a_{0}\right) \cdot \mathbf{C}+j\left(1-a_{0}\right) \cdot \mathbf{S} \tag{32}
\end{align*}
$$

By $\mathbf{E}$ we denoted the $5 \times 5$ zero matrix with the central element of value 1. The mapping of the biquad function $H_{b}(z)$ to $H_{2 D}\left(z_{1}, z_{2}\right)$ can be written as:

$$
\begin{equation*}
H_{b}(z) \rightarrow H_{2 D}\left(z_{1}, z_{2}\right)=\left(\mathbf{Z}_{1} \times \mathbf{B} \times \mathbf{Z}_{2}^{T}\right) /\left(\mathbf{Z}_{1} \times \mathbf{A} \times \mathbf{Z}_{2}^{T}\right) \tag{33}
\end{equation*}
$$

The reason for which the filter templates result complex is that both functions $C\left(\omega_{1}, \omega_{2}\right)$ and $S\left(\omega_{1}, \omega_{2}\right)$ have even parity in $\omega_{1}$ and $\omega_{2}$, therefore both can be developed in a series of functions of the form $\cos \left(m \omega_{1}+n \omega_{2}\right)$.

### 2.3.1 Design example

Let us consider the elliptic low-pass prototype filter function $H(z)=\frac{0.1539 \cdot z^{4}+0.482 \cdot z^{3}+0.6734 \cdot z^{2}+0.482 \cdot z+0.1539}{z^{4}+0.155 \cdot z^{3}+0.7649 \cdot z^{2}-0.0376 \cdot z+0.079}(34)$ of order $N=4, R_{p}=0.7 \mathrm{~dB}$ pass-band ripple, a minimum stopband attenuation $R_{S}=40 \mathrm{~dB}$, and pass-band edge frequency $\omega_{S}=0.5$, having the frequency response magnitude plotted in Fig.4. We notice that it is maximally-flat, with a relatively steep descent. We will design a diamond shaped filter starting from this prototype. $H(z)$ can be factorized as follows:
$H(z)=0.1539 \cdot \frac{\left(z^{2}+1.2884 z+1\right)}{\left(z^{2}+0.2554 z+0.6732\right)} \cdot \frac{\left(z^{2}+1.8425 z+1\right)}{\left(z^{2}-0.1004 z+0.1173\right)}$ (35)
For the first biquad from (35), we identify the coefficients of general


Figure 4 - Magnitude of the elliptic low-pass prototype filter
form (14): $b_{2}=1, b_{1}=1.2884, b_{0}=1, a_{1}=0.2554, a_{0}=0.6732$.
Since we have $b_{0}=b_{2}$, the matrix $\mathbf{B}$ from (31) results real (the imaginary part is cancelled) and will be:

$$
\begin{equation*}
\mathbf{B}_{1}=1.2884 \cdot \mathbf{E}+2 \cdot \mathbf{C} \tag{36}
\end{equation*}
$$

while matrix A results complex:

$$
\begin{equation*}
\mathbf{A}_{1}=0.2554 \cdot \mathbf{E}+1.6732 \cdot \mathbf{C}+0.3268 j \cdot \mathbf{S} \tag{37}
\end{equation*}
$$

For the second biquad from (35) we get as well:

$$
\begin{align*}
& \mathbf{B}_{2}=1.8425 \cdot \mathbf{E}+2 \cdot \mathbf{C} \\
& \mathbf{A}_{2}=-0.1004 \cdot \mathbf{E}+1.1173 \cdot \mathbf{C}+0.8827 j \cdot \mathbf{S} \tag{38}
\end{align*}
$$

The final 2D filter templates $\mathbf{B}$ and $\mathbf{A}$ result as convolutions of the templates for the two biquads from:

$$
\begin{equation*}
\mathbf{B}=0.1359 \cdot \mathbf{B}_{1} * \mathbf{B}_{2}, \quad \mathbf{A}=\mathbf{A}_{1} * \mathbf{A}_{2} \tag{39}
\end{equation*}
$$

The coefficient in front of $H(z)$ from (35) was included in $\mathbf{B}$.

### 2.4 Design of zero-phase diamond-shaped filters

In this section we briefly describe a design method for zero-phase diamond filters.
We consider an IIR discrete filter of order $N$ defined by:

$$
\begin{equation*}
H(z)=\frac{P(z)}{Q(z)}=\sum_{i=1}^{M} p_{i} \cdot z^{i} / \sum_{j=1}^{N} q_{j} \cdot z^{j} \tag{40}
\end{equation*}
$$

The first step is to obtain a zero-phase prototype filter (with a realvalued transfer function). Starting from a general IIR discrete filter like (40), we derive a filter which preserves only its magnitude characteristic, while its phase will be zero throughout the frequency domain, namely the range $[-\pi, \pi]$.
We consider the magnitude characteristics, defined by the absolute value of $H(z)=H\left(e^{j \omega}\right)$, taken from (40):

$$
\begin{equation*}
\left|H\left(e^{j \omega}\right)\right|=\left|\sum_{n=0}^{M} p_{n} \exp (j n \omega)\right| /\left|\sum_{m=0}^{N} q_{m} \exp (j m \omega)\right| \tag{41}
\end{equation*}
$$

We use again the method from section 2.3 , where we make the change of variable $\omega=\arccos x \Leftrightarrow x=\cos \omega$ and using a symbolic computation software we derive immediately a ChebyshevPadé rational approximation of the magnitude $\left|H\left(e^{j \omega}\right)\right|$, in which the numerator and denominator are polynomials in variable $\cos \omega$, preferably of the same order $N$, which finally leads to a filter implementation with templates of equal size:

$$
\begin{equation*}
\left|H\left(e^{j \omega}\right)\right| \cong \sum_{n=1}^{N} b_{n} \cos ^{n} \omega / \sum_{m=1}^{N} a_{m} \cos ^{m} \omega=B(\omega) / A(\omega) \tag{42}
\end{equation*}
$$

The numerator and denominator can be factorized into first and second order polynomials in $\cos \omega$. For instance, $A(\omega)$ becomes:

$$
\begin{equation*}
A(\omega)=k \cdot \prod_{i=1}^{n}\left(\cos \omega+a_{i}\right) \cdot \prod_{j=1}^{m}\left(\cos ^{2} \omega+a_{1 j} \cos \omega+a_{2 j}\right) \tag{43}
\end{equation*}
$$

with $n+2 m=N$, the filter order.
To obtain a 2D diamond-shaped filter from the factorized function, we simply substitute in (43) $\cos \omega$ with the function $C_{Z}\left(z_{1}, z_{2}\right)$, corresponding to matrix $\mathbf{C}$ in (29). For the general factors $\left(\cos \omega+a_{i}\right)$ and $\left(\cos ^{2} \omega+a_{1 j} \cos \omega+a_{2 j}\right)$, the templates $\mathbf{A}_{1 i}$ $(3 \times 3)$ and $\mathbf{A}_{2 j}(5 \times 5)$ result as:

$$
\begin{gather*}
\mathbf{A}_{1 i}=\mathbf{C}+a_{i} \cdot \mathbf{A}_{01}  \tag{45}\\
\mathbf{A}_{2 j}=\mathbf{C} * \mathbf{C}+a_{1 j} \cdot \mathbf{C}_{0}+a_{2 j} \cdot \mathbf{A}_{02} \tag{46}
\end{gather*}
$$

where $\mathbf{A}_{01}$ is a $3 \times 3$ zero template and $\mathbf{A}_{02}$ a $5 \times 5$ zero template with central element one; $\mathbf{C}_{0}$ is a $5 \times 5$ template obtained by bordering $\mathbf{C}(3 \times 3)$ with zeros.

### 2.4.1 Design example

Let us consider the same digital prototype filter as in section 2.3.1, given by (34). Using the procedure described above, we get:
$\left|H\left(e^{j \omega}\right)\right| \cong$
$\underline{(\cos \omega+0.9154)(\cos \omega+0.6684)\left(\cos ^{2} \omega+0.1899 \cos \omega+0.5499\right)}$
$\left(\cos ^{2} \omega+0.3574 \cos \omega+0.1503\right)\left(\cos ^{2} \omega-0.2271 \cos \omega+1.2679\right)$
This approximates very accurately the exact filter magnitude plotted in Fig.4. Once expressed $\left|H\left(e^{j \omega}\right)\right|$ as a factorized function of $\cos \omega$, we will now apply the real frequency transformation already derived in section 2.3, namely $\cos \omega \rightarrow C_{Z}\left(z_{1}, z_{2}\right)$. Following the steps described earlier, we finally obtain the filter templates $\mathbf{B}$ and $\mathbf{A}$. The frequency response and contour plot for the designed zerophase diamond-shaped filter are shown in Fig.5(a) and (b). In Fig.5(c) and (d) a narrower diamond filter is shown, also designed with the second method. Both filters show an accurate square shape and a relatively steep transition band and the stop-band regions present a negligible ripple.

### 2.5 Discussion

The proposed design method results as a combination of an analytical approach involving frequency transformations and a numerical approximation step, therefore it is very efficient. An advantage of this method is that it avoids using the bilinear transform, which is known to introduce relatively large distortions unless a pre-warping is included. Pre-warping would increase the order of the obtained 2D filter. The filters have low complexity and are quite efficient.
Although the analytical procedure described before may seem complicated, it leads to accurate approximations of the functions $C\left(\omega_{1}, \omega_{2}\right)$ and $S\left(\omega_{1}, \omega_{2}\right)$ which correspond to the matrices $\mathbf{C}$ and S. Once calculated these matrices, the design consists in substituting them in the expressions (31), (32) for a particular set of parameters, corresponding to a given prototype filter.
Thus the method is versatile in the sense that it can be applied to any desired prototype. However, the range of useful prototypes for this application is limited, since we generally need a maximally-flat filter with a relatively narrow transition band. For instance we have used an efficient elliptic filter with very small pass-band ripple.
Moreover, for the 2D diamond filter we can choose a narrower lowpass prototype, or we can obtain diamond filters rotated in the frequency plane with an arbitrary angle $\varphi_{0}$, as suggested in Fig.2.


Figure 5 - Frequency response and contour plot for two zero-phase diamond filters

Since the prototype filter can be factorized, the templates of the 2D filter will result as convolutions of $5 \times 5$ matrices, which is an advantage in implementation. The drawback of the first design method is that the filter results with complex coefficients, at least at the denominator, as can be seen in the presented design example.
Stability of the resulted 2D filter is an important issue as well. We cannot make here a detailed analysis of stability, but it can be shown that the proposed frequency transformations preserve the stability of the 1 D prototype filter. Therefore, the only issue is to ensure the stability of the prototype filter. The derived 2D filter could become unstable only if the numerical approximations used introduce large errors. In this case we would have to increase the precision of approximation by taking more higher order terms, which would increase the filter complexity.

## 3. CONCLUSION

We proposed a design method for recursive diamond-shaped filters, based on the specification of their shape in polar coordinates in the frequency plane. The method however is more general and applies to any 2D filter which can be described in this way. The design starts from a prototype filter function, from which we can derive 2D diamond-shaped filters using specific frequency transformations. The method combines an analytical approach with a numerical approximation. We approached both the general case and the zero-phase filters. Another advantage of the method is that it is more general and allows for the design of diamond filters with a specified rotation angle. The designed 2D filters are efficient and have high selectivity at a relatively low complexity. Further research may focus on an efficient implementation of this type of filters.

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