# EIGENFUNCTIONS, EIGENVALUES, AND FRACTIONALIZATION OF THE QUATERNION AND BIQUATERNION FOURIER TRANSFORMS 

Soo-Chang Pei, Jian-Jiun Ding, and Kuo-Wei Chang<br>Department of Electrical Engineering, National Taiwan University,<br>No. 1, Sec. 4, Roosevelt Rd., 10617, Taipei, Taiwan, R.O.C<br>TEL: 886-2-23635251-321, Fax: 886-2-23671909, Email: pei@cc.ee.ntu.edu.tw, djj@cc.ee.ntu.edu.tw, b89901130@ntu.edu.tw


#### Abstract

The discrete quaternion Fourier transform (DQFT) is useful for signal analysis and image processing. In this paper, we derive the eigenfunctions and eigenvalues of the DQFT. We also extend our works to the reduced biquaternion case, i.e., the discrete reduced biquaternion Fourier transform (DRBQFT). We find that an even or odd symmetric eigenvector of the 2-D DFT will also be an eigenvector of the DQFT and the DRBQFT. Moreover, both the DQFT and the DRBQFT have 8 eigenspaces, which correspond to the eigenvalues of $1,-1, i,-i, j,-j, k$, and $-k$. We also use the derived eigenvectors to fractionalize the DQFT and the DRBQFT and define the discrete fractional quaternion transform and the discrete fractional reduced biquaternion Fourier transform.


## 1. Introduction

The quaternion algebra is a generalization of the complex algebra [1]. A number in the quaternion field has three imaginary parts and can be expressed as:

$$
\begin{equation*}
q=q_{r}+q_{i} \cdot i+q_{j} \cdot j+q_{k} \cdot k, \tag{1}
\end{equation*}
$$

where $i, j$, and $k$ satisfy the following rules:

$$
\begin{align*}
& i^{2}=j^{2}=k^{2}=-1, \quad i \cdot j=k, \quad j \cdot k=i, \quad k \cdot i=j, \\
& j \cdot i=-k, \quad k \cdot j=-i, \quad i \cdot k=-j . \tag{2}
\end{align*}
$$

Based on the quaternion algebra, the discrete quaternion Fourier transform (DQFT) [2] is defined as

$$
\begin{align*}
& \operatorname{DQFT}(x[m, n]) \\
= & \sqrt{\frac{1}{M N}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \exp \left(-i \frac{2 \pi}{M} p m\right) x[m, n] \exp \left(-j \frac{2 \pi}{N} q n\right) . \tag{3}
\end{align*}
$$

The DQFT is useful for color image analysis, spectral analysis, and filter design [2][3][4].

There is another algebra that also has four elements, i.e., the reduced biquaternion algebra [5][6]. A number in the reduced biquaternion field can be expressed as:

$$
\begin{equation*}
q=q_{r}+q_{i} \cdot i+q_{j} \cdot j+q_{k} \cdot k, \tag{4}
\end{equation*}
$$

where $i^{2}=k^{2}=-1, \quad j^{2}=1, \quad i j=j i=k$,

$$
\begin{equation*}
i k=k i=-j, \quad j k=k j=i \tag{5}
\end{equation*}
$$

There are two differences between the reduced biquaternion and the quaternion algebras. First, in the reduced biquaternion algebra, $j^{2}=1$. However, in the quaternion algebra, $j^{2}=-1$. Moreover, the reduced biquaternion algebra is always commutative (i.e., $x y=y x$ is always satisfied), but in the quaternion algebra, $i j=-j i, i k=-k i$, and $j k=-k j$. The discrete reduced biquaternion Fourier transform (DRBQFT) [6] can also be defined as the following form

$$
\begin{align*}
& \operatorname{DRBQFT}(x[m, n]) \\
= & \sqrt{\frac{1}{M N}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \exp \left(-i \frac{2 \pi}{M} p m\right) \exp \left(-k \frac{2 \pi}{N} q n\right) x[m, n] . \tag{6}
\end{align*}
$$

As the DQFT, the DRBQFT is also useful for color image analysis and filter design [5][6]. The DRBQFT is easier to implement and suitable for multiple channel signal analysis.

In this paper, we derive the eigenvectors and eigenvalues of the DQFT and the DRBQFT. We find that if $e_{f}[m, n]$ is an eigenvector of the 2-D DFT in the complex filed and $e_{f}[m, n]= \pm e_{f}[m$, $-n]$ or $e_{f}[m, n]= \pm e_{f}[-m, n]$, then it is also an eigenvector of the DQFT and the DRBQFT. Moreover, the original DFT has 4 distinct eigenvalues ( $\pm 1$ and $\pm i$ ), but both the DQFT and the DRBQFT have 8 distinct eigenvalues ( $\pm 1, \pm i, \pm j$, and $\pm k$ ). See Sections 2 and 3.

Moreover, in Section 4, we use the derived eigenvectors and eigenvalues of the DQFT and the DRBQFT to define the discrete fractional quaternion transform (DFRQFT) and the discrete fractional reduced biquaternion Fourier transform (DFRRBQFT). They are analogous to the discrete fractional Fourier transform [10][11] in the complex field and generalize the DQFT and the DRBQFT.

Furthermore, our results can be easily extended to the continuous case, i.e., deriving the eigenfunctions and eigenvalues of the continuous quaternion and reduced biquaternion Fourier transforms. See Section 5

## 2. Eigenvectors and Eigenfunctions of Discrete Quaternion Fourier Transforms

Since the quaternion algebra does not have the commutative rule, there are two different ways to define the eigenvectors and eigenvalues of the DQFT:

$$
\begin{array}{ll}
\text { (right-sided form) } & \operatorname{DQFT}(e[m, n])=e[m, n] \lambda, \\
\text { (left-sided form) } & \operatorname{DQFT}(e[m, n])=\lambda e[m, n] . \tag{8}
\end{array}
$$

We first discuss the right sided form eigenvectors and eigenvalues of the DQFT. We find that they can be derived from those of the original 2-D discrete Fourier transform (2-D DFT) in the complex field.
[Theorem 1] Suppose that $e_{f}[m, n]$ is an eigenvector of the 2-D DFT:

$$
\begin{equation*}
\operatorname{DFT}\left(e_{f}[m, n]\right)=e_{f}[m, n] \lambda, \tag{9}
\end{equation*}
$$

where $\operatorname{DFT}(x[p, q])=\sqrt{\frac{1}{M N}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} e^{-i \frac{2 \pi}{M} p m-i \frac{2 \pi}{N} q n} x[m, n]$.

If $e_{f}[m, n]$ is even along $n$, i.e.,

$$
\begin{equation*}
e_{f}[m, n]=e_{f}[m, N-n], \tag{11}
\end{equation*}
$$

then is also the eigenvector of the DQFT and the corresponding eigenvalue is also $\lambda$.

$$
\begin{equation*}
\operatorname{DQFT}\left(e_{f}[m, n]\right)=e_{f}[m, n] \lambda . \tag{12}
\end{equation*}
$$

(Proof): Since $e_{f}[m, n]$ is even along $n$,

$$
\begin{equation*}
\sum_{n=0}^{N-1} e_{f}[m, n] \sin \left(\frac{2 \pi}{N} q n\right)=0 \tag{13}
\end{equation*}
$$

the DQFT of $e_{f}[m, n]$ is

$$
\begin{align*}
& \operatorname{DQFT}\left(e_{f}[m, n]\right) \\
&= \sqrt{\frac{1}{M N}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} e^{-i \frac{2 \pi}{M} p m} e_{f}[m, n]\left(\cos \left(\frac{2 \pi}{N} q n\right)-j \sin \left(\frac{2 \pi}{N} q n\right)\right) \\
&=\sqrt{\frac{1}{M N}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} e^{-i \frac{2 \pi}{M} p m} e_{f}[m, n] \cos \left(\frac{2 \pi}{N} q n\right) \\
&=\sqrt{\frac{1}{M N}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} e^{-i \frac{2 \pi}{M} p m} e_{f}[m, n]\left(\cos \left(\frac{2 \pi}{N} q n\right)-i \sin \left(\frac{2 \pi}{N} q n\right)\right) \\
&=\sqrt{\frac{1}{M N}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} e^{-i \frac{2 \pi}{M} p m} e^{-i \frac{2 \pi}{N} q n} e_{f}[m, n]=\operatorname{DFT}\left(e_{f}[m, n]\right) . \tag{14}
\end{align*}
$$

Then, from (9), we obtain (12).
[Theorem 2] Similarly, if $e_{f}[m, n]$ is an eigenvector of the 2-D DFT that satisfies (9) and $e_{f}[m, n]$ is odd along $\boldsymbol{n}$ :

$$
\begin{equation*}
e_{f}[m, n]=-e_{f}[m, N-n], \tag{15}
\end{equation*}
$$

then it is also an eigenvector of the QDFT, but the eigenvalue is changed into $-\lambda k$.

$$
\begin{equation*}
\operatorname{DQFT}\left(e_{f}[m, n]\right)=e_{f}[m, n](-\lambda k) . \tag{16}
\end{equation*}
$$

(Proof): Since

$$
\begin{equation*}
\sum_{n=0}^{N-1} e_{f}[m, n] \cos \left(\frac{2 \pi}{N} q n\right)=0(\text { from }(15)), \tag{17}
\end{equation*}
$$

$\operatorname{DQFT}\left(e_{f}[m, n]\right)$
$=\sqrt{\frac{1}{M N}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} e^{-i \frac{2 \pi}{M} p m} e_{f}[m, n] \sin \left(\frac{2 \pi}{N} q n\right)(-j)$
$=\sqrt{\frac{1}{M N}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} e^{-i \frac{2 \pi}{M} p m} e_{f}[m, n]\left(\cos \left(\frac{2 \pi}{N} q n\right)-i \sin \left(\frac{2 \pi}{N} q n\right)\right)(-i j)$
$=\operatorname{DFT}\left(e_{f}[m, n]\right)(-k)=e_{f}[m, n] \cdot(-\lambda k)$.
[Theorem 3] By contrast, if $e_{f}[m, n]$ is an eigenvector of the 2-D DFT that satisfies (9) and $e_{f}[m, n]$ is even along $\boldsymbol{m}$ :

$$
\begin{equation*}
e_{f}[m, n]=e_{f}[M-m, n], \tag{18}
\end{equation*}
$$

then $e_{f}[m, n]$ is also an eigenvector of the DQFT but the eigenvalue is changed into $\lambda_{q}$ :

$$
\begin{gather*}
\operatorname{DQFT}\left(e_{f}[m, n]\right)=e_{f}[m, n] \lambda_{q},  \tag{19}\\
\text { where } \lambda_{q}=\lambda \text { if } \lambda= \pm 1, \quad \lambda_{q}= \pm j \text { if } \lambda= \pm i . \tag{20}
\end{gather*}
$$

(Proof): From (18), the inner product of $e_{f}[m, n]$ and $\sin (2 \pi p m / N)$ is zero. Therefore,

$$
\begin{aligned}
& \operatorname{DQFT}\left(e_{f}[m, n]\right)=\sqrt{\frac{1}{M N}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \cos \left(\frac{2 \pi}{M} p m\right) e_{f}[m, n] e^{-j \frac{2 \pi}{N} q n} \\
& =\sqrt{\frac{1}{M N}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} e_{f}[m, n]\left(\cos \left(\frac{2 \pi}{M} p m\right)-j \sin \left(\frac{2 \pi}{M} p m\right)\right) e^{-j \frac{2 \pi}{N} q n}
\end{aligned}
$$

$$
\begin{align*}
\operatorname{DQFT}\left(e_{f}[m, n]\right) & =\sqrt{\frac{1}{M N}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} e_{f}[m, n] e^{-j \frac{2 \pi}{M} p m} e^{-j \frac{2 \pi}{N} q n} \\
& =e_{f}[m, n] \lambda_{q} .
\end{align*}
$$

[Theorem 4] If $e_{f}[m, n]$ satisfies (9) and is odd along $\boldsymbol{m}$ :

$$
\begin{equation*}
e_{f}[m, n]=-e_{f}[M-m, n], \tag{21}
\end{equation*}
$$

then

$$
\begin{equation*}
\operatorname{DQFT}\left(e_{f}[m, n]\right)=e_{f}[m, n]\left(-k \lambda_{q}\right), \tag{22}
\end{equation*}
$$

where $\lambda_{q}$ is defined in (20).
(Proof): Since the inner product of $e_{f}[m, n]$ and $\cos (2 \pi p m / N)$ is zero, if we use $e_{f, r}[m, n]$ and $e_{f ; i}[m, n]$ to denote the real part and the imaginary part of $e_{f}[m, n]$, then
$\operatorname{DQFT}\left(e_{f}[m, n]\right)$
$=\sqrt{\frac{1}{M N}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1}-i \sin \left(\frac{2 \pi}{M} p m\right) e_{f}[m, n] e^{-j \frac{2 \pi}{N} q n}$
$=\sqrt{\frac{1}{M N}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1}(-k)(-j) \sin \left(\frac{2 \pi}{M} p m\right)\left(e_{f, r}[m, n]+i e_{f, i}[m, n] e^{-j \frac{2 \pi}{N} q n}\right.$
$=\sqrt{\frac{1}{M N}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1}(-k)\left(e_{f, r}[m, n]-i e_{f, i}[m, n]\right)(-j) \sin \left(\frac{2 \pi}{M} p m\right) e^{-j \frac{2 \pi}{N} q n}$
$=\sqrt{\frac{1}{M N}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1}\left((-k) e_{f, r}[m, n]+j e_{f, i}[m, n]\right) e^{-j \frac{2 \pi}{M} p m} e^{-j \frac{2 \pi}{N} q n}$
From the fact that both $e_{f, r}[m, n]$ and $e_{f, i}[m, n]$ are the eigenvectors of the DFT, we obtain

$$
\begin{align*}
& \operatorname{DQFT}\left(e_{f}[m, n]\right)=\left(e_{f, r}[m, n](-k) \lambda_{q}+e_{f, i}[m, n] j \lambda_{q}\right) \\
& =\left(e_{f, r}[m, n](-k) \lambda_{q}+e_{f, i}[m, n] i(-k) \lambda_{q}\right) \\
& =\left(e_{f, r}[m, n]+e_{f, i}[m, n] i\right)(-k) \lambda_{q}=e_{f}[m, n](-k) \lambda_{q} .
\end{align*}
$$

[Corollary 1] From Theorems 1-4, we can conclude that if $e_{f}[m, n]$ is a 2-D DFT eigenvector in the complex field and $e_{f}[m, n]= \pm e_{f}[m$, $N-n]$ or $e_{f}[m, n]= \pm e_{f}[M-m, n]$, then it is also an eigenvector of the DQFT.

By contrast, if $e_{f}[m, n]$ is a 2-D DFT eigenvector, but none of the symmetry relations in (11), (15), (18), and (21) is satisfied, then $e_{f}[m, n]$ is not an eigenvector of the DQFT. This can be proven from the fact that the even part and the odd part of $e_{f}[m, n]$ will be separated into different eigenspaces of the DQFT.
[Corollary 2] Moreover, since the DFT has four eigenvalues: 1, $-1, i$, and $-i$ [7], from (12), (16), (19), and (22), we can conclude that the DQFT has $\mathbf{8}$ possible eigenvalues, which are $1,-1, i,-i, j$, $-j, k$, and $-k$.

Specially, in Theorems 1-4, we can choose the 2-D DFT eigenvectors as the discrete Hermite-Gaussian functions:

$$
\begin{equation*}
e_{f}[m, n]=h_{M, a}[m] h_{N, b}[n], \tag{24}
\end{equation*}
$$

where $h_{M, a}[m]$ is the $a^{\text {th }}$ discrete Hermite-Gaussian function of the 1-D $M$-point DFT. It can be derived from the commuting matrix method as in [7][8][11]. $\left\{h_{M, a}[m] h_{N, b}[n], a=0,1,2, \ldots, M-2, M_{1}\right.$, $\left.b=0,1,2, \ldots, N-2, N_{1}\right\}$ forms a complete and orthogonal eigenvector set of the 2-D DFT, where

$$
\begin{equation*}
M_{1}=M-1 \text { if } M \text { is odd, } \quad M_{1}=M \text { if } M \text { is even, } \tag{25}
\end{equation*}
$$

and $N_{1}$ is defined in the similar way. The eigenvalue of the 2-D DFT corresponding to $h_{M, a}[m] h_{N, b}[n]$ is $(-i)^{a+b}$ :

$$
\begin{equation*}
\operatorname{DFT}\left(h_{M, a}[m] h_{N, b}[n]\right)=h_{M, a}[m] h_{N, b}[n](-i)^{a+b} . \tag{26}
\end{equation*}
$$

Table 1 The eigenvalues of the DQFT corresponding to the discrete Hermite-Gaussian eigenvectors $h_{M, a}[m] h_{N, b}[n]$.

| Conditions | Eigenvalues of the DFT | Eigenvalues of the DQFT |
| :---: | :---: | :---: |
| $\begin{aligned} & ((a))_{4}=0,((b))_{4}=0 \text { or } \\ & \left.((a))_{4}=2,(b)\right)_{4}=2 \end{aligned}$ | 1 | 1 |
| $\begin{aligned} & ((a))_{4}=1,((b))_{4}=3 \text { or } \\ & \left.((a))_{4}=3,(b)\right)_{4}=1 \end{aligned}$ |  | -k |
| $\begin{aligned} & ((a))_{4}=1,((b))_{4}=0 \text { or } \\ & \left.((a))_{4}=3,(b)\right)_{4}=2 \end{aligned}$ | $-i$ | $-i$ |
| $\begin{aligned} & ((a))_{4}=0,((b))_{4}=1 \text { or } \\ & \left.((a))_{4}=2,(b)\right)_{4}=3 \end{aligned}$ |  | -j |
| $\begin{aligned} & ((a))_{4}=0,((b))_{4}=2 \text { or } \\ & \left.((a))_{4}=2,(b)\right)_{4}=0 \end{aligned}$ | -1 | -1 |
| $\begin{aligned} & ((a))_{4}=1,((b))_{4}=1 \text { or } \\ & \left.((a))_{4}=3,(b)\right)_{4}=3 \end{aligned}$ |  | $k$ |
| $\begin{aligned} & ((a))_{4}=1,((b))_{4}=2 \text { or } \\ & ((a))_{4}=3,((b))_{4}=0 \end{aligned}$ | $i$ | $i$ |
| $\begin{aligned} & ((a))_{4}=0,((b))_{4}=3 \text { or } \\ & \left.((a))_{4}=2,(b)\right)_{4}=1 \end{aligned}$ |  | $j$ |

Since [7][8][11],

$$
\begin{equation*}
h_{N, b}[n]=(-1)^{b} h_{N, b}[N-n], \tag{27}
\end{equation*}
$$

i.e., $e_{f}[m, n]=h_{M, a}[m] h_{N, b}[n]$ is even or odd symmetric along $n$, thus, from Theorems 1 and 2 , we have:
[Theorem 5] The separable discrete Hermite-Gaussian functions in (24) are also the eigenvectors of the DQFT and

$$
\begin{gather*}
\operatorname{DQFT}\left(h_{M, a}[m] h_{N, b}[n]\right)=h_{M, a}[m] h_{N, b}[n](-i)^{a+b}  \tag{28}\\
\text { when } b \text { is even, } \\
\operatorname{DQFT}\left(h_{M, a}[m] h_{N, b}[n]\right)=h_{M, a}[m] h_{N, b}[n](-i)^{a+b}(-k)  \tag{29}\\
\text { when } b \text { is odd. }
\end{gather*}
$$

Moreover, as the case of the DFT, $\left\{h_{M, a}[m] h_{N, b}[n], a=0,1,2, \ldots\right.$, $\left.M-2, M_{1}, b=0,1,2, \ldots ., N-2, N_{1}\right\}$ also forms a complete and orthogonal eigenvector set for the DQFT.

From (28) and (29), the DQFT has 8 distinct eigenvalues, which include $\pm 1, \pm i, \pm j$, and $\pm k$.

We list the relation among $a, b$, and the eigenvalues of the DQFT in Table 1, where $(())_{4}$ means the remainder of a number after being divided by 4 (e.g., $\left.((15))_{4}=((3 \cdot 4+3))_{4}=3\right)$. Note that each eigenspace of the original DFT corresponds to two eigenspaces of the DQFT.

We can use the similar way to derive the "left-sided" eigenvectors and eigenvalues of the DQFT (see (8)). From the similar process as those in Theorems 1-4, we obtain:
[Theorem 6] Suppose that $e_{f}[m, n]$ is an eigenvector of the 2-D DFT in the complex filed, as in (9). Then
(a) $\operatorname{DQFT}\left(e_{f}[m, n]\right)=\lambda e_{f}[m, n]$
when $e_{f}[m, n]=e_{f}[m, N-n]$,
(b) $\operatorname{DQFT}\left(e_{f}[m, n]\right)=(-\lambda k) e_{f}[m, n]$
when $e_{f}[m, n]=-e_{f}[m, N-n]$ and $e_{f}[m, n]$ is real,
(c) $\operatorname{DQFT}\left(e_{f}[m, n]\right)=\lambda_{q} e_{f}[m, n]\left(\lambda_{q}\right.$ is defined in (20))
when $e_{f}[m, n]=e_{f}[M-m, n]$ and $e_{f}[m, n]$ is real or $\lambda= \pm 1$.
(d) $\operatorname{DQFT}\left(e_{f}[m, n]\right)=\left(-k \lambda_{q}\right) e_{f}[m, n]$
when $e_{f}[m, n]=-e_{f}[M-m, n]$ and $e_{f}[m, n]$ is real or $\lambda= \pm i$.

Note that, in the right-sided case, $e_{f}[m, n]$ is not constrained to be a real function. However, in the left-sided case, when (1) $e_{f}[m, n]=$ $-e_{f}[m, N-n]$, (2) $e_{f}[m, n]=e_{f}[M-m, n]$ and $\lambda= \pm i$, and (3) $e_{f}[m, n]=$ $-e_{f}[M-m, n]$ and $\lambda= \pm 1, e_{f}[m, n]$ should be a real function. Otherwise, it is not an eigenvector of the DQFT.
[Corollary 3] As the right-sided case, we can also prove that the separable discrete Hermite-Gaussian functions in (24) also form a complete and orthogonal eigenvector set for the DQFT in the leftsided case. Moreover, the eigenvalues listed in Table 2 are also valid for the left-sided case.

## 3. Eigenvectors and Eigenfunctions of Discrete reduced biquaternion Fourier Transforms

In the reduced biquaternion algebra, the idempotent elements $\mathbf{E}_{\mathbf{1}}$ and $\mathbf{E}_{2}$ as follows play very important roles

$$
\begin{equation*}
E_{1}=(1+j) / 2, \quad E_{2}=(1-j) / 2 . \tag{34}
\end{equation*}
$$

They satisfy the properties of

A reduced biquaternion number in (4) can be re-expressed by the idempotent element form as:

$$
\begin{gather*}
q=q_{1} E_{1}+q_{2} E_{2}, \text { where }  \tag{37}\\
q_{1}=q_{r}+q_{j}+i\left(q_{i}+q_{k}\right), \quad q_{2}=q_{r}-q_{j}+i\left(q_{i}-q_{k}\right) . \tag{38}
\end{gather*}
$$

Therefore

$$
\begin{align*}
& \exp \left(-k \frac{2 \pi}{N} q n\right)=\cos \left(\frac{2 \pi}{N} q n\right)-k \sin \left(\frac{2 \pi}{N} q n\right)  \tag{39}\\
= & {\left[\cos \left(\frac{2 \pi}{N} q n\right)-i \sin \left(\frac{2 \pi}{N} q n\right)\right] E_{1}+\left[\cos \left(\frac{2 \pi}{N} q n\right)+i \sin \left(\frac{2 \pi}{N} q n\right)\right] E_{2} . }
\end{align*}
$$

Thus, the DRBQFT in (6) can be rewritten as

$$
\begin{align*}
& \operatorname{DRBQFT}(x[m, n]) \\
& =\sqrt{\frac{1}{M N}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \exp \left(-i \frac{2 \pi}{M} p m\right) \exp \left(-i \frac{2 \pi}{N} q n\right) x_{1}[m, n] E_{1} \\
&  \tag{40}\\
& +\sqrt{\frac{1}{M N}} \sum_{m=0}^{M-N} \sum_{n=0}^{N-1} \exp \left(-i \frac{2 \pi}{M} p m\right) \exp \left(i \frac{2 \pi}{N} q n\right) x_{2}[m, n] E_{2},
\end{align*}
$$

where $x_{1}[m, n]=x_{r}[m, n]+x_{j}[m, n]+i\left(x_{i}[m, n]+x_{k}[m, n]\right)$ and $x_{2}[m$, $n]=x_{r}[m, n]-x_{j}[m, n]+i\left(x_{i}[m, n]-x_{k}[m, n]\right)$. Therefore, the DRBQFT also has a close relation with the 2-D DFT in the complex field and we can use the eigenvectors and eigenvalues of the 2-D DFT to derive those of the DRBQFT.
[Theorem 7] Suppose that $e_{f}[m, n]$ is the eigenvector of the 2-D DFT in the complex field, as in (9). If

$$
\begin{equation*}
e_{f}[m, n]=e_{f}[m, N-n], \tag{41}
\end{equation*}
$$

then it satisfies (13) and the DRBQFT of $e_{f}[m, n]$ is

$$
\operatorname{DRBQFT}\left(e_{f}[m, n]\right)
$$

$$
=\sqrt{\frac{1}{M N}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \exp \left(-i \frac{2 \pi}{M} p m\right) \cos \left(\frac{2 \pi}{N} q n\right) e_{f}[m, n] E_{1}
$$

$$
+\sqrt{\frac{1}{M N}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \exp \left(-i \frac{2 \pi}{M} p m\right) \cos \left(\frac{2 \pi}{N} q n\right) e_{f}[m, n] E_{2}
$$

$$
=\sqrt{\frac{1}{M N}} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \exp \left(-i \frac{2 \pi}{N} p m-i \frac{2 \pi}{N} q n\right)\left(e_{f}[m, n] E_{1}+e_{f}[m, n] E_{2}\right)
$$

$$
\begin{equation*}
=\lambda\left(e_{f}[m, n] E_{1}+e_{f}[m, n] E_{2}\right)=\lambda e_{f}[m, n] . \tag{42}
\end{equation*}
$$

That is, $e_{f}[m, n]$ is also an eigenvector of the DRBQFT and the corresponding eigenvalue is also $\lambda$.

$$
\begin{align*}
& E_{1} E_{2}=0, E_{1}=E_{1}^{2}=\ldots . .=E_{1}^{n-1}=E_{1}^{n},  \tag{35}\\
& E_{2}=E_{2}^{2}=\ldots . .=E_{2}^{n}=E_{2}^{n-1} . \tag{36}
\end{align*}
$$

Table2 The eigenvalues of the DRBQFT corresponding to the discrete Hermite-Gaussian eigenvectors $h_{M, a}[m] h_{N, b}[n]$.
$\begin{array}{|c|c|c|}\hline \text { Conditions } & \begin{array}{c}\text { Eigenvalues of the } \\ \text { DFT }\end{array} & \begin{array}{c}\text { Eigenvalues of } \\ \text { the DRBQFT }\end{array} \\ \hline \begin{array}{c}((a))_{4}=0,((b))_{4}=0 \text { or } \\ ((a))_{4}=2,((b))_{4}=2\end{array} & & 1 \\$\cline { 1 - 1 }$\left.((a))_{4}=1,((b))_{4}=3 \text { or } & 1 & j \\ ((a))_{4}=3,((b))_{4}=1\end{array}\right)$
[Theorem 8] Similarly, if $e_{f}[m, n]$ is the eigenvector of the 2-D DFT and

$$
\begin{equation*}
e_{f}[m, n]=e_{f}[M-m, n], \tag{43}
\end{equation*}
$$

then since the inner product of $e_{[ }[m, n]$ and $\sin (2 \pi p m / N)$ is zero, the DRBQFT of $e_{[ }[m, n]$ is

$$
\begin{align*}
& \operatorname{DRBQFT}\left(e_{f}[m, n]\right) \\
& =\sqrt{\frac{1}{M N}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \cos \left(\frac{2 \pi}{M} p m\right) \exp \left(-i \frac{2 \pi}{N} q n\right) e_{f}[m, n] E_{1} \\
& +\sqrt{\frac{1}{M N}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \cos \left(\frac{2 \pi}{M} p m\right) \exp \left(i \frac{2 \pi}{N} q n\right) e_{f}[m, n] E_{2} \\
& =\sqrt{\frac{1}{M N}} \sum_{m=0}^{M-N} \sum_{n=0}^{N-1}\left(e^{-i \frac{2 \pi}{M} p m} e^{-i \frac{2 \pi}{N} q n} e_{f}[m, n] E_{1}+e^{i \frac{i \pi}{M} p m} e^{i \frac{2 \pi}{N} q n} e_{f}[m, n] E_{2}\right) \\
& =\operatorname{DFT}\left(e_{f}[m, n]\right) E_{1}+I D F T\left(e_{f}[m, n]\right) E_{2} \\
& =\left(\lambda E_{1}+\lambda^{-1} E_{2}\right) e_{f}[m, n] . \tag{44}
\end{align*}
$$

That is, $e_{[ }[m, n]$ is still the eigenvector of the DRBQFT, but the eigenvalues is changed into $\lambda E_{1}+\lambda^{-1} E_{2}$.
[Theorem 9] Using the similar ways, we can also prove that if $e_{f}[m, n]$ is an eigenvector of the 2-D DFT that satisfies (9) and $e_{f}[m$, $n]=-e_{f}[m, N-n]$, then

$$
\begin{equation*}
\operatorname{DRBQFT}\left(e_{f}[m, n]\right)=j \lambda e_{f}[m, n] . \tag{45}
\end{equation*}
$$

If $e_{f}[m, n]$ satisfies (9) and $e_{f}[m, n]=-e_{f}[M-m, n]$, then

$$
\begin{equation*}
\operatorname{DRBQFT}\left(e_{f}[m, n]\right)=\left(\lambda E_{1}-\lambda^{-1} E_{2}\right) e_{f}[m, n] . \tag{4}
\end{equation*}
$$

Therefore, from Theorems 7, 8, and 9, if $e_{d}[m, n]$ is an eigenvector of the 2-D DFT in the complex field and $e_{f}[m, n]= \pm e_{f}[m, N-n]$ or $e_{f}[m, n]= \pm e_{f}[M-m, n]$, then it is also an eigenvector of the DRBQFT. Furthermore, from (42), (44), (45), and (46), the DRBQFT will have 8 possible eigenvalues ( $\pm 1, \pm i, \pm j$, and $\pm k$ ).
[Theorem 10] As the quaternion case, the separable discrete Her-mite-Gaussian functions $\left\{h_{M, a}[m] h_{N, b}[n], a=0,1,2, \ldots, M-2, M_{1}\right.$, $\left.b=0,1,2, \ldots, N-2, N_{1}\right\}$ also form a complete and orthogonal eigenvector set for the DRBQFT. Moreover, from (26), (42), and (45), we can conclude that

$$
\begin{gather*}
\operatorname{DRBQFT}\left(h_{M, a}[m] h_{N, b}[n]\right)=(-i)^{a+b} h_{M, a}[m] h_{N, b}[n] \\
\text { if } b \text { is even, }  \tag{47}\\
\operatorname{DRBQFT}\left(h_{M, a}[m] h_{N, b}[n]\right)=j(-i)^{a+b} h_{M, a}[m] h_{N, b}[n] \\
\text { if } b \text { is odd. } \tag{48}
\end{gather*}
$$

We list the relations among $a, b$, and the eigenvalues of the DRBQFT in Table 2. Note that the 2-D DFT in the complex field has 4 eigenspaces ( $\pm 1$ and $\pm i$ ). By contrast, the DRBQFT will have 8 eigenspaces ( $\pm 1, \pm i \pm j$, and $\pm k$ ).

## 4. Discrete Fractional Quaternion and Biquaternion Fourier Transforms

In [9], Xu et al. derived the continuous fractional quaternion Fourier transform based on generalizing the integral kernel. In this section, we will derive its discrete counterpart, i.e., the discrete fractional quaternion Fourier transform (DFRQFT) and the discrete fractional reduced biquaternion Fourier transform. (DFRRBQFT).

In [10][11], the conventional DFT was generalized into the discrete fractional Fourier transform (DFRFT) based on eigenvector decomposition. Since the eigenvectors and the eigenvalues of the DQFT and the DRBQFT have been derived in this paper, we can also use the method of eigenvector decomposition to derive the DFRQFT and the DFRRBQFT successfully. As the DFRFT [10][11], the DFRQFT and the DFRRBQFT will be useful in signal and image processing.

To derive the DFRQFT, first, since the discrete HermiteGaussian functions $h_{M, a}[m] h_{N, b}[n]$ in (24) form a complete and orthogonal eigenvector set for the DQFT, any function in the quaternion field can be decomposed as a summation of $h_{M, a}[m] h_{N, b}[n]$ :

$$
\begin{equation*}
x[m, n]=\sum_{a=0}^{M_{1}} \sum_{b=0}^{N_{1}} \tau_{a, b} h_{M, a}[m] h_{N, b}[n], \tag{49}
\end{equation*}
$$

where $m, n=0,1,2, \ldots, N-1, M_{1}$ and $N_{1}$ are defined as in (16),

$$
\begin{align*}
& \tau_{M-1}=0 \text { if } M \text { is even, } \tau_{N-1}=0 \text { if } N \text { is even, }  \tag{50}\\
& \tau_{a, b}=\sum_{m=0}^{M-1} \sum_{n=0}^{N-1} x[m, n] h_{M, a}[m] h_{N, b}[n] . \tag{51}
\end{align*}
$$

Here, we suppose that $h_{M, a}[m] h_{N, b}[n]$ has been normalized:

$$
\begin{equation*}
\sum_{m=0}^{M-1} h_{M, a}^{2}[m]=\sum_{n=0}^{N-1} h_{N, b}^{2}[n]=1 . \tag{52}
\end{equation*}
$$

After substituting (49) into (3), we obtain

$$
\begin{equation*}
\operatorname{DQFT}(x[m, n])=\sum_{a=0}^{M_{1}} \sum_{b=0}^{N_{1}}(-i)^{a} \tau_{a, b}(-j)^{b} h_{M, a}[m] h_{N, b}[n] . \tag{53}
\end{equation*}
$$

Therefore, we suggest that the DFRQFT can be defined as the following process:
(Step 1) First, we decompose the input $x[m, n]$ by (49) and determine the coefficients $\tau_{a, b}$ from (51).
(Step 2) Calculate

$$
\begin{equation*}
e^{-i a \phi}=\cos (a \phi)-i \sin (a \phi), e^{-j b \theta}=\cos (b \theta)-j \sin (b \theta) . \tag{54}
\end{equation*}
$$

(Step 3) Then, the DFRQFT can be defined as

$$
\begin{equation*}
\operatorname{DFRQFT}_{\phi, \theta}(x[m, n])=\sum_{a=0}^{M_{1}} \sum_{b=0}^{N_{1}} e^{-i a \phi} \tau_{a, b} e^{-j b \theta} h_{M, a}[m] h_{N, b}[n] . \tag{55}
\end{equation*}
$$

There are some interesting properties that can be noticed. First, when $\phi=\theta=\pi / 2$, the DFRQFT becomes the original DQFT. When $\phi=\theta=-\pi / 2$, it becomes the inverse DQFT. When $\phi=\pi / 2, \theta=0$ and $\phi=0, \theta=\pi / 2$, it becomes the 1-D DQFT along the $m$-axis and the $n$-axis, respectively.

Moreover, it is no hard to prove that the DFRQFT has the additivity property as follows:

$$
\begin{align*}
& \operatorname{DFRQFT} T_{\phi_{1}, \theta_{1}}\left\{D F R Q F T_{\phi_{2}, \theta_{2}}(x[m, n])\right\} \\
= & D F R Q F T_{\phi_{1}+\phi_{2}, \theta_{1}+\theta_{2}}(x[m, n]) . \tag{56}
\end{align*}
$$

From the additivity property, we can conclude that the DFRQFT with parameters $-\phi$ and $-\theta$ is the inverse operation of the DFRQFT with parameters $\phi$ and $\theta$.

To define the DFRRBQFT, we can also use the fact that the discrete Hermite-Gaussian functions in (24) form a complete and orthogonal eigenvector set for the DRBQFT. From (40), (26), and

$$
\begin{equation*}
\sqrt{\frac{1}{M N}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} e^{-i \frac{2 \pi}{M} p m} e^{i \frac{2 \pi}{N} q n} h_{a}[m] h_{b}[n]=(-i)^{a-b} h_{a}[m] h_{b}[n], \tag{57}
\end{equation*}
$$

we obtain

$$
\begin{align*}
& \operatorname{DRBQFT}\left(h_{a}[m] h_{b}[n]\right) \\
= & \left.\left.\sum_{a=0}^{M_{1}} \sum_{b=0}^{N_{1}} h_{a}[m] h_{b}[n]\right\}(-i)^{a+b} E_{1}+(-i)^{a-b} E_{2}\right\} . \tag{58}
\end{align*}
$$

Therefore, the DFRRBQFT can be defined as

$$
\begin{align*}
& \operatorname{DFRRBQFT}_{\phi, \theta}(x[m, n]) \\
= & \sum_{a=0}^{M_{1}} \sum_{b=0}^{N_{1}} h_{a}[m] h_{b}[m]\left\{e^{-i(a \phi+b \theta)} E_{1}+e^{-i(a \phi-b \theta)} E_{2}\right\} . \tag{59}
\end{align*}
$$

Note that

$$
\begin{align*}
& e^{-i(a \phi+b \theta)} E_{1}+e^{-i(a \phi-b \theta)} E_{2}=e^{-i a \phi}\left(e^{-i b \theta} E_{1}+e^{i b \theta} E_{2}\right) \\
= & e^{-i a \phi}\left(\cos (b \theta)\left(E_{1}+E_{2}\right)+i \sin (b \theta)\left(E_{2}-E_{1}\right)\right) \\
= & e^{-i a \phi}(\cos (b \theta)+i \sin (b \theta)(-j))=e^{-i a \phi} e^{-k b \theta} . \tag{60}
\end{align*}
$$

Thus, the DFRRBQFT can be defined as the following way
(Step 1) First, decompose the input $x[m, n]$ by (49) and determine the coefficients $\tau_{a, b}$ from (51).
(Step 2) Calculate

$$
\begin{equation*}
e^{-i a \phi}=\cos (a \phi)-i \sin (a \phi), e^{-k b \theta}=\cos (b \theta)-k \sin (b \theta) . \tag{61}
\end{equation*}
$$

(Step 3) Then, the DFRRBQFT can be defined as

$$
\begin{align*}
& D F R R B Q F T_{\phi, \theta}(x[m, n]) \\
= & \sum_{a=0}^{M_{1}} \sum_{b=0}^{N_{1}} \tau_{a, b} h_{a}[m] h_{b}[m] \exp (-i a \phi) \exp (-k b \theta) . \tag{62}
\end{align*}
$$

As the case of the DFRFT and the DFRQFT, when $\phi=\theta=\pi / 2$, the DFRRBQFT becomes the original DRBQFT. Moreover, the DFRRBQFT also has the additivity property:

$$
\begin{align*}
& D F R R B Q F T_{\phi_{1}, \theta_{1}}\left\{D F R R B Q F T_{\phi_{2}, \theta_{2}}(x[m, n])\right\} \\
= & D F R R B Q F T_{\phi_{1}+\phi_{2}, \theta_{1}+\theta_{2}}(x[m, n]) \tag{63}
\end{align*}
$$

and the DFRRBQFT with parameters $-\phi$ and $-\theta$ is the inverse operation of the DFRRBQFT with parameters $\phi$ and $\theta$.

## 5. Extension to the Continuous Case

In fact, the results in Sections 2 and 3 can be extended to the continuous case. We can use the similar way to find the eigenfunctions and eigenvalues of the continuous quaternion Fourier transform (QFT) and the continuous reduced biquaternion Fourier transform (RBQFT). Theorems 1 to 10 can all be applied to the continuous
case, except that the 2-D FT, the DQFT, and the DRBQFT are replaced by their continuous counterparts.

As the discrete case, if $e_{f}(x, y)$ is an eigenfunction of the continuous 2-D FT and

$$
\begin{equation*}
e_{f}(x, y)= \pm e_{f}(x,-y) \text { or } e_{f}(x, y)= \pm e_{f}(-x, y), \tag{64}
\end{equation*}
$$

then $e_{f}(x, y)$ is also the right-sided eigenfunction of the continuous QFT and the eigenfunction of the continuous RBQFT. Moreover, the continuous 2-D Hermite-Gaussian functions (i.e., the continuous counterpart of (24)) form a complete and orthogonal eigenfunction set for the QFT and the RBQFT. Furthermore, both the continuous QFT and the continuous RBQFT also have 8 distinct eigenvalues ( $\pm 1, \pm i, \pm j$, and $\pm k$ ).

## 6. Conclusions

In this paper, we derived the eigenfunctions and eigenvectors of the DQFT (including the right-sided and the left-sided forms) and the DRBQFT. We find that the 2-D discrete Hermite-Gaussian functions, which are the eigenvectors of the 2-D DFT in the complex field, also form a complete and orthogonal eigenvector set for the DQFT and the DRBQFT . Furthermore, both the DQFT and the DRBQFT have 8 eigenspaces, which correspond to the eigenvalues of $\pm 1, \pm i, \pm j$, and $\pm k$. We also use the eigenvectors and eigenvalues we found to derive the discrete fractional quaternion and reduced biquaternion Fourier transforms.

## References

[1] W. R. Hamilton, Elements of Quaternions, Longmans, Green and Co., London, 1866.
[2] S. J. Sangwine, "The discrete quaternion Fourier transform," IEE Conf. Pub., vol. 2, pp. 790-793, 1997.
[3] T. A. Ell and S. J. Sangwine, "Decomposition of 2D hypercomplex Fourier transforms into pairs of complex Fourier transforms", EUSIPCO, pp. 151-154, 2000.
[4] Z. Lu, Y. Xu, X. Yang, L. Song, and L. Traversoni, "2D quaternion Fourier transform: the spectral properties and its application in color image representation," IEEE International Conference on Multimedia and Expo., pp. 1715-1718, July 2007.
[5] V. S. Dimitrov, T. V. Cooklev, and B. D. Donevsky, "On the multiplication of reduced biquaternions and applications," Inf. Process. Lett., vol. 43, pp. 161-164, 1992.
[6] S. C. Pei, J. H. Chang, and J. J. Ding, "Commutative reduced biquaternions and their Fourier transform for signal and image processing," IEEE Trans. Signal Processing, vol. 52, no. 7, pp. 2012-2031, July 2004.
[7] B. W. Dickinson and K. Steiglitz, "Eigenvectors and functions of the discrete Fourier transform," IEEE Trans. Acoust, Speech, Signal Process., vol. 30, pp. 25-31, 1982.
[8] F. A. Grünbaum, "The eigenvectors of the discrete Fourier transform: A version of the Hermite functions", J. Math. Anal. Appl., vol. 88, pp. 355-363, 1982.
[9] G. Xu, X. Wang, and X. Xu, "Fractional quaternion Fourier transform, convolution and correlation," Signal Processing, vol. 88. pp. 2511-2517, 2008.
[10] O. Arikan, M. A. Kutay, H. M. Ozaktas, and O. K. Akdemir, "The discrete fractional Fourier transformation," Proceedings of the IEEE-SP International Symposium on Time-Frequency and Time-Scale Analysis, pp. 205-207, 1996.
[11] S. C. Pei, W. L. Hsue, and J. J. Ding, "Discrete fractional Fourier transform based on new nearly tridiagonal commuting matrices," IEEE Trans. Signal Processing, vol. 54, no. 10, pp. 3815-3828, Oct. 2006.

