SYMMETRIC CORRELATION AND ITS PROPERTIES

Izumi ITO† and Hitoshi KIYA ‡

† Tokyo Institute of Technology, 2-12-1-W9-108 Ookayama, Megro-ku, Tokyo, 152-8552, Japan
phone: +(81) 3-5734-3232, fax: +(81) 3-5734-2560, email: izumii@ss.em-net.ne.jp
‡ Tokyo Metropolitan University, 6-6 Asahigaoka, Hino-shi, Tokyo, 191-0065, Japan
phone: + (81) 42-585-8419, fax: + (81) 42-585-8454, email: kiya@sd.tmu.ac.jp

ABSTRACT

In this paper, symmetric correlation that is a correlation between symmetrically extended signals is proposed to estimate shifts between two signals. Symmetrical extension creates smooth boundaries of endpoints of signals to avoid the discontinuity of endpoints. The symmetric correlation is performed by discrete cosine transform (DCT) without the increase of the number of samples. Moreover, it is shown that whitening of signals removes the interference effect of symmetry. That is, the correlation between signals can be directly estimated by symmetric correlation with whitening. Using the signs of DCT coefficients can be approximated as whitening, which contributes to lower computational complexity.

1. INTRODUCTION

The symmetric extension creates symmetry at the endpoints for smooth boundaries to mitigate the end effects encountered in convolving finite-length signals. One area in which symmetric extension is particularly useful is image filtering. Symmetric extension has been mainly studied for convolution [1, 2, 3].

Correlation, on the other hand, is a measure of similarity of two signals, which is used in many areas such as in communications, pattern recognition, and cryptanalysis. In signal matching, the shift between two signals is estimated by maximum correlation values [4]. The correlation can be calculated in either the spatial domain or the frequency domain by discrete Fourier transform (DFT). Although the periodicity is effective for calculation, finite-length signals may create the discontinuity of endpoints, which impairs the estimation. In this case, window functions are generally used in order to avoid the discontinuity of endpoints. However, when the length of signals is not enough, the estimated value is not reliable because window functions distort signals.

In the present paper, we define symmetric correlation and describe its properties. Symmetric correlation is used for signal matching and motivated by low computational complexity and creating smooth boundaries of endpoints of signals. Symmetric correlation is performed by discrete cosine transform (DCT) without the increase of the number of samples. Moreover, whitening of signals removes the interference effect of symmetry, which suggests the correlation between the original signals are estimated by symmetric correlation. We show that using the positive and negative signs of DCT coefficients can be approximated as whitening, which also contributes to lower computational complexity. Some experimental results are presented for the appropriateness and effectiveness of symmetric correlation.

2. PRELIMINARY

Circular correlation, phase correlation, and correlation coefficients matrix are described. One-dimensional notation is used for the sake of brevity.

2.1 Circular correlation

The circular correlation between $x(n)$ and $y(n)$, both of length N, is defined as

$$r(n) = x(n) \odot y(n) = \sum_{k=0}^{N-1} x(k)y(((n+k))_N) \tag{1}$$

where the notation $((n))_N$ denotes (n modulo N). Circular correlation can be also calculated using DFT as

$$r(n) = \frac{1}{N}\sum_{k=0}^{N-1} X^*(k)Y(k)W_N^{-nk} \tag{2}$$

where $X(k)$ and $Y(k)$ are the DFT coefficients of $x(n)$ and $y(n)$, respectively, and W_N denotes $\exp(-j2\pi/N)$.

2.2 Phase correlation

Phase correlation is defined as

$$r_\phi(n) = \frac{1}{N}\sum_{k=0}^{N-1} \phi_X^*(k)\phi_Y(k)W_N^{-nk} \tag{3}$$

where $\phi_X(k)$ and $\phi_Y(k)$ are phase factor of $X(k) = |X(k)|\phi_X(k)$ and $Y(k) = |Y(k)|\phi_Y(k)$, respectively. That is, phase correlation is a kind of weighted cross correlation:

$$r_\phi(n) = \frac{1}{N}\sum_{k=0}^{N-1} W_X X^*(k) W_Y Y(k)W_N^{-nk} \tag{4}$$

where $W_X = 1/|X(k)|$ and $W_Y = 1/|Y(k)|$.

Phase correlation can be also expressed as a cross correlation between signals whose spectral magnitude is normalized. The normalized spectral magnitude signal is defined as the inverse DFT of phase factor by

$$x_\phi(n) = \frac{1}{N}\sum_{k=0}^{N-1} \phi_X(k)W_N^{-nk}. \tag{5}$$

2.3 Correlation coefficients matrix

Correlation coefficient, $r_{XX}(l)$, of N-point signal $x(n)$ is defined as

$$r_{XX}(l) = \frac{\sum_{n=0}^{N-1}(x(n)-\bar{x})(x(n+l)-\bar{x})}{\sum_{n=0}^{N-1}(x(n)-\bar{x})^2} \tag{6}$$

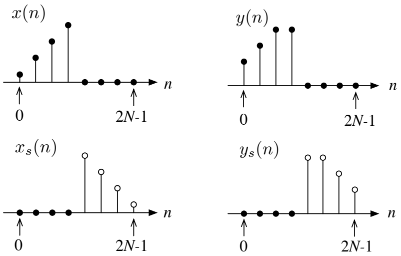


Figure 1: Composition of signals.

where l denotes a lag and $\bar{x} = \sum_{n=0}^{N-1} x(n)/N$. Correlation between samples which are away from l on a signal is evaluated by correlation coefficient. It is well known that the correlation coefficients of natural images are approximated with $\rho < 1$ as

$$r_{XX}(l) \simeq \rho^{|l|}. \tag{7}$$

That is, the nearest samples are highly correlated.

The correlation coefficients matrix is defined as

$$\mathbf{R}_{XX} = \mathbf{R}\mathbf{R}^t \tag{8}$$

where

$$\mathbf{R} = [r_{XX}(0), r_{XX}(1), r_{XX}(2), \cdots, r_{XX}(M-1)]^t. \tag{9}$$

The correlation coefficients matrix in AR(1) process is given as a Toeplitz matrix [5], i.e.,

$$\mathbf{R}_{XX} = \mathbf{R}\mathbf{R}^t = \begin{pmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{M-1} \\ \rho & 1 & \rho & \cdots & \\ \rho^2 & \rho & 1 & & \\ \vdots & & & \ddots & \\ \rho^{M-1} & & & & 1 \end{pmatrix}. \tag{10}$$

3. SYMMETRIC CORRELATION

Symmetric correlation is proposed herein. The motivation of symmetric correlation is to estimate the shifts between signals with low computational complexity and smooth boundary of end points.

3.1 Symmetric correlation and its type

Let $y(n)$ be a shifted signal of $x(n)$, both of length N. Let $\hat{x}(n)$ and $\hat{y}(n)$ be symmetrically extended signal of $x(n)$ and $y(n)$, respectively, as illustrated in Fig. 1. That is,

$$\hat{x}(n) = x(n) + x_s(n) \tag{11}$$
$$\hat{y}(n) = y(n) + y_s(n) \tag{12}$$

where

$$x_s(n) = x(2N-n-1), \tag{13}$$
$$y_s(n) = y(2N-n-1). \tag{14}$$

Symmetric correlation, $\hat{r}(n)$, between $x(n)$ and $y(n)$ is defined as

$$\hat{r}(n) = \hat{x}(n) \odot \hat{y}(n) = \sum_{k=0}^{2N-1} \hat{x}(k)\hat{y}(((n+k))_{2N}). \tag{15}$$

Symmetric correlation can be also calculated using DFT, i.e.,

$$\hat{r}(n) = \frac{1}{2N} \sum_{k=0}^{2N-1} \hat{X}^*(k)\hat{Y}(k)W_{2N}^{-nk}, \quad n = 0, 1, \cdots, 2N-1$$

where $\hat{X}(k)$ and $\hat{Y}(k)$ are the DFT coefficients of $\hat{x}(n)$ and $\hat{y}(n)$. The weighted symmetric correlation is hereby defined as

$$\hat{r}_w(n) = \frac{1}{2N} \sum_{k=0}^{2N-1} W_{\hat{X}}(k)\hat{X}^*(k)W_{\hat{Y}}(k)\hat{Y}^*(k)W_{2N}^{-nk} \tag{16}$$

where $W_{\hat{X}}(k)$ and $W_{\hat{Y}}(k)$ are the weights.

3.2 Efficient calculation of symmetric correlation

It is well known that there is a relationship between the DFT and DCT. From the relationship, symmetric correlation is calculated by N-point DCT in stead of $2N$-point DFT, i.e.,

$$\hat{r}_w(n) = \frac{\alpha}{N} \sum_{k=0}^{N-1} (k_k)^2 W_{X_C}(k)X_C(k)W_{Y_C}(k)Y_C(k) \cos\left(\frac{\pi nk}{N}\right),$$
$$n = 0, 1, \cdots, N-1 \tag{17}$$

where α is the scale factor, $W_{X_C}(k)$ and $W_{Y_C}(k)$ are the weights, and $X_C(k)$ and $Y_C(k)$ are N-point DCT coefficients of $x(n)$ and $y(n)$, respectively. The DCT coefficients, $X_C(k)$, of $x(n)$ is defined as

$$X_C(k) = \sqrt{\frac{2}{N}} k_k \sum_{n=0}^{N-1} x(n) \cos\left(\frac{\pi k(n+1/2)}{N}\right) \tag{18}$$

where

$$k_k = \begin{cases} 1, & k \neq 0 \\ 1/\sqrt{2}, & k = 0 \end{cases}. \tag{19}$$

That is, the use of DCT achieves a symmetric correlation without the increase of the number of samples.

There are some types of DCT on the basis of the type of symmetry. With respect to other type of DCT, the study of symmetric convolution [3] can be applied to symmetric correlation.

4. PROPERTIES OF SYMMETRIC CORRELATION

The properties of symmetric correlation is shown and the effect of weighting is discussed.

4.1 Basic properties

Since circular correlation is a linear operator, symmetric correlation $\hat{r}(n)$ is developed as

$$\begin{aligned} \hat{r}(n) &= \hat{x}(n) \odot \hat{y}(n) = (x(n) + x_s(n)) \odot (y(n) + y_s(n)) \\ &= x(n) \odot y(n) + x(n) \odot y_s(n) \\ &\quad + x_s(n) \odot y(n) + x_s(n) \odot y_s(n) \\ &= x(n) \odot y(n) + x(n) \circledast y(n-1) \\ &\quad + x(n-1) \circledast y(n) + y(-n-1) \odot x(-n-1) \end{aligned} \tag{20}$$

where the operator '$\circledast$' denotes circular convolution of $x(n)$ and $y(n)$:

$$x(n) \circledast y(n) = \sum_{k=0}^{2N-1} x(((k))_{2N})y(((n-k))_{2N}). \tag{21}$$

That is, symmetric correlation consists of two correlation terms and two convolution terms.

When, the two convolution terms in (20) are

$$\begin{cases} x(n) \circledast y(n-1) = 0 \\ x(n-1) \circledast y(n) = 0 \end{cases}, \tag{22}$$

then the symmetric correlation shows only the correlation terms.

$$\hat{r}(n) = \hat{x}(n) \odot \hat{y}(n) + y(-n-1) \odot x(-n-1) \tag{23}$$

To satisfy (22), the correlation coefficients matrix in (8) is an identity matrix, i.e.,

$$R_{XX} = I. \tag{24}$$

4.2 Special case of weighted symmetric correlation

If the weights $W_{\hat{X}}(k)$ and $W_{\hat{Y}}(k)$ in (16) are

$$\begin{cases} W_{\hat{X}}(k) = 1/|\hat{X}(k)| \\ W_{\hat{Y}}(k) = 1/|\hat{Y}(k)| \end{cases}, \tag{25}$$

then

$$W_{\hat{X}}(k)\hat{X}^*(k) \cdot W_{\hat{Y}}(k)\hat{Y}(k) = \alpha\sigma_X(k) \cdot \sigma_Y(k) \tag{26}$$

where $\sigma_X(k)$ and $\sigma_Y(k)$ are the sign of N-point DCT coefficients of $X_C(k)$ and $Y_C(k)$, respectively. Like phase correlation, the weighted symmetric correlation in this case is a cross correlation between the symmetrically extended signals whose spectral magnitude is normalized. In condition (25), the symmetric correlation corresponds to a cross correlation between normalized DCT-magnitude signals.

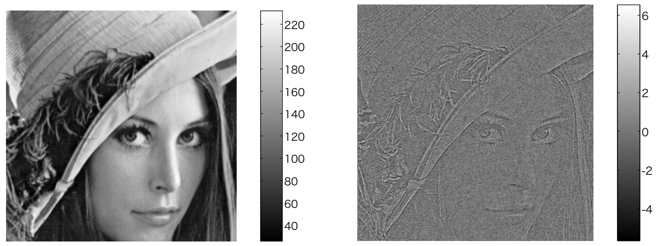
The normalized DCT-magnitude signal is defined as the inverse DCT of the signs of DCT coefficients of a signal by

$$x_\sigma(n) = \sqrt{\frac{2}{N}} \sum_{k=0}^{N-1} k_k \sigma_X(k) \cos\left(\frac{\pi k(n+1/2)}{N}\right). \tag{27}$$

Figure 2(b) shows the normalized DCT-magnitude signal of the original signal, Lena. Although the structure, such as edge, of the original signal is preserved in the normalized DCT-magnitude signal, the brightness of the signal is homogeneous. In this case, the weighted symmetric correlation reduces to DCT sign correlation [6, 7]. However, the discussion was limited to the relationship between the sign of DCT coefficients and the phase factor of DFT coefficients.

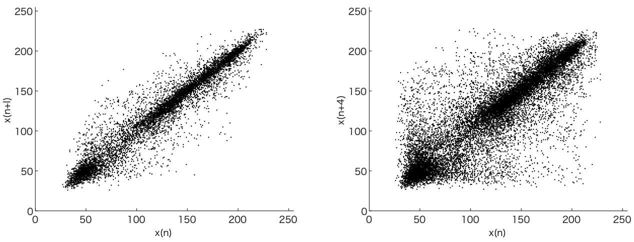
Figure 3(a) shows correlation between samples ($x(n)$, $x(n+l)$) on a line of Lena. In the case for which $l = 1$, the correlation of each set is high, while in the case for which $l = 4$, the correlation is lower than the case for which $l = 1$. Figure 3(b) shows the correlation between samples on a line of the normalized DCT-magnitude signal of Lena. Regardless of l, the correlation of the normalized DCT-magnitude signal of Lena is lower than that of the original image Lena.

Figures 4(a), 4(b), and 4(c) show the correlation coefficient matrix R_{XX} in (8) of the original signal Lena, the normalized DCT-magnitude signal of Lena, and the normalized magnitude signal of Lena, respectively. The correlation coefficients matrix of the original signal is approximated as the Toepliz matrix in (10) with $\rho = 0.96$. Conversely, the



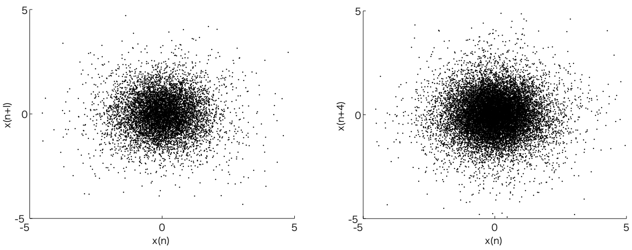
(a) the original signal (b) normalized DCT-magnitude signal

Figure 2: Normalized DCT-magnitude signal. Although the structure, such as edge, of the original signal is preserved, the brightness of the signal is homogeneous.



$l = 1$ $l = 4$

(a) the original signal



$l = 1$ $l = 4$

(b) normalized DCT-magnitude signal

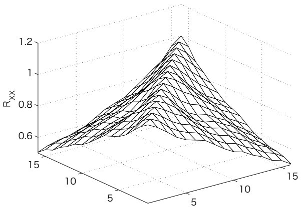
Figure 3: Correlation between samples, ($x(n)$, $x(n+l)$). Regardless of l, the correlation of the normalized DCT-magnitude signal of Lena is lower than that of the original signal Lena.

correlation coefficients matrix of the normalized DCT magnitude signal and normalized magnitude signal is approximated as an identity matrix. In Fig. 4(b), the mean and variance of the absolute error between R_{XX} and I are 0.0246 and 6.44×10^{-4}, respectively. In Fig. 4(c), the mean and variance of the absolute error between R_{XX} and I are 0.0263 and 3.94×10^{-4}, respectively.

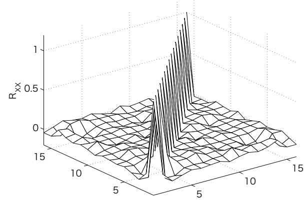
4.3 Steps of correlation between the original signals

The calculation steps of correlation between the original signals are as follows.

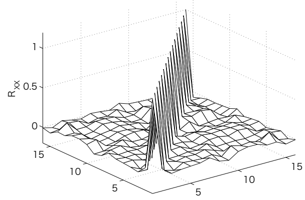
1. DCT in (18) is applied to two signals.
2. The DCT coefficients are weighted.
3. The weighted DCT coefficients of two signals are multiplied element by element.
4. The inverse transform in (17) is applied to the result.



(a) the original signal



(b)normalized DCT-magnitude signal



(c) normalized spectral-magnitude signal

Figure 4: Correlation coefficients matrix. The correlation coefficients matrix of the original signal is approximated as the Toepliz matrix with $\rho = 0.96$. Conversely, the correlation coefficients matrix of the normalized DCT magnitude signal and the normalized magnitude signal is approximated as an identity matrix.

In Step 2, when the weights are the reciprocal of the absolute value of DCT coefficients, the weighted DCT coefficients become the signs, which reduces the computational complexity.

5. SIMULATIONS

5.1 Computational complexity

We evaluated the computational complexity of symmetric correlation. There are fast calculation algorithms of DCT as well as DFT. Wang's algorithm achieves N-point DCT with

$$\mu_N = \frac{N}{2}\log_2 N + 1, \tag{28}$$

$$\alpha_N = \frac{3N}{2}\log_2 N - N + 1 \tag{29}$$

where μ_N denotes the number of multiplications of real numbers, and α_N denotes the number of additions [8]. The N-point FFT, on the other hand, is achieved by

$$M_N = \frac{N}{2}\log_2 N, \tag{30}$$

$$A_N = N\log_2 N \tag{31}$$

where M_N and A_N denote the number of multiplications and additions of complex numbers, respectively [9].

Table 1 summarizes the number of real number operations for circular correlation between two signals, both of length N. The complex multiplication is commuted to three real multiplication and three real additions [10]. The number of real number multiplications is shown in Fig. 5.

5.2 The effect of weighting (1D)

Shift estimation is performed to show the effect of whitening. Two signals, both of length 400, are shifted by 50 samples. A line of Lena was used for the two signals.

Table 1: The number of real number operations for circular correlation between two signals, both of length N. CC and SC denote circular correlation and symmetric correlation, respectively.

method	multiplications	additions
CC by (1)	N^2	$N(N-1)$
CC by (2)	$3(3(N\log_2 N)/2+N)$	$2(3N\log_2 N)$
SC by DCT	$3((N\log_2 N)/2+1)+N$	$3((N\log_2 N)/2-N+1)$

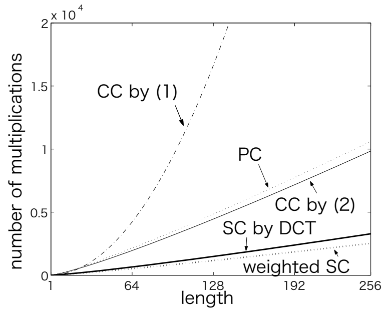


Figure 5: The number of multiplications for correlation between two signals. CC, PC, and SC denote circular correlation, phase correlation, and symmetric correlation, respectively.

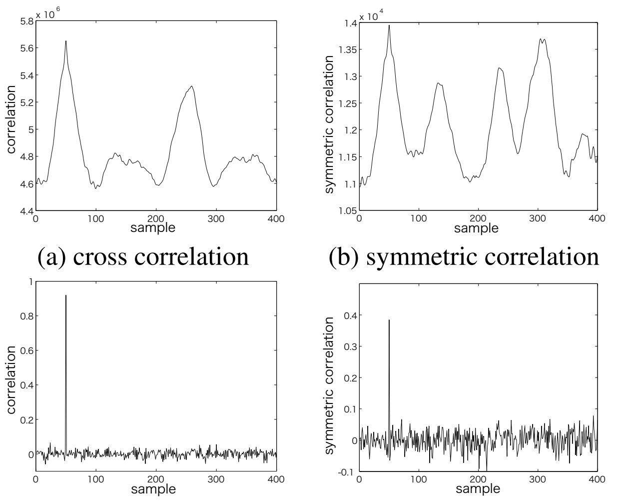
Figures 6(a) and 6(b) show the correlation according to (2) and symmetric correlation, respectively, between two signals. Although the location which has the highest correlation value shows the shift between signals, the effect of convolution terms are shown in symmetric correlation. Figures 6(c) and 6(d) show the phase correlation according to (3) and weighted symmetric correlation according to (17) where the signs of DCT coefficients are used, respectively, between two signals. The effect of whitening is shown by weighting.

5.3 The effect of weighting (2D)

Figures 7(a), 7(b), and 7(c) show the symmetric correlation, weighted symmetric correlation where signs of DCT coefficients of one of the signals are used, and weighted symmetric correlation in condition (25). Image Lena was used for signals in which they are shifted by 20 pixels in the horizontal and vertical directions. Even the case in which signs of DCT coefficients in one of the signals are used, the effect of whitening is confirmed.

5.4 Effect of symmetry

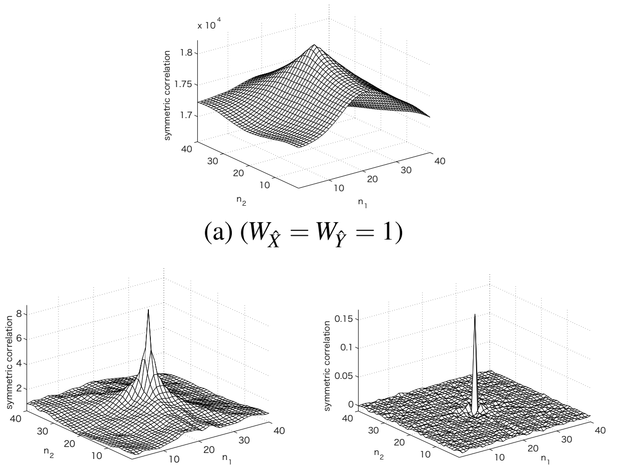
We show the effect of symmetric correlation on end effects encountered in calculating finite length signals. Figure 8(a) shows two signals, the signal $x(n)$ of length 64, and the signal $y(n)$ of length 32, in which $y(n) = x(n+n_0)$ and $n_0 = 32$. Although the estimated shift between the two signals is incorrect in phase correlation shown in Fig. 8(b), the shift is correctly estimated by symmetric correlation shown in Fig. 8(c). This difference is caused by the periodicity of signals in which the DFT creates the discontinuity of signals and the DCT creates smooth boundaries.



(a) cross correlation (b) symmetric correlation

(c) phase correlation (d) weighted symmetric correlation

Figure 6: Shift estimation (1D). Weighting by (25) provides the effect of whitening.



(a) $(W_{\hat{X}} = W_{\hat{Y}} = 1)$

(b) $(W_{\hat{X}} = 1, W_{\hat{Y}} = R)$ (c) $(W_{\hat{X}} = R, W_{\hat{Y}} = R)$

Figure 7: Correlation of Lena images. R denotes the reciprocal of the absolute value of DFT coefficients in (16).

6. CONCLUSION

We have proposed a symmetric correlation and discussed its properties. Symmetric correlation is performed by DCT without increasing samples. We have defined weighted symmetric correlation in which weights whiten a signal, and as an example, a correlation between signs of DCT coefficients has been shown. Some experimental results have demonstrated the appropriateness of symmetric correlation.

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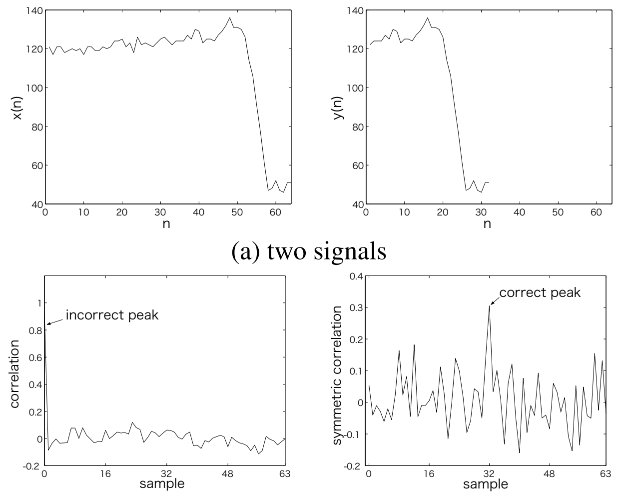

(a) two signals

(b) phase correlation (c) weighted symmetric correlation

Figure 8: The effect of symmetry. The shift can be correctly estimated by symmetric correlation.

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