MORPHOLOGICAL REGULARIZATION FOR ADAPTATION OF IMAGE OPENING

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ABSTRACT

In this paper, an adaptation method for structuring elements of morphological filters is proposed. Morphological filters provide set theoretic image processing with structuring elements. The structuring element specifies a shape of a local image structure that is eliminated or preserved in a filter output. The adaptation of the structuring element is crucial problem for morphological image processing. For existing adaptation methods of structuring element, the training images that contain the pairs of a supposed input image and an ideal output image are required. In this paper, we propose an adaptation method for the structuring element without preliminary training for image opening. Our approach is based on a regularization technique for inverse problems of image recovery. The morphological regularization is defined as a minimization of the cost function which consists of a fidelity term and a regularization term that corresponds to a smoothness criteria of the structuring element. For this minimization problem, the morphological filters are approximated as differentiable functions in order to employ a gradient descent method. In experiments, we demonstrate the impulsive noise reduction from texture images by using the proposed morphological regularization.

1. INTRODUCTION

The mathematical morphology is a methodology of signal and image processing based on set operations. Originally, the morphological filters have been proposed for analysis of binary images[1][2][3]. Later, the morphological filters have been extended to gray-level image processing methods and contribute to low-level image processing tasks including image enhancement and denoising[1][2]Maragos2.

For the morphological image processing, a gray-scale image is transformed into a subset of the three-dimensional space, which is spanned by the spatial and the intensity axes. The gray-scale morphological operations are interpreted as set operations between the subset of the image and *structuring elements* (SEs). In the gray-scale image processing, an image is assumed to be a union of the set of the SEs, which are translated in the three dimensional space. The morphological opening[1][2][3], which is realized by a dilation after an erosion operation, approximates the image as a union of the SE while eliminating the small subsets of the image that cannot include the translated SE. The morphological closing that is the dual operator of the opening approximates the complement of the image with the SE. The small subsets of the complement that cannot include the translated SE are eliminated. Therefore, the opening and closing can eliminate pits and valleys of the intensity surface of an image and has been used for the impulsive noise reduction. In this paper, the adaptation method for the SE of image opening is discussed. Obviously, the adaptation method for the opening can be also applied to adaptation of the closing.

In morphological image opening for image denoising, the image is assumed to be generated by a union of the translated SEs. Since the SE specifies which image local structures are eliminated or preserved, the choice of the SE is a crucial problem for the morphological denoising. However, the SEs of the morphology are usually specified as simple structures, such as squares, circles and rhombuses in many applications. For binary images, the optimum SE is obtained by simple and small structures with morphological operations[4]. In order to get a good denoising result for a gray-scale image, the gray-scale SE have to be adapted to the local structures of the image.

For the SE adaptation, several approaches have been proposed. In Ref. [5] and [8], the optimization methods for a class of the rank-order filters have been proposed. Erosion and dilation can be optimized by using these methods, since the both filters are included to the class of the rank-order filters. In these methods, the rank function is introduced to approximate the filters as a differentiable functions. In Ref. [6], the iterative adaptation method that is similar to the leas mean square (LMS) algorithms is proposed. The SE for the image opening is adapted for image denoising by using the LMS-like algorithm. In Ref. [7], the genetic algorithm is employed for the optimization of the SE.

In these existing approach, the SE adaption is performed with a training set that consist of pairs of a supposed corrupted image and an ideal output image of the morphological filter. The objective function that is minimized during the preliminary training is defined as an error between the ideal output and the filtered image. The supposed corrupted image is generated by adding a noise to the ideal image. Noisy images, of which characteristics are similar to the ideal image that is employed for the preliminary training, can be well denoised by the adapted SE. However, the training set that consists of the noisy and ideal images is not always obtained for actual denoising application.

In this paper, we propose an adaptation method for the SE of the image opening without preliminary training. Our approach is based on a regularization method for linear inverse problem for image recovery. In the regularization of images, the smoothness of the image is assumed and is represented by a penalty in the cost function[11][12]. The linear inverse problem is solved by the minimization of the cost function that is defined as a sum of two terms, a fidelity term and a regularization term. In our approach, the regularization term is defined for the morphological image model as well as the regularization for the linear image model. The regularization term is imposed on the recovered image for linear image generative models. For the morphological regularization, the regularization term is defined as a function of the SE, since the SE is supposed to represent the local intensity surface of the clean image.

In the next section, the image opening is briefly explained. In Sect. 3, we introduce the morphological regularization for image opening. In order to apply a gradient-based minimization for the morphological regularization, the morphological filters are approximated as differentiable functions is Sect. 4. Finally, we show several examples of the texture denoising to demonstrate the advantage of the SE adaptation by using our approach.

2. DENOISING BY MORPHOLOGICAL OPENING

In this paper, the observed image $\{f_{\mathbf{x}}\}_{\mathbf{x}\in\mathbb{I}}$, where $\mathbb{I}$ denotes the set of two-dimensional coordinates of image pixels, is assumed to be a sum of a clean original image g and an additive noise e as

$$f_{\mathbf{x}} = g_{\mathbf{x}} + e_{\mathbf{x}}. \tag{1}$$

The problem of the denoising is to estimate g from the observation f. In the morphological opening, the approximation $\hat{g}$ of the clean

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image g is generated by the dilation[1][2][3] of an image d,

$$\hat{g}_{\mathbf{x}} = \bigvee_{\mathbf{y}\in\mathbb{A}} d_{\mathbf{x}+\mathbf{y}} + s_{\mathbf{y}} \tag{2}$$

where $\bigvee_{\mathbf{y}\in\mathbb{A}}$ denotes the maximum value of the elements with respect to the set $\mathbb{A}$. $\{s_{\mathbf{y}}\}_{\mathbf{y}\in\mathbb{A}}$ is the SE for the opening. $\mathbb{A}$ denotes the set of the coordinates that are supported by the SE. In this operation, the SE corresponds to the smallest component of the image. From the set-theoretic point of view, the original image g is assumed to be composed as a union of the SEs that are translated in the three-dimensional space, which is spanned by the horizontal and vertical axes of the pixel coordinates and the intensity axis. For the estimation of $\hat{g}$, d specifies the intensity offset of the SEs that are allocated to each coordinate. For the morphological opening, the intensity offset d is estimated by a morphological erosion [1][2][3] of the observed image f as

$$d_{\mathbf{x}} = \bigwedge_{\mathbf{y}\in\mathbb{A}} f_{\mathbf{x}-\mathbf{y}} - s_{\mathbf{y}} \tag{3}$$

where $\bigwedge_{\mathbf{y}\in\mathbb{A}}$ denotes the minimum value of the elements with respect to the set $\mathbb{A}$. Hereinafter, the image obtained by the opening of $\{f_{\mathbf{x}}\}_{\mathbf{x}\in\mathbb{I}}$ with the SE $\{s_{\mathbf{y}}\}_{\mathbf{y}\in\mathbb{A}}$ is denoted as $\{O_s f_{\mathbf{x}}\}_{\mathbf{x}\in\mathbb{I}}$. If the SE, which cannot be included in the additive noise e and can represent the original image $\mathbf{g}$ with the dilation in (2) exactly, is known, the noise components are eliminated by the erosion process in (3), and the following dilation process (2) can only approximate the original image components. However, the ideal SE is usually unknown for image denoising. In this paper, we propose the adaptation method for the SE for image denoising with the morphological opening from the noisy observation.

3. MORPHOLOGICAL REGULARIZATION

In this section, we briefly overview the regularization for the linear inverse problem. Let us suppose that an image $\mathbf{f}$, which is a vector that consists of image pixels, is obtained as

$$\mathbf{f} = \mathbf{H}\mathbf{g} + \mathbf{e} \tag{4}$$

where $\mathbf{H}$ denotes an observation process. $\mathbf{g}$ is a clean original image. $\mathbf{e}$ indicates a noise that appears in the observed image. In the inverse problem estimating $\mathbf{g}$ from the observation $\mathbf{f}$, the noise components $\mathbf{e}$ have to be eliminated. Moreover, this inverse problem is usually ill-posed, some constraints that are obtained from the prior information about the original image $\mathbf{g}$ are imposed on the estimation. In many image recovery applications, the inverse problem of estimation $\mathbf{g}$ is reduced to the regularization problem

$$\min_{\mathbf{g}} \| \mathbf{f} - \mathbf{H}\mathbf{g} \|_2^2 + \lambda R(\mathbf{g}). \tag{5}$$

The first term of the objective function is a data fidelity term that is defined as a squared error between the recovered image $\mathbf{g}$ and the observation $\mathbf{f}$. The second term is a regularization penalty that is imposed on the recovered image $\mathbf{g}$. $R(\mathbf{g})$ is a regularization penalty in order to avoid overfitting of $\mathbf{H}\mathbf{g}$ to $\mathbf{f}$. λ is a regularization parameter. In many image recovery algorithm, the regularization penalty that is defined based on a smoothness assumption of images.

When the penalty function is a quadratic function $\| \Gamma\mathbf{g} \|_2^2$, this regularization corresponds to a Tikhonov regularization. In the image processing, Γ is chosen as a Laplacian operator to reduce fluctuations that include noises in the recovered image. In the total variation regularization[11][12], the penalty term is defined from the absolute sum of discrete gradients at every coordinates. In these regularization, the surface of the image intensity is assumed to be smooth, and the penalty term is small for the clean image.

In this paper, we also define the minimization problem of the denoising with the opening as well as the regularization for the linear inverse problems. The objective function of the morphological opening is defined as

$$Q(\{s_{\mathbf{y}}\}_{\mathbf{y}\in\mathbb{A}}) = E(\{s_{\mathbf{y}}\}_{\mathbf{y}\in\mathbb{A}}) + \lambda P(\{s_{\mathbf{y}}\}_{\mathbf{y}\in\mathbb{A}}). \tag{6}$$

The first term of the right side of Q denotes the fidelity term that is defined as a difference between the opened image $O_s f$ and the noisy observation f. The fidelity term is hence the function with respect to the SE s. The second term is a regularization penalty that is imposed on the approximation to avoid overfitting of the opened image $O_s f$.

In this study, the approximation error is measured by an absolute error between the opened image $O_s f$ and the input image f as

$$E(\{s_{\mathbf{y}}\}_{\mathbf{y}\in\mathbb{A}}) = \sum_{\mathbf{x}\in\mathbb{I}} |f_{\mathbf{x}} - O_s f_{\mathbf{x}}|. \tag{7}$$

Due to the antiextensivity of the opening, the opened image satisfies $O_s f_{\mathbf{x}} \leq f_{\mathbf{x}}$ at every coordinates $\mathbf{x}$. So, the absolute error can be reduced to

$$E(\{s_{\mathbf{y}}\}_{\mathbf{y}\in\mathbb{A}}) = \sum_{\mathbf{x}\in\mathbb{I}} f_{\mathbf{x}} - O_s f_{\mathbf{x}}. \tag{8}$$

For morphological opening, we also assume the smoothness of the opened image $O_s f$ that is the estimation of the clean original image. As seen in the dilation part of the morphological opening in (2), an image is approximated by the superimposition of the translated SEs. Local intensity variations of the image are hence approximated by the SE of the opening. In order to simplify the penalty function, we impose the smoothness penalty on the SE instead of the opened image itself. To define the penalty of the smoothness on the SE, the relationship between the SE and the opened image $O_s f$ should be regarded to restrict the feasible solution space. First, the opened image $O_s f$ is invariant with respect to the average over all elements of the SE. So, we suppose that each element of the SE is non-positive as

$$s_{\mathbf{y}} \leq 0 \ \ \forall \mathbf{y} \in \mathbb{A}. \tag{9}$$

This non-positivity constraint of the SE does not restrict the opened image $O_s f$ and contributes to restrict the solution space of the SE. Second, the opened image $O_s f$ is invariant with respect to the translation of the SE. In order to restrict the solution space of the SE, we suppose that the maximum element of the SE always appears at the center of the region $\mathbb{A}$ that is supported by the SE. Under the non-positivity constraint, all elements of the smoothest SE are zero. The smoothness of the SE is hence measured as the distance from the smoothest SE as

$$P(\{s_{\mathbf{y}}\}_{\mathbf{y}\in\mathbb{A}}) = \sum_{\mathbf{y}\in\mathbb{A}} s_{\mathbf{y}}^2. \tag{10}$$

In this definition, the distance between the SE and the smoothest SE is defined as an Euclidean norm. Since the maximum element is supposed to appear at the center of the SE and is zero for minimization of this penalty, this penalty approximates the sum of the squared differences between the neighboring pixels.

By addition of the approximation error term and the smoothness penalty of the SE, we have an image approximation problem with the opening as

$$\min_{\{s_{\mathbf{y}}\}_{\mathbf{y}\in\mathbb{A}}} \sum_{\mathbf{x}\in\mathbb{I}} (f_x - O_s f_x) + \lambda \sum_{\mathbf{y}\in\mathbb{A}} s_{\mathbf{y}}^2$$
$$\text{subject to } \ s_{\mathbf{y}} \leq 0 \ \forall \mathbf{y} \in \mathbb{A},\ s_{\mathbf{y}_c} = 0 \tag{11}$$

where $\mathbf{y}_c \in \mathbb{A}$ denotes the supposed center of the SE. In the above morphological regularization, the regularization parameter λ is the real number in $[0, +\infty]$. In order to restrict the range of the parameter, the penalty of the smoothness is realized as a constraint for the minimization as

$$\min_{\{s_{\mathbf{y}}\}_{\mathbf{y}\in\mathbb{A}}} \sum_{\mathbf{x}\in\mathbb{I}} (f_x - O_s f_x)$$
$$\text{subject to } \sum_{\mathbf{y}\in\mathbb{A}} s_{\mathbf{y}}^2 \leq \sigma^2,\ s_{\mathbf{y}} \leq 0 \ \forall \mathbf{y} \in \mathbb{A}, s_{\mathbf{y}_c} = 0. \tag{12}$$

In this problem, the parameter σ that specifies the tolerable variation of the SE is introduced instead of the parameter λ. The image approximation is the problem that seeks the SE that minimizes the approximation error under the restriction for the variance of the SE. Obviously, the approximation error of the opening is zero, when σ is large enough. Let us suppose that the image intensity $f_{\mathbf{x}}$ is a value in the range $[0, M]$, where M is an integer. If $s_{\mathbf{y}_c} = 0$, which is the center of the SE, and the other elements of the SE are $-M$, the approximation error of the opening becomes zero. In this case, the squared norm of the SE is $(N-1)M^2$ where N is the number of the elements in the SE. Therefore, σ can be restricted to a value in the range $[0, \sqrt{N-1}M]$. For denoising, the appropriate σ is obtained by seeking a value in this range. In next section, the gradient-based minimization is applied to minimize the fidelity term under the constraints of the SE.

4. GRADIENT DESCENT ALGORITHM FOR MORPHOLOGICAL REGULARIZATION

4.1 Approximation of the morphological opening

One of the major difficulties that arise in the optimization of the morphological filters is that the max ($\bigvee$) and min ($\bigwedge$) are not differentiable with respect to the elements of the SE. Since the cost function that is minimized during the optimization of the SE is not also differentiable with respect to the SE, a gradient based optimization is difficult to apply the optimization directly. In Ref. [8], the rank functions that include the max and min functions are approximated by the differentiable functions for optimization of the rank filters. For construction of the morphological filters, only the max and min functions are required. In Ref. [9] and [10], the max and min functions are approximated by generalized–f means. In this paper, we also approximate the max and min functions as differentiable functions with the generalized–f mean. The erosion (3), which specifies the offsets of the opening, is approximated as

$$\hat{d}_{\mathbf{x}} = -T \log \sum_{\mathbf{y} \in \mathbb{A}} \exp\left(-(f_{\mathbf{x}-\mathbf{y}} - s_{\mathbf{y}})/T\right). \tag{13}$$

When $T \to 0$, $\hat{d}_{\mathbf{x}}$ converges to the true eroded image $d_{\mathbf{x}}$. Consequently, the opening is approximated as

$$\hat{O}_s f_{\mathbf{x}} = T \log \sum_{\mathbf{y} \in \mathbb{A}} \exp\left(\left(\hat{d}_{\mathbf{x}+\mathbf{y}} + s_{\mathbf{y}}\right)/T\right). \tag{14}$$

The approximated opening $\hat{O}_s f$ is differentiable with respect to any element of the SE. For the bounded input f and SE, the error between the true opening $O_s f$ and $\hat{O}_s f$ is bounded as

$$| \hat{O}_s f_{\mathbf{x}} - O_s f_{\mathbf{x}} | < T \log N \tag{15}$$

where N is the number of the elements of the SE. This bound can be found from the bounds of the approximation errors of the max and min functions. By using the approximation of the opening, the cost function in (12) can be approximated as

$$\hat{E} = \sum_{\mathbf{x} \in \mathbb{I}} \left(f_x - \hat{O}_s f_x\right). \tag{16}$$

Since the approximation of the opened image is differentiable with respects to the elements of the SE, the approximation of the objective function is also differentiable. The partial differential of the objective function can be derived and is used for the gradient based optimization.

4.2 Gradient descent under the convex constraints

The update rule of the gradient descent for the SE adaptation is

$$\mathbf{s}^{i+1} = P_B P_C \left(\mathbf{s}^i - h \nabla \hat{E}\right) \tag{17}$$

where h is a step size for i-th iteration. $\nabla \hat{E}$ is the gradient of the approximated cost function $\hat{E}$ that is derived in the previous subsection. $\mathbf{s}^i$ denotes the vector representation of the SE at i-th iteration. P_C denotes the operator that projects an SE on the convex set, of which any element $\{c_{\mathbf{y}}\}_{\mathbf{y} \in \mathbb{A}}$ satisfies

$$c_{\mathbf{y}} \leq 0 \text{ for } \forall \mathbf{y} \in \mathbb{A}, c_{\mathbf{y}_c} = 0. \tag{18}$$

This projection is realized by replacing the center of the SE and the elements of which values are larger than zero with zero.

P_B denotes the operator that projects the vector of SE onto the l_2 ball

$$B = \{\mathbf{x} | \parallel \mathbf{x} \parallel_2 \leq \sigma\} \tag{19}$$

where $\parallel \mathbf{x} \parallel_2$ denotes the l_2 norm of the vector $\mathbf{x}$. This projection can be realized as

$$\mathbf{s} \leftarrow \begin{cases} \sigma \frac{\mathbf{s}}{\|\mathbf{s}\|_2} & \text{for } \parallel \mathbf{s} \parallel_2 \geq \sigma \\ \mathbf{s} & \text{otherwise.} \end{cases} \tag{20}$$

In our implementation of the gradient descent, the direct search method is employed for the line search of the optimum step size h at each iteration. The iteration searches the SE that yields smaller objective function at each step, however, this iteration converges to a local minimum or saddle point due to the non-convexity of the objective function. After the convergence, the trained SE is employed for the denoising.

5. EXAMPLES OF TEXTURE DENOSING

In this section, we provide several examples of the regularization-based SE adaptation for image denoising. We employ texture images that are chosen from the Brodatz texture database. Since the texture image consists of micro structures that repeatedly appear, the morphological image generative model can well represent the texture images. The chosen textures D15, D16 and D84 are shown in Fig. 1, 2 and 3, respectively.

The morphological opening can eliminate positive noises that cannot include the translated SE. In order to generate the nosies, the intensities of the pixels, which are in the range $[0, 255]$, are represented in binary numbers of 8 bits. Each bit that indicates zero is flipped with the error probability p. The generated noise is equivalent to the noise due to error in a binary asymmetric channel. The error probability p is specified as $1/8$ for the experiments.

The size of the SE is specified as 3×3 pixels for all images. The appropriate choice of the weight σ will depend on the noise level and the textures. In order to observe the relationship between the trained SE and σ, σ is initially specified as zero and is increased to the limit while performing the optimization iteratively. Each optimization of the SE is started with the initial SE that is obtained by the previous optimization. Obviously, when σ is specified as zero, the optimization yields a flat square SE, of which elements are zero. The approximation is repeated while increasing σ by 50 until the largest $\sigma = 800$. The SE that can perfectly recover the original image consists of the element that is zero at the center, other elements are smaller than or equal to -255. In this case, the l_2 norm of the SE is larger than 721. So, it is expected that the SE at the maximum $\sigma = 800$ recovers the input image without errors. Prior to denoisng, a mean absolute error (MAE) between a noisy input image and an original image is supposed to be estimated. Under this prior, we seek the appropriate σ for denoising. We assume that noise components will perfectly eliminated while preserving the original image at appropriate σ. Under this assumption, we accept the opening result, of which absolute error to the noisy input is closest to the MAE that is prior estimated, as the denoising result. The denosing is hence performed without the clean original image, with estimated MAE of the noisy image.

In Fig. 1, 2 and 3, the opening results with three tolerable variations, which include $\sigma = 0$, quasi-optimum σ estimated from the MAE. For comparison, the denoisng results that are obtained by a

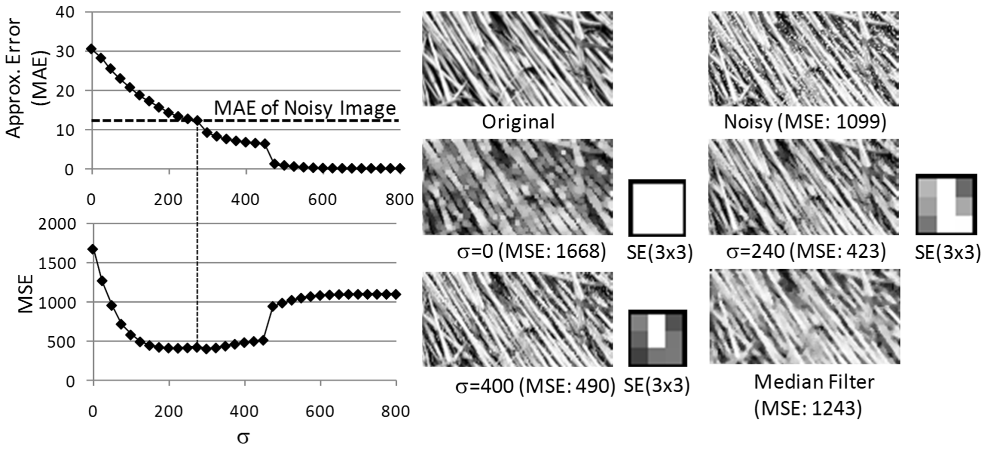

Figure 1: Denoising of the texture D15. Upper left: Relationship between the approximation error and the tolerable deviation σ. Lower left: Relationship between MSE of opened image and σ. Right: Original, noisy and opened images. SEs that obtain the opened images are also shown.

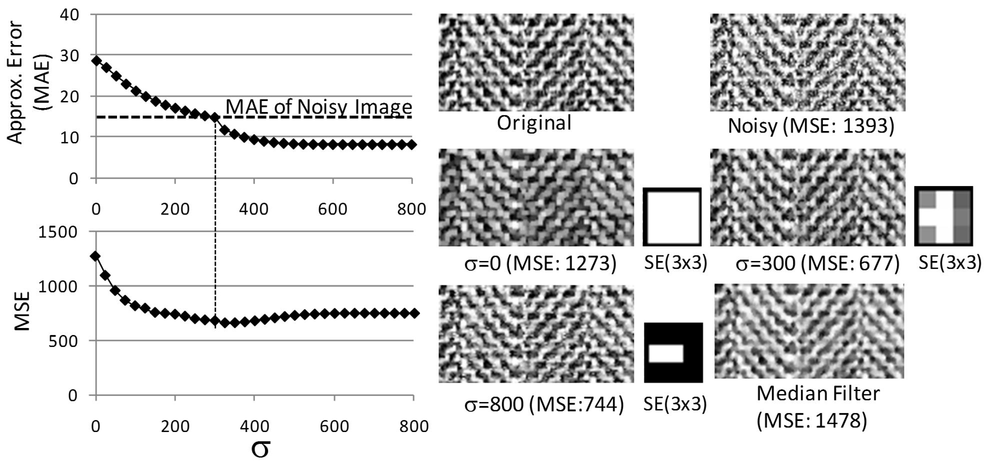

Figure 2: Denoising of the texture D16. Upper left: Relationship between the approximation error and the tolerable deviation σ. Lower left: Relationship between MSE of opened image and σ. Right: Original, noisy and opened images. SEs that obtains opened images are also shown.

median filter, which is widely applied to impulsive noise reduction, are also shown. The window size of the median filter is specified as 3×3 pixels. In each figure, the relationships between σ and the approximation error, which is defined as the difference between the input noisy image and the result of the opening is also depicted. The relationship between σ and the mean squared error (MSE) of the opening result is also shown. Each MSE is measured between an opening result and the original image. In each case, the estimated quasi-optimum σ exists close to the optimum parameter σ that yields the smallest MSE. We see that the approximation error decreases along with the increment of σ, which specifies tolerable variation of the SE. Ideally, the input image is perfectly reconstructed with the largest σ. However, the approximation error does not converge to zero in the case of D16 in Fig. 2, since the iteration of the gradient descent converges to the local minimum.

Comparing with the opening results with flat square SEs and the adapted SEs, the MSEs of the output results obtained by the adapted SEs are smaller than the results with the flat square SE. We see that the SEs that are adapted to the input images reflect the shape of the micro structures of the texture images. The adapted SEs well approximates the local structures of the images while eliminating impulsive noises for three texture image. Comparing with the results obtained by the median filter, the small details, which are preserved in the results of the adapted SE, are corrupted by the median filtering. The adaptation of the SE successfully decreases the errors of the denoising for all texture images while preserving the micro structures of the textures.

When the parameter σ exceeds the optimum value, the SEs shrink along with increment of σ. In the opening results with σ that is larger than the optimum value, we see the small noise components that is approximated with the adapted SE due to the overfitting to the noisy images.

6. CONCLUSIONS

In this paper, we propose an adaptation method for the SE of the opening filter. In the proposed method, the adaptation is achieved by the minimization of the cost function that is defined as the sum of the fiderity term and the smooth penalty for the image, as well as the regularization for linear inverse problems. The regularization

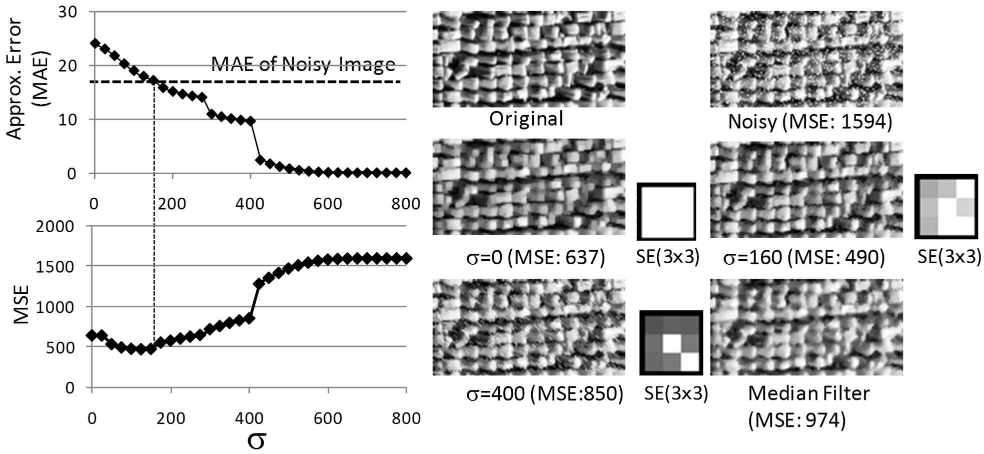

Figure 3: Denoising of the texture D84. Upper left: Relationship between the approximation error and the tolerable deviation σ. Lower left: Relationship between MSE of opened image and σ. Right: Original, noisy and opened images. SEs that obtains opened images are also shown.

penalty, which is defined to represent the smoothness of images, is imposed on the SE of the opening. In order to minimize the objective function, the opening and its objective function are approximated as the differentiable functions. By using these approximations, the minimization task is achieved by the gradient descent. In experiments, we demonstrate that the proposed approach can adapt the SEs by using only noisy input images and improve the denoising capability of the opening.

The proposed approach can also be applied to the morphological closing, that is complementary operation of the morphological opening. The experiments that are demonstrated in Sect. 5 are limited in denoising for positive noises. The pair of the closing and the opening can be applied for noises that include both of the positive and the negative values. The sequence of erosion and dilation operations is referred to as an alternative sequential filter[3]. The adaptation of the SE for the alternative sequential filter is one of future topics. In many existing image processing methods based on the mathematical morphology, the SEs are specified as combinations of several simple SEs. The SEs that are adapted through the proposed approach will contribute to the improvement of the performance of the existing morphological image analysis. The applications of the adaptation of the SE, including the image analysis and image recovery are future topics.

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