Extension of Time-Difference-of-Arrival Self Calibration Solutions Using Robust Multilateration

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***Abstract*—Recent advances in robust self-calibration have made it possible to estimate microphone positions and at least partial sound source positions using ambient sound. However, there are limits on how well sound source paths can be recovered using state-of-the-art techniques. In this paper we develop and evaluate several techniques to extend partial and incomplete solutions. We present minimal solvers for sound source positioning using non-overlapping pairs of microphone positions and their respective time-difference measurements, and show how these new solvers can be used in a hypothesis and test setting. We also investigate techniques that exploit temporal smoothness of the sound source paths. The different techniques are evaluated on both real and synthetic data, and compared to several state-of-the-art techniques for time-difference-of-arrival multilateration.**

***Index Terms*—TDOA, multilateration, minimal problems, RANSAC, self-calibration**

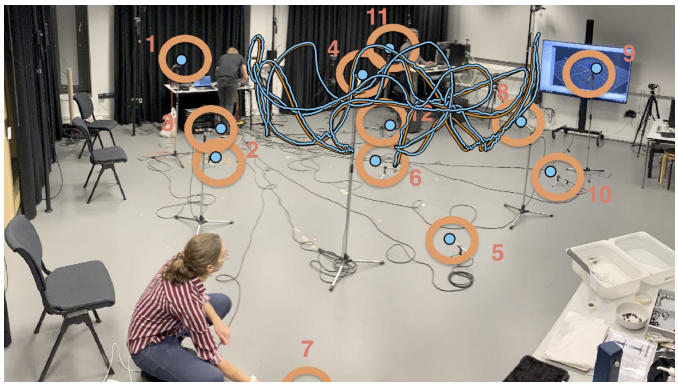

Fig. 1. A setup consisting of 12 omni-directional microphones used to collect TDOA measurements from a moving sound source. In the figure is shown 3D reconstructions (light blue) and ground truth (orange) of sound source path.

I. Introduction

Precise localization of sender/receiver node positions using radio or sound signals is a key enabler in numerous applications such as microphone array calibration, speaker diarization, beamforming, radio antenna array calibration, mapping and positioning [1]. In this paper we study the problem of self-calibration of sender/receiver positions using time-difference-of-arrival (TDOA) measurements from a set of fixed and synchronised microphones. The problem is simpler if the sound source has distinct sound events, which are easy to detect [2], or if the sound profile is known [3]. Recent advances in robust parameter estimation has made it possible to solve such problems even for the relatively difficult scenario of unknown ambient sound [4]–[8]. In many cases it is possible to achieve at least partial estimates of sound source positions and microphone positions. However, these methods typically do not provide good estimates of sound source positions for all time instants, at least not for difficult situations.

In this paper we develop improved robust multilateration methods and show how such methods can improve on sender/receiver node position calibration systems. While focusing on the problem of multilateration, we envision that the proposed method works as a part of a larger self-calibration system, in order to increase robustness. For this reason, our experiments are focused on this scenario.

The contributions of this paper are (i) new datasets for robust TDOA multilateration[1], (ii) a new fast solver for the minimal problem of TDOA multilateration, (iii) new methods for robust TDOA multilateration and (iv) evaluation of state-of-the-art methods for robust TDOA multilateration.

II. Structure from sound pipeline

For the solution of the structure from sound problem, we use the following structure, inspired by [5]. The input to the system is a number m of synchronised sound recordings (Fig. 2a). For each pair (i, j) of recordings we use a detector to generate a set of putative time-difference-of-arrival measurements z for a number of time instants $t_k, k = 1, \ldots, n$ (Fig. 2b). After a heuristic step for removing outliers (Fig. 2c), the data is used as input to a system for robust structure from sound (Fig. 2d). When successful, the system outputs all or a subset of the microphone positions, but often only a subset of the sound source positions. In this paper we study methods for extending an initial solution to additional sound source positions (Fig. 2e). The idea is that the set of putative matches (Fig. 2b), contains valuable information that could be better exploited using the sound source positions in (Fig. 2d).

This work was partially supported by the strategic research projects ELLIIT and eSSENCE, the Swedish Foundation for Strategic Research project, Semantic Mapping and Visual Navigation for Smart Robots (grant no. RIT15-0038) and Wallenberg Artificial Intelligence, Autonomous Systems and Software Program (WASP) funded by Knut and Alice Wallenberg Foundation. The authors gratefully acknowledge Lund University Humanities Lab.

[1]We provide both dataset https://vision.maths.lth.se/sfsdb/ and code https://github.com/kalleastrom/StructureFromSound.

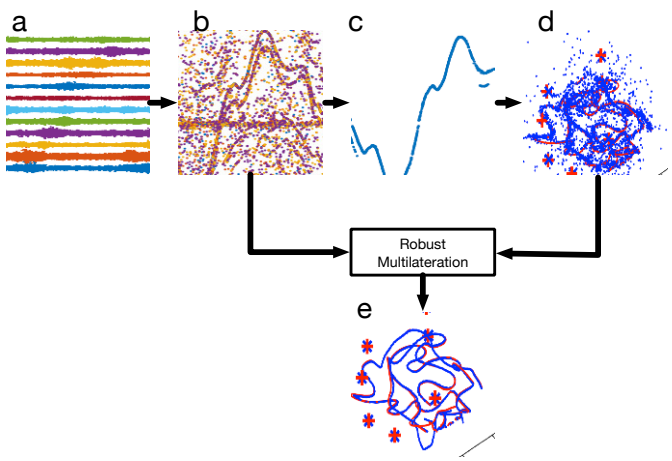

Fig. 2. System overview: The input consists of a number of sound recordings (a). Using GCC-PHAT, a number of putative TDOA estimates for each time instant and for each pair of microphones are produced (b). Heuristics are used to prune these matches (c), which then are used to estimate microphone and source positions (d). In this paper we study methods for improved sound source localization using microphone positions from (d) and putative matches from (b). The aim is to achieve robust sound source localization (e).

III. Multilateration Methods

Using sound to measure distances has been exploited for a long time, see for example [9]. Multilateration of sound source positions from a set of known microphones has been utilized, at least since World War I, to locate the source of artillery fire using sound waves [10].

For the multilateration problem, we assume that the microphone positions $(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_m)$ are known. As an example, these microphone posiitons could have been estimated using a self-calibration system, e.g., [8]. At each time instant t_k we estimate time-differences τ_{ij} of the arrival of sound to the two microphones $\boldsymbol{r}_i$ and $\boldsymbol{r}_j$. When multiplied with the speed of sound c, each such time-difference τ_{ij} gives a distance-difference estimate

$$z_{ij} = \tau_{ij} c \approx \|\boldsymbol{r}_i - \boldsymbol{s}\| - \|\boldsymbol{r}_j - \boldsymbol{s}\| + \epsilon, \tag{1}$$

where $\boldsymbol{s}$ is the unknown sound source position and $\|\cdot\|$ denotes the ℓ^2-norm. The noise ϵ is either an *inlier*, assumed to be normally distributed with a relatively small standard deviation, or an *outlier*, assumed to be drawn from a uniform distribution with a significantly larger standard deviation. Henceforth, we will use the term TDOA (time-difference-of-arrival) for the measurements z_{ij}, even though they actually represent distances and not time. Early algorithms were constructed for solving for $\boldsymbol{s}$ in (1), often assuming a planar geometry, and further assuming that the TDOA measurements are outlier free and without missing data. Such algorithms were often iterative and assumed that an initial guess of $\boldsymbol{s}$ was given.

In this paper we assume that we have a pool of hypotheses for the measurements z_{ij}. Each measurement is a collection of tuples $M = \begin{pmatrix} i & j & z_{ij} \end{pmatrix}$. The pool $\mathcal{P}$ consists of all of these putative measurements $\mathcal{P} = \{M_1, \ldots, M_N\}$, where several measurements could be to the same (i, j) combination.

In the experiments the measurements $\mathcal{P}$ were obtained by taking the top $K = 4$ peaks in the GCC-PHAT score [11], for each microphone pair (i, j). In [12], a system was proposed that uses a few top peaks in the GCC-PHAT score and tracks through time using continuity constraint using the Viterbi algorithm. Another method was proposed in [5], where RANSAC together with continuity constraints was used to track peaks over time. Collecting distance-difference measurements in an $m \times m$ matrix

$$\boldsymbol{Z} = \begin{pmatrix} z_{11} & \cdots & z_{1m} \\ \vdots & \ddots & \vdots \\ z_{m1} & \cdots & z_{mm} \end{pmatrix} \tag{2}$$

we obtain a *TDOA matrix*. In the suggested computational pipeline, we can view the step in Fig. 2b as having a TDOA matrix at each time instant. Again, note that for each element of these matrices there are several putative entries.

The true TDOA matrix is at most rank 2, [13], and can be written as

$$\boldsymbol{Z} = \boldsymbol{v}\boldsymbol{1}^T - \boldsymbol{1}\boldsymbol{v}^T, \tag{3}$$

where

$$\boldsymbol{v} = \begin{pmatrix} v_1 & \cdots & v_m \end{pmatrix}^T \tag{4}$$

is a vector of distance-differences, which will be called a *TDOA vector*. Adding a constant to $\boldsymbol{v}$ will not change the matrix $\boldsymbol{Z}$. Notice that each column of the TDOA matrix could be used as a TDOA vector, if they are outlier-free and without missing data. In the TDOA vector formulation, the measurement equation is

$$v_i = \|\boldsymbol{r}_i - \boldsymbol{s}\| + o, \tag{5}$$

where the unknown o can be interpreted as the unknown offset of the TDOA vector as discussed above. Alternatively, it can be interpreted as the unknown starting point of the sound. If the vector is obtained by measuring time-differences to a fixed microphone, e.g. $\boldsymbol{r}_1$, then we have $v_i = z_{i1} = \|\boldsymbol{r}_i - \boldsymbol{s}\| + o$ with $o = -\|\boldsymbol{r}_1 - \boldsymbol{s}\|$.

A common trick is to use four or more equations of type (5) to derive three or more equations of the form

$$(v_i - o)^2 - (v_1 - o)^2 = \boldsymbol{r}_i^T \boldsymbol{r}_i - \boldsymbol{r}_1^T \boldsymbol{r}_1 - (\boldsymbol{r}_i - \boldsymbol{r}_1)^T \boldsymbol{s}, \tag{6}$$

where two equations of type (5) are used for microphone i and 1. Note that the square terms $\boldsymbol{s}^T\boldsymbol{s}$ and o^2 disappear, and the constraints become linear in $\boldsymbol{s}$ and o.

In terms of the computational pipeline, we can view the step in Fig. 2c as having a TDOA vector at each time instant, although, possibly with missing data and outliers.

Several closed-form solutions exist for multilateration using the TDOA vector formulation and the elimination in (6), e.g., [14]–[18]. Thus, all of these methods assume that all time-differences are given to the same microphone. The minimal problem for the 3D case is to use four microphones and it has in general two solutions (counted with complex solutions and multiplicity of solutions). This can be seen as using three linear constraints of type (6) to reduce the four unknowns in $\boldsymbol{s}$ and

o. This parameterizes the solution affinely with one parameter. Inserting this into the first equation

$$v_1 = \|\boldsymbol{r}_1 - \boldsymbol{s}\| + o,$$

gives a quadratic constraint, which has at most two solutions.

An initial solution can be refined iteratively by minimizing

$$f_{\boldsymbol{v}}(\boldsymbol{s}, o) = \sum_{i=1}^{m} L(v_i - (\|\boldsymbol{r}_i - \boldsymbol{s}\| + o)), \tag{7}$$

for the TDOA vector formulation or

$$f_{\boldsymbol{Z}}(\boldsymbol{s}) = \sum_{i=1}^{m}\sum_{j=1}^{m} L(z_{ij} - (\|\boldsymbol{r}_i - \boldsymbol{s}\| - \|\boldsymbol{r}_j - \boldsymbol{s}\|)), \tag{8}$$

for the TDOA matrix formulation. Here L is a loss function, e.g., the ℓ^2-loss $L(x) = x^2$. Other common choices are the ℓ^1-loss $L(x) = |x|$ or a robust version such as the Huber loss or truncated versions of ℓ^1or ℓ^2. We will also assume that L removes datapoints that are missing or known to be outliers (e.g. from our proposed bootstrapping in Section V-A).

Building on previous results [19], [20], Velasco et al. used the redundancy of measurements in the TDOA matrix to perform denoising, detect outliers and fill in missing data [13]. The output from their approach is a TDOA vector, which can be used for trilateration using, e.g., [18]. Unlike the proposed method, [13] does not exploit the known microphone positions when denoising and allows for at most one TDOA measurement for each microphone pair. Additionally, the number of expected outliers is a nuisance parameter that must be specified prior to denoising.

IV. Minimal Solvers

The closed-form solution for determining $\boldsymbol{s}$ using TDOA measurements, as presented in previous papers, e.g., [14]–[18], all assume (for the 3D case) that four elements of the TDOA vector are given, or that four elements of the TDOA matrix from the same row (or column) are given. The trick that is used in (6) does not work for the minimal case of any three measurements of the TDOA matrix. Here we introduce a fast and numerically stable solver for this minimal case. We derive this for the general N-dimensional case although in practice we most often use it for 2D and 3D problems.

Let N be the dimension of the space, i.e., $\boldsymbol{s} \in \mathbb{R}^N$. Suitably rearranging and squaring (1) twice results in the quadratic constraint

$$\boldsymbol{s}^T \boldsymbol{A} \boldsymbol{s} + \boldsymbol{b}^T \boldsymbol{s} + c = 0, \tag{9}$$

where

$$\boldsymbol{A} = 4(\boldsymbol{r}_i - \boldsymbol{r}_j)(\boldsymbol{r}_i - \boldsymbol{r}_j)^T - 4z_{ij}^2 \boldsymbol{I}, \tag{10}$$

$$\boldsymbol{b} = 4z_{ij}^2(\boldsymbol{r}_i + \boldsymbol{r}_j) - 4(\boldsymbol{r}_i^T\boldsymbol{r}_i - \boldsymbol{r}_j^T\boldsymbol{r}_j)(\boldsymbol{r}_i - \boldsymbol{r}_j), \tag{11}$$

$$c = (\boldsymbol{r}_i^T\boldsymbol{r}_i - \boldsymbol{r}_j^T\boldsymbol{r}_j)^2 - 2(\boldsymbol{r}_i^T\boldsymbol{r}_i + \boldsymbol{r}_j^T\boldsymbol{r}_j)z_{ij}^2 + z_{ij}^4. \tag{12}$$

Constructing N quadratic combinations for different $(\boldsymbol{r}_i, \boldsymbol{r}_j, z_{ij})$ results in a polynomial system in $\boldsymbol{s}$. Using methods from algebraic geometry [21, p. 235] we conclude that there are at most four solutions for $N = 2$ and eight when $N = 3$. Some of the solutions may be complex and some may not satisfy (1) since we have lost the sign of z_{ij} due to the squaring. These solutions are however easily discarded. To produce a solver for the system we use an automatic solver generator [22]. Although there are dependencies between the polynomial coefficients, the problem does not admit smaller elimination template sizes (see [22]) than the case of independent coefficients (6×10 and 26×34 for $N = 2$ and $N = 3$, respectively).

An efficient method for solving three quadratics in three variables, corresponding to $N = 3$, was presented in [23]. There, the problem was reduced to a single univariate polynomial of degree eight whose real solutions were found using Sturm sequences [24]. We implemented their solver but found no clear improvement in execution time or numerical stability over our generated solver.

V. Robust Multilateration Algorithms

A. Proposed RANSAC scheme using minimal pairwise solver

We propose to use random sampling consensus (RANSAC) [25]. In the hypothesis and test loop we randomly choose three measurements from the pool of putative matches $\mathcal{P}$. From these three TDOA measurements we use the fast minimal solver to obtain hypotheses for the sound source position $\boldsymbol{s}$ and choose the one whose inlier set is maximal. Inliers are measurements for which

$$\big|z - \|\boldsymbol{r}_i - \boldsymbol{s}\| - \|\boldsymbol{r}_j - \boldsymbol{s}\|\big| < T, \tag{13}$$

where T is a threshold chosen to distinguish between inliers and outliers. This initial estimate is then improved by optimizing truncated ℓ^2-loss according to (8).

B. Using smoothness over time

In the experiments we also consider using smoothness priors on the sound source path. The solution (including the microphone positions) is refined by local minimization of

$$f_{reg}(\boldsymbol{s}, \boldsymbol{r}) = \sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{n} L\big(z_{ijk} - (\|\boldsymbol{r}_i - \boldsymbol{s}_k\| - \|\boldsymbol{r}_j - \boldsymbol{s}_k\|)\big) + \\ + \lambda \sum_{k=2}^{n-1} \|\boldsymbol{s}_{k-1} - 2\boldsymbol{s}_k + \boldsymbol{s}_{k+1}\|^2. \tag{14}$$

This requires a good initial estimate of the sound source path.

VI. Experimental Validation

A. Real data

We collected one dataset consisting of seven recordings with different songs and different sound source motion. The setup consisted of 12 omni-directional microphones (the T-bone MM-1) spanning a volume of $4.0 \times 4.6 \times 1.5$ meters (see Fig. 1). Ground truth positions for the microphones and speaker positions were found using a Qualisys motion capture system. The microphones were all internally synchronized, but we assume that the time of sound emission from the speaker is unknown. For each recording a song was played as the speaker

was moved around in the room and approximately one minute was recorded using a soundcard with sampling rate $96,000$ Hz. The temperature in the room was measured to be $20.1\,^{\circ}\mathrm{C}$ which indicates a speed of sound of $c = 343\,\mathrm{m/s}$.

For each pair of microphones, the GCC-PHAT score [11] was calculated. We used a window of $2,048$ samples centered at every $1,000$:th sample points. The search width for the GCC-PHAT score was cropped to ± 800 sample points. Thus we are able to find time-difference-of-arrival measurements corresponding to $\pm 2.85\,\mathrm{m}$ distance-difference to microphone pairs. For each time instant and each pair of microphones we selected at the four strongest local maxima in the GCC-PHAT score, resulting in a pool of putative measurements. In Fig. 2.b these are shown for microphone pair 6 och 8.

Thus, for each recording and for each time window we had time-difference-of-arrival measurements and ground truth microphone and sound source positions. In total there were 46,066 such examples to validate the algorithms on.

B. Simulated/real data

The real dataset is quite challenging. It contains outliers, missing data and multiple hypotheses. In order to understand the behaviour of the algorithms we also constructed simulated data. This was done using the ground truth positions of the microphones and the sound source for the seven datasets above. In this way we could make datasets that were similar in geometry, but for which there was less noise, less outliers and/or less missing data. Together with the real recordings, this resulted in the following four datasets:

(a) Simulated TDOA measurements. One hypothesis. Gaussian noise with $\sigma = 2$ sample points. No missing data. No outliers. Ground truth microphone positions.
(b) Simulated TDOA measurements. One hypothesis. Gaussian noise with $\sigma = 2$ sample points. Missing data: 20%. Outliers: 20%. Ground truth microphone positions.
(c) Real TDOA measurements. Four hypotheses. Inlier noise estimated to have $\sigma \approx 5$ sample points. Outliers: $\approx 86\%$. Ground truth microphone positions.
(d) Real TDOA measurements. Four hypotheses. Inlier noise estimated to have $\sigma \approx 5$ sample points. Outliers: $\approx 86\%$. Estimated microphone positions from a state-of-the-art self-calibration system, [8].

C. Evaluation of the multilateration methods

We first evaluate methods that only use one individual time instant. We tested two state-of-the-art routines. For both methods we initially, from the pool of putative matches $\mathcal{P}$, generate the TDOA matrix $\boldsymbol{Z}$ by selecting the measurement for each microphone pair for which the GCC-PHAT score is the strongest. For (i) Chan and Ho [18], we then use one of the microphones (no 6 in our experiment) to calculate the TDOA vector $\boldsymbol{v}$ from $\boldsymbol{Z}$. The 6th microphone was considered to be best for this purpose since it was in the centre of the room. Finally we estimate the sound source position $\boldsymbol{s}$ using microphone positions and the TDOA vector according to [18]. For the second method (ii) Velasco et al., we use [13] to robustly estimate the TDOA vector $\boldsymbol{v}$ from $\boldsymbol{Z}$. Finally, we estimate the sound source position $\boldsymbol{s}$ using microphone positions and the TDOA vector according to [18] as suggested in [13]. We also compared our method to four search based methods. These were (iii) ℓ^2-optimization from a random starting point, (iv) ℓ^1-optimization from a random starting point, (v) truncated ℓ^1-optimization from a random starting point and (vi) truncated ℓ^1-optimization from ten random starting points, choosing the solution with the lowest truncated ℓ^1-loss, as well as the proposed algorithm based on (vii) RANSAC loop to select starting point followed by truncated ℓ^2-optimization.

For each scenario above we calculated the percentage of times the estimated sound source came within 15 cm of the ground truth position. The results are shown in Fig. 3. Notice that most methods work well for the outlier free dataset (a), except the truncated ℓ^1-loss optimization with one single random starting point. This shows that finding a good starting point is critical for robust loss optimization. With more outliers (dataset (b)), we see that Chan and Ho and several other methods struggle to find a good solution. For the real data problem with ground truth microphone positions (dataset (c)) the proposed method clearly outperforms the other methods. The final (dataset (d)), is even more challenging, but the overall trend is the same.

D. Applying motion priors

We used the result from the different multilateration methods to optimize over the whole sound recording using the motion prior as described in Section V-B. This optimization needs a fairly good initial estimate in order to converge to the global optimum. The result is also shown in Fig. 3. In Table I we show the results for dataset (d), with a breakdown to the individual seven recordings in the dataset. As can be seen in the table, there are four songs for which no methods work well. This is clearly a result of having a poor estimate of the microphone positions, since the result from dataset (c) works significantly better, indicating a need for further research. Finally we visualize how the suggested improvements affect the reconstructed 3D path. In Fig. 4 we show the improvement to the 3D reconstruction with the proposed system for recording nr 6. It is clear that the proposed improvements (Fig. 4-right) reduce the noise and improves the estimation of the sound source path as compared to the current state-of-the-art self-calibration system (Fig. 4-left).

VII. Conclusions

In this paper we have made several improvements on robust multilateration using TDOA matrices with multiple hypotheses for each entry. We have developed a fast and efficient solver for the minimal problem of disjoint pairwise TDOA measurements. We have combined this solver with RANSAC algorithms and robust nonlinear estimation and obtained better results than state-of-the-art algorithms for robust multilateration. The resulting system has been tested on both synthetic

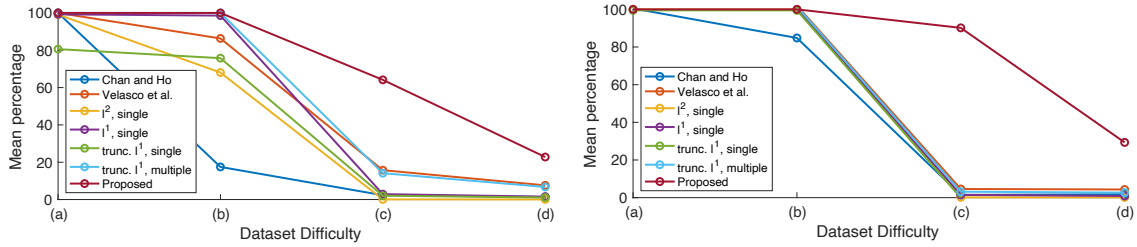

Fig. 3. Performance aggregated over all datasets, as a function of varying difficulty in experimental setup, without (left) and with (right) temporal smoothing. The performance is measured by computing how often the source position was estimated within 15 cm from the ground truth position.

TABLE I
THE RESULTS FOR THE SEVEN DIFFERENT RECORDINGS FROM DATASET (d) MEASURED AS THE PERCENTAGE OF TIMES THE SOUND SOURCE POSITION WAS ESTIMATED WITHIN 15 CM OF THE GROUND TRUTH POSITION.

Recording	1	2	3	4	5	6	7
Chan and Ho	0	0	0	1	0	4	4
Velasco et al.	0	0	0	3	0	23	28
ℓ^2, single	0	0	0	0	0	0	0
ℓ^1, single	0	0	0	0	0	4	6
trunc. ℓ^1, single	0	0	0	0	0	3	4
trunc. ℓ^1, multiple	0	0	0	1	0	20	25
Proposed	0	0	0	10	0	70	78

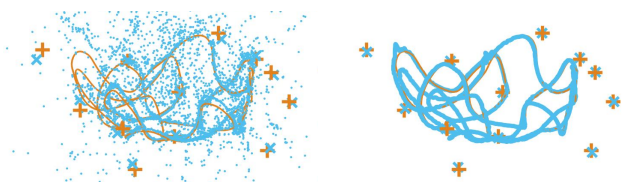

Fig. 4. 3D reconstruction (blue) and ground truth (orange) of sound source path from calibration system (left) and from the proposed method (right).

and real data, producing high quality solutions even in the presence of missing data and high amount of outliers.

REFERENCES

[1] A. Plinge, F. Jacob, R. Haeb-Umbach, and G. A. Fink, "Acoustic microphone geometry calibration: An overview and experimental evaluation of state-of-the-art algorithms," *IEEE Signal Processing Magazine*, vol. 33, no. 4, pp. 14–29, 2016.

[2] T. Janson, C. Schindelhauer, and J. Wendeberg, "Self-localization based on ambient signals," in *Proceedings of the 6th international conference on Algorithms for Sensor systems, Wireless Adhoc Networks, and Autonomous Mobile Entities*. Springer-Verlag, 2010, pp. 176–188.

[3] K. Batstone, G. Flood, T. Beleyur, V. Larsson, H. R. Goerlitz, M. Oskarsson, and K. Åström, "Robust self-calibration of constant offset time-difference-of-arrival," in *ICASSP 2019-2019 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*. IEEE, 2019, pp. 4410–4414.

[4] P. Pertila, M. Mieskolainen, and M. Hamalainen, "Passive self-localization of microphones using ambient sounds," in *Proc. European Signal Processing Conference (EUSIPCO)*. IEEE, 2012.

[5] Z. Simayijiang, S. Segerblom Rex, Y. Kuang, F. Andersson, and K. Åström, "An automatic system for acoustic microphone geometry calibration based on minimal solvers," *Arxiv*, 10 2016.

[6] T. K. Le and N. Ono, "Robust TDOA-based joint source and microphone localization in a reverberant environment using medians of acceptable recovered TOAS," in *2016 International Workshop on Acoustic Signal Enhancement, IWAENC 2016*. Institute of Electrical and Electronics Engineers Inc., 10 2016.

[7] M. Larsson, G. Flood, M. Oskarsson, and K. Åström, "Upgrade methods for stratified sensor network self-calibration," in *ICASSP 2020-2020 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*. IEEE, 2020, pp. 4851–4855.

[8] M. Larsson, G. Flood, M. Oskarsson, and K. Åström, "Fast and robust stratified self-calibration using time-difference-of-arrival measurements," in *ICASSP, IEEE International Conference on Acoustics, Speech and Signal Processing - Proceedings*. IEEE, 2021.

[9] J. Meldercreutz, "Om längders mätning genom dåns tilhielp," *Vetenskapsakademiens Handlingar*, vol. 2, pp. 73–77, 1741.

[10] A. P. Bogatskiy, F. S. Kuznetsov, and A. F. Shapovalov, Словарь ракетных и артиллерийских терминов. Voyennoye, Moscow, 1968.

[11] C. Knapp and G. Carter, "The generalized correlation method for estimation of time delay," *Acoustics, Speech and Signal Processing, IEEE Transactions on*, vol. 24, no. 4, pp. 320 – 327, aug 1976.

[12] X. Anguera, C. Wooters, and J. Hernando, "Acoustic beamforming for speaker diarization of meetings," *IEEE Transactions on Audio, Speech and Language Processing*, vol. 15, pp. 2011–2022, 9 2007.

[13] J. Velasco, D. Pizarro, J. Macias-Guarasa, and A. Asaei, "TDOA matrices: Algebraic properties and their application to robust denoising with missing data," *IEEE Transactions on Signal Processing*, vol. 64, pp. 5242–5254, 10 2016.

[14] W. Hahn and S. Tretter, "Optimum processing for delay-vector estimation in passive signal arrays," *IEEE Transactions on Information Theory*, vol. 19, no. 5, pp. 608–614, 1973.

[15] W. R. Hahn, "Optimum signal processing for passive sonar range and bearing estimation," *The Journal of the Acoustical Society of America*, vol. 58, no. 1, pp. 201–207, 1975.

[16] S. Bancroft, "An algebraic solution of the GPS equations," *IEEE transactions on Aerospace and Electronic Systems*, vol. 21, no. 7, pp. 56–59, 1985.

[17] B. T. Fang, "Simple solutions for hyperbolic and related position fixes," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 26, pp. 748–753, 1990.

[18] Y. T. Chan and K. C. Ho, "A simple and efficient estimator for hyperbolic location," *IEEE Transactions on Signal Processing*, vol. 42, pp. 1905–1915, 1994.

[19] H. C. So, Y. T. Chan, and F. K. W. Chan, "Closed-form formulae for time-difference-of-arrival estimation," *IEEE Transactions on Signal Processing*, vol. 56, no. 6, pp. 2614–2620, 2008.

[20] R. Schmidt, "Least squares range difference location," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 32, no. 1, pp. 234–242, 1996.

[21] D. Cox, J. Little, and D. O'Shea, *Ideals, Varieties, and Algorithms*. Springer Verlag, 2007.

[22] V. Larsson, K. Åström, and M. Oskarsson, "Efficient solvers for minimal problems by syzygy-based reduction," in *IEEE Conference on Computer Vision and Pattern Recognition (CVPR)*, 2017, pp. 2383–2392.

[23] Z. Kukelova, J. Heller, and A. Fitzgibbon, "Efficient intersection of three quadrics and applications in computer vision," in *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition*, 2016, pp. 1799–1808.

[24] C. F. Sturm, "Résolution des équations algébriques," *Bulletin de Férussac*, vol. 11, pp. 419–425, 1829.

[25] M. A. Fischler and R. C. Bolles, "Random sample consensus: a paradigm for model fitting with applications to image analysis and automated cartography," *Communications of the ACM*, vol. 24, no. 6, pp. 381–95, 1981.