

Error analysis of a demodulation procedure for multicarrier signals with slowly-varying carriers

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Abstract—We propose a procedure to demodulate analog signals encoded by a multicarrier modulator, with slowly-varying carrier shapes. We prove that the asymptotic demodulation error can be made arbitrarily small. The intended application is the “sensorless” control of AC electric motors at or near standstill, through the decoding of the PWM-induced current ripple.

Index Terms—analog demodulation, multicarrier signals, slowly-varying carriers, multiple access methods, PWM injection

I. INTRODUCTION

We consider a composite signal y of the form

$$y(t) := \sum_{i=1}^n z_i(t) s_i(t, \frac{t}{\varepsilon}) + d(t, \frac{t}{\varepsilon}) + O(\varepsilon^k), \quad (1)$$

where the s_i ’s are (known) 1-periodic functions in the second variable; ε being a (known) “small” parameter, the s_i ’s can be seen as rapidly oscillating carriers with slowly varying shapes modulating the (unknown) z_i ’s. The function d is a disturbance, also 1-periodic in the second variable, about which little is known except that for each t the support of $d(t, \cdot)$ is contained in a “well-behaved” known subset D_t of $[0, 1]$. In other words, on each period of the carriers, part of the signal y is garbled and considered useless. Finally, the $O(\varepsilon^k)$ term corresponds to “small” disturbances, where O denotes the (uniform) “big O” symbol of analysis, i.e. $f(t, \varepsilon) = O(\varepsilon^k)$ if $\|f(t, \varepsilon)\| \leq K\varepsilon^k$ for some K independent of t and ε .

The objective is to recover by an implementable causal process the unknown z_i ’s with an accuracy of up to $O(\varepsilon^k)$ from the known y and s_i ’s, provided the s_i ’s and z_i ’s satisfy some suitable regularity assumptions.

The motivation for this problem is the following. When operating an AC electric motor through a PWM inverter with period ε , an analysis based on the theory of averaging reveals that the currents in the motor have the form

$$y(t) = y_a(t) + \varepsilon y_v(t) s(t, \frac{t}{\varepsilon}) + O(\varepsilon^2) + d(t, \frac{t}{\varepsilon}),$$

which is a particular instance of (1) with $z_1 := y_a$, $z_2 := \varepsilon y_v$, $s_1 := 1$, and $s_2 := s$, where s is determined by the PWM process [1], [2]. The $O(\varepsilon^2)$ term corresponds to a small higher-order ripple which can be ignored. The disturbance d consists of short spikes appearing at each PWM commutation, due to stray capacitances in the power electronics. A typical (synthetic) signal y is shown in Fig. 4, see also [2, Fig. 9]

for experimental data. In “sensorless” industrial drives, these currents are the only measurements, and controlling the motor at or near standstill with this sole information is a difficult problem for several theoretical and technological reasons. A way to achieve this is to extract y_a and εy_v from the modulated currents y ; a suitable processing of y_v then gives access to the motor angular position [2], which is instrumental in controlling the motor. It is therefore very important to ensure the demodulation error is at most $O(\varepsilon^2)$.

The demodulation procedure proposed in this paper, essentially consisting of multiplications by known signals followed by low-pass filters, is reminiscent of various schemes in communication theory and signal processing. Nevertheless, nothing really close seems to exist in the literature, let alone a quantitative analysis of the demodulation error:

- it is of course a generalization of coherent demodulation in quadrature carrier multiplexing, with more than two carriers not restricted to sine and cosine, see e.g. [3, section 4.4]; but even in this simple case, no analysis of the error is usually performed, the challenges being more on carrier reconstruction
- it somewhat looks like synchronous decorrelating detection in Code-Division Multiple Access communication systems, where the s_i ’s would play the roles of the signature waveforms and the z_i ’s the role of the symbols, see e.g. [4, section 5.1]; but the encoded signals being there digital, the issues and analysis are very different
- it is also akin to multicarrier reception, with or without multiple access, see e.g. [5, section 12.2] and [6, section 2.2]; but once again that field is exclusively concerned with digital encoded signals
- finally, it bears some resemblance for its filtering part with the interpolation/compensation filters used in $\Delta\Sigma$ analog-to-digital converters, see e.g. [7, chapter 14].

The paper extends the previous work [8] in two ways that are paramount for the intended application: on the one hand, it considers carriers with slowly-varying shapes, which makes the error analysis much more difficult; on the other hand, the procedure is not restricted to “orthogonal” demodulation, hence can directly handle the disturbance d without ad-hoc prefiltering as in [2].

The paper runs as follows: in section II, we collect notations

and definitions, in particular the \mathcal{A}_k regularity property; in section III we state and prove the main result; in section IV we illustrate this result and confirm the error estimates with numerical experiments.

II. NOTATIONS AND DEFINITIONS

We collect here definitions used throughout the paper. The most important notion is the \mathcal{A}_k regularity property introduced in proposition 1, which is needed in lemma 4 to repeatedly integrate by parts; this property, which is paramount for handling carriers with slowly-varying shapes, is trivially satisfied for fixed-shape carriers as in [8].

Let $g(t, \sigma)$ be a function of two variables; informally speaking, t represents the slow timescale and σ the fast timescale. We will often use the convenient notation $g_\varepsilon(t) := g(t, \frac{t}{\varepsilon})$.

The function g is 1-periodic in the second variable if $g(t, \sigma + 1) = g(t, \sigma)$ for all t . Its mean in the second variable is the function $\bar{g}(t) := \int_0^1 g(t, \sigma) d\sigma$. For brevity, we will usually omit the phrase “in the second variable”. If g is 1-periodic with zero mean, any of its primitives (in the second variable) is also 1-periodic, in particular its zero-mean primitive $g^{(-1)}(t, \sigma) := \int_0^\sigma g(t, \tau) d\tau - \int_0^1 \int_0^\zeta g(t, \tau) d\tau d\zeta$. Likewise, $g^{(-k-1)}$ denotes the zero-mean primitive of $g^{(-k)}$.

We say g is Lipschitz if $\|g(t_1, \sigma) - g(t_2, \sigma)\| \leq L\|t_1 - t_2\|$ for some L independent of t_1, t_2 and σ .

Finally, we introduce the \mathcal{A}_k regularity property.

Definition 1 (\mathcal{A}_k property). *Let $g(t, \sigma)$ be 1-periodic with zero mean. It is said to be \mathcal{A}_k , $k \geq 1$, if $g^{(-k)}$ is $k-1$ times differentiable in the first variable, with bounded derivatives at all orders, and $\partial_1^{k-1} g^{(-k)}$ Lipschitz.*

A typical \mathcal{A}_k function encountered in practice is $g(t, \sigma) = \text{sign}(u(t) - \sigma) - 2u(t) + 1$ where $\sigma := \sigma \bmod 1$; $u(t) \in (0, 1)$ represents the PWM duty cycle and is assumed $k-1$ times differentiable, with bounded derivatives at all orders, and $u^{(k-1)}$ Lipschitz.

It is easy to show that if on the one hand $g(t, \sigma)$ is \mathcal{A}_k , and on the other hand $z(t)$ is $k-1$ times differentiable, with bounded derivatives at all orders, and $z^{(k-1)}$ Lipschitz, then the product zg is also \mathcal{A}_k .

III. THE DEMODULATION PROCEDURE

The demodulation procedure for an error of order ε^k consists of multiplications by a suitable demodulating basis $R := (r_1, \dots, r_n)^T$, followed by a bank of low-pass finite impulse response filters with kernel \tilde{K}_k ; see section III-A for a discussion of how to select R . The kernel \tilde{K}_k is a “compensated” k -times iterated moving average, namely a suitable linear combination of shifted instances of K_k , where the kernel K_k is defined recursively by $K_1 := \frac{1}{\varepsilon} \mathbb{1}_{[0, \varepsilon]}$ and $K_k := K_{k-1} * K$, see e.g. [9, chapter 6.7] for explicit expressions. For instance for $k = 3$, the linear combination is

$$\tilde{K}_3(t) := \frac{17}{4} K_3(t) - 5K_3(t - \varepsilon) + \frac{7}{4} K_3(t - 2\varepsilon),$$

see section III-B for more details.

Fig. 1 illustrates the whole demodulation procedure:

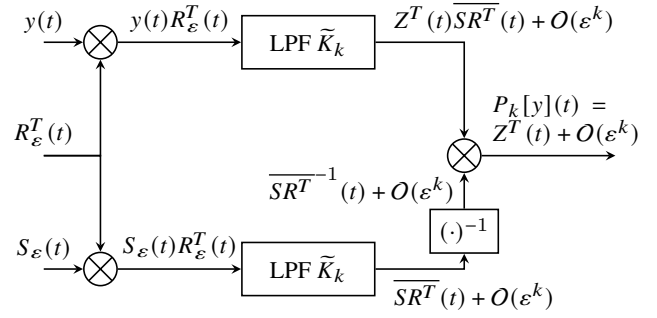


Figure 1. The demodulation procedure.

- $y(t)$ is multiplied by $R_\varepsilon^T(t)$, and filtered by \tilde{K}_k ; the result, $\left(\tilde{K}_k * (yR_\varepsilon^T)\right)(t)$, turns out to be $Z^T(t)\bar{S}^T(t) + O(\varepsilon^k)$, where $Z := (z_1, \dots, z_n)^T$ is the vector signal to recover
- the modulating basis $S := (s_1, \dots, s_n)^T$ is also multiplied by $R_\varepsilon^T(t)$, and filtered by \tilde{K}_k ; the result, $\left(\tilde{K}_k * (S_\varepsilon R_\varepsilon^T)\right)(t)$, turns out to be $\bar{S}^T(t) + O(\varepsilon^k)$
- finally, $\left(\tilde{K}_k * (yR_\varepsilon^T)\right)(t)$ is multiplied by the inverse of the matrix $\left(\tilde{K}_k * (S_\varepsilon R_\varepsilon^T)\right)(t)$; the result, $\left(\tilde{K}_k * (yR_\varepsilon^T)\right)(t) \times \left(\tilde{K}_k * (S_\varepsilon R_\varepsilon^T)\right)^{-1}(t)$, is as desired $Z^T(t) + O(\varepsilon^k)$.

As pointed out in the introduction, this demodulation scheme is at first sight not completely surprising. What is much less obvious is that the overall demodulation error is indeed of order ε^k .

A. Main result

We assume that the s_i 's are independent outside the subset D_t containing the support of the disturbance $d(t, \cdot)$, i.e. that the \check{s}_i 's defined by $\check{s}_i(t, \sigma) := (1 - \mathbb{1}_{D_t})(\sigma)s_i(t, \sigma)$ are linearly independent. We can thus choose the demodulating basis $R := (r_1, \dots, r_n)^T$ such that $Rd = 0$ and \bar{S}^T is invertible, where $S := (s_1, \dots, s_n)^T$ is the modulating basis; one simple choice is for instance $R(t, \sigma) := (1 - \mathbb{1}_{D_t}(\sigma))S(t, \sigma)$. A delicate point is to select R also such that $SR^T - \bar{S}^T$ is \mathcal{A}_k , provided of course that D_t is “well-behaved” (for instance a finite union of intervals with sufficiently regular moving bounds). For simplicity, we just assume this is the case (and check it a posteriori in the numerical experiments of section IV). Finally, we assume the z_i 's are $k-1$ times differentiable, with bounded derivatives at all orders, and $z_i^{(k-1)}$ Lipschitz, so that $z_i(SR^T - \bar{S}^T)$ is also \mathcal{A}_k .

Theorem 1. $Z := (z_1, \dots, z_n)^T$ can be recovered to order ε^k from y by the causal process P_k defined by

$$P_k[y](t) := \left(\tilde{K}_k * (yR_\varepsilon^T)\right)(t) \times \left(\tilde{K}_k * (S_\varepsilon R_\varepsilon^T)\right)^{-1}(t).$$

In other words, $Z^T(t) = P_k[y](t) + O(\varepsilon^k)$.

B. Proof of theorem 1

Rewriting (1) as

$$y(t) = Z^T(t)S(t, \frac{t}{\varepsilon}) + d(t, \frac{t}{\varepsilon}) + O(\varepsilon^k),$$

right-multiplying by R_ε^T and convolving with K_k yields

$$\begin{aligned} (K_k * (yR_\varepsilon^T))(t) &= (K_k * (Z^T S_\varepsilon R_\varepsilon^T))(t) + O(\varepsilon^k) \\ &= \left[K_k * \left(Z^T (S_\varepsilon R_\varepsilon^T - \overline{SR^T}) \right) \right](t) \\ &\quad + (K_k * (Z^T \overline{SR^T}))(t) + O(\varepsilon^k) \\ &= (K_k * (Z^T \overline{SR^T}))(t) + O(\varepsilon^k); \end{aligned}$$

to obtain the last line, we have applied Lemma 4 with $g(t, \sigma) := Z^T(t)(S(t, \sigma)R^T(t, \sigma) - \overline{SR^T}(t))$, which is by construction zero-mean and \mathcal{A}_k . The result obviously holds also if $K_k(t)$ is replaced by the shifted kernel $\tau_T K_k(t) := K_k(t - T)$.

On the other hand, [8, Theorem 1] asserts that a C^k -function φ with bounded $\varphi^{(k)}$ is left unchanged to order ε^k by a suitable linear combination \tilde{K}_k of the shifted kernels $\tau_{i\varepsilon} K_k$, $i = 0, \dots, k-1$, i.e. $(\tilde{K}_k * \varphi)(t) = \varphi(t) + O(\varepsilon^k)$. For instance,

$$\begin{aligned} \tilde{K}_1(t) &:= K_1(t) \\ \tilde{K}_2(t) &:= 2K_2(t) - K_2(t - \varepsilon), \\ \tilde{K}_3(t) &:= \frac{17}{4}K_3(t) - 5K_3(t - \varepsilon) + \frac{7}{4}K_3(t - 2\varepsilon). \end{aligned}$$

Actually, we must slightly extend the result to the case where φ is $k-1$ times differentiable with $\varphi^{(k-1)}$ Lipschitz, which we omit by lack of space. As a consequence,

$$\begin{aligned} (\tilde{K}_k * (yR_\varepsilon^T))(t) &= (\tilde{K}_k * (Z^T \overline{SR^T}))(t) + O(\varepsilon^k) \\ &= Z^T(t) \overline{SR^T}(t) + O(\varepsilon^k). \end{aligned}$$

Since $\overline{SR^T}(t)$ is invertible, $Z(t)$ can be recovered to order ε^k .

To make the process truly implementable in practice, notice $\overline{SR^T}(t)$ can be computed to order ε^k by

$$(\tilde{K}_k * (S_\varepsilon R_\varepsilon^T))(t) = \overline{SR^T}(t) + O(\varepsilon^k),$$

which is an instance of the previous equation with $Z(t) = I_n$.

In conclusion, $Z(t)$ is recovered to order ε^k by

$$\begin{aligned} P_k[y](t) &:= (\tilde{K}_k * (yR_\varepsilon^T))(t) \times (\tilde{K}_k * (S_\varepsilon R_\varepsilon^T))^{-1}(t) \\ &= Z^T(t) + O(\varepsilon^k), \end{aligned}$$

where the process P_k is causal since the kernel \tilde{K}_k is supported on $[0, k\varepsilon] \subset \mathbb{R}^+$.

C. Technical lemmas

This section is quite technical and can be skipped without disturbing the flow of ideas. Its goal is to establish Lemma 4, which is instrumental in the proof of Theorem 1. Lemma 4 relies on Lemma 3, which itself relies on Lemma 2. Lemmas 4 and 3 are in some sense properties of the convolution kernel K_k , whereas Lemma 2 extends to our context a classical result of finite-differences calculus. Notice the use of the \mathcal{A}_k property when integrating by parts in Lemma 4, which is the main trick to extend the ideas of [8] to slowly-moving carriers.

Define the k^{th} -order backward difference $\Delta_k g_\varepsilon$ of the function $g_\varepsilon(t) := g(t, \frac{t}{\varepsilon})$ by

$$(\Delta_k g_\varepsilon)(t) := \sum_{i=0}^k (-1)^i \binom{k}{i} g_\varepsilon(t - i\varepsilon).$$

On the other hand, recall that K_k is $k-1$ times differentiable, with compact support for all the derivatives. As for $K_k^{(k)}$, it can be defined in the distributional sense, and is a linear combination of Dirac delta functions, and in particular also has compact support; for instance, $K_1^{(1)} = \frac{1}{\varepsilon}(\delta_0 - \delta_\varepsilon)$.

Lemma 2. *Let $g(t, \sigma)$ be 1-periodic, and $k-1$ times differentiable in the first variable with $\partial_1^{(k-1)} g$ Lipschitz. Then $(\Delta_k g_\varepsilon)(t) = O(\varepsilon^k)$.*

Proof. By the Lipschitz form of Taylor's formula [10, (2.1)],

$$g(t + \mu, \sigma) = \sum_{j=0}^{k-1} \frac{\mu^j}{j!} \partial_1^j g(t, \sigma) + \mu^k \rho_t(\mu, \sigma),$$

where the remainder ρ_t is $O(1)$ since it satisfies

$$\begin{aligned} \mu \rho_t(\mu, \sigma) &= \frac{1}{(k-2)!} \int_0^1 (1-\tau)^{k-2} \\ &\quad \times (\partial_1^{k-1} g(t + \mu\tau, \sigma) - \partial_1^{k-1} g(t, \sigma)) d\tau. \end{aligned}$$

Applying this to $g(t - i\varepsilon, \frac{t-i\varepsilon}{\varepsilon}) = g(t - i\varepsilon, \frac{t}{\varepsilon})$ since g is 1-periodic yields

$$\begin{aligned} (\Delta_k g_\varepsilon)(t) &= \sum_{i=0}^k (-1)^i \binom{k}{i} g(t - i\varepsilon, \frac{t-i\varepsilon}{\varepsilon}) \\ &= \sum_{i=0}^k (-1)^i \binom{k}{i} \left(\sum_{j=0}^{k-1} \frac{(-i\varepsilon)^j}{j!} \partial_1^j g(t, \frac{t}{\varepsilon}) + O(\varepsilon^k) \right) \\ &= \sum_{j=0}^{k-1} \frac{(-\varepsilon)^j}{j!} \partial_1^j g(t, \frac{t}{\varepsilon}) \sum_{i=0}^k (-1)^i \binom{k}{i} i^j + O(\varepsilon^k) \end{aligned}$$

As $\sum_{i=0}^k (-1)^i \binom{k}{i} i^j = 0$, see [11, Cor. 2], this gives the desired result. \square

Lemma 3. *Let $g(t, \sigma)$ be 1-periodic, and $k-1$ times differentiable in the first variable with $\partial_1^{(k-1)} g$ Lipschitz. Then $(K_k^{(k)} * g_\varepsilon)(t) = O(1)$.*

Proof. We first prove by induction that $K_k^{(k)} * g_\varepsilon = \frac{1}{\varepsilon^k} \Delta_k g_\varepsilon$. Indeed, $K_1' * g_\varepsilon = \frac{1}{\varepsilon}(\delta_0 - \delta_\varepsilon) * g_\varepsilon = \frac{1}{\varepsilon} \Delta_1 g_\varepsilon$. Assuming the property holds at rank k ,

$$\begin{aligned} K_{k+1}^{(k+1)} * g_\varepsilon &= (K_k * K_1)^{(k+1)} * g_\varepsilon \\ &= K_k^{(k)} * K_1' * g_\varepsilon \\ &= \frac{1}{\varepsilon^k} \Delta_k (K_1' * g_\varepsilon) \\ &= \frac{1}{\varepsilon^k} \Delta_k \left(\frac{\Delta_1 g_\varepsilon}{\varepsilon} \right) \\ &= \frac{1}{\varepsilon^{k+1}} \Delta_{k+1} g_\varepsilon. \end{aligned}$$

To obtain the second line, we have repeatedly used $(T * S)' = T' * S = T * S'$.

Applying Lemma 2, we eventually find $(K_k^{(k)} * g_\varepsilon)(t) = \frac{1}{\varepsilon^k} \Delta_k g_\varepsilon(t) = \frac{1}{\varepsilon^k} O(\varepsilon^k) = O(1)$. \square

Lemma 4. *Let g be \mathcal{A}_k . Then $(K_k * g_\varepsilon)(t) = O(\varepsilon^k)$*

Proof. We sketch the proof for $k = 3$, the general result following by induction. Notice that by assumption $g^{(-3)}$ is twice differentiable in the first variable with $\partial_1^2 g^{(-3)}$ Lipschitz, which will be used each time Lemma 3 is invoked.

We first prove $(K_3'' * (g^{(-2)})_\varepsilon)(t) = O(\varepsilon)$. Starting from

$$\begin{aligned} \left((g^{(-3)})_\varepsilon \right)' &= (\partial_1 g^{(-3)})_\varepsilon + \frac{1}{\varepsilon} (\partial_2 g^{(-3)})_\varepsilon \\ &= (\partial_1 g^{(-3)})_\varepsilon + \frac{1}{\varepsilon} (g^{(-2)})_\varepsilon, \end{aligned}$$

we find after convolving with K_3'' and integrating by parts

$$K_3'' * (g^{(-2)})_\varepsilon = \varepsilon K_3''' * (g^{(-3)})_\varepsilon - \varepsilon K_3'' * (\partial_1 g^{(-3)})_\varepsilon;$$

the boundary terms vanish since K_3'' has compact support. The first term is $O(\varepsilon)$ by Lemma 3. Using $K_3'' = (K_1 * K_2)'' = K_1 * K_2''$, the second term reads $\varepsilon K_1 * (K_2'' * (\partial_1 g^{(-3)})_\varepsilon)$, hence is also $O(\varepsilon)$ by Lemma 3. The sum of the two terms is therefore also $O(\varepsilon)$.

We next prove $(K_3' * (g^{(-1)})_\varepsilon)(t) = O(\varepsilon^2)$. Indeed, using successively

$$\begin{aligned} (g^{(-1)})_\varepsilon &= \varepsilon (g^{(-2)})'_\varepsilon - \varepsilon (\partial_1 g^{(-2)})_\varepsilon \\ (\partial_1 g^{(-2)})_\varepsilon &= \varepsilon (\partial_1 g^{(-3)})'_\varepsilon - \varepsilon (\partial_1^2 g^{(-3)})_\varepsilon \end{aligned}$$

yields

$$(g^{(-1)})_\varepsilon = \varepsilon (g^{(-2)})'_\varepsilon - \varepsilon^2 (\partial_1 g^{(-3)})'_\varepsilon + \varepsilon^2 (\partial_1^2 g^{(-3)})_\varepsilon.$$

Convolving with K_3' and integrating by parts,

$$\begin{aligned} K_3' * (g^{(-1)})_\varepsilon &= \varepsilon K_3'' * (g^{(-2)})_\varepsilon - \varepsilon^2 K_3'' * (\partial_1 g^{(-3)})_\varepsilon \\ &\quad + \varepsilon^2 K_3' * (\partial_1^2 g^{(-3)})_\varepsilon; \end{aligned}$$

the boundary terms vanish since K_3' has a bounded support. We already know the first two terms are $O(\varepsilon^2)$. Using $K_3' = (K_2 * K_1)' = K_2 * K_1'$, the last term reads $\varepsilon^2 K_2 * (K_1' * (\partial_1^2 g^{(-3)})_\varepsilon)$, hence is also $O(\varepsilon^2)$ by lemma 3. The sum of the three terms is therefore also $O(\varepsilon^2)$.

We finally prove $(K_3 * g_\varepsilon)(t) = O(\varepsilon^3)$. Indeed, using successively

$$\begin{aligned} g_\varepsilon &= \varepsilon (g^{(-1)})'_\varepsilon - \varepsilon (\partial_1 g^{(-1)})_\varepsilon \\ (\partial_1 g^{(-1)})_\varepsilon &= \varepsilon (\partial_1 g^{(-2)})'_\varepsilon - \varepsilon (\partial_1^2 g^{(-2)})_\varepsilon \\ (\partial_1^2 g^{(-2)})_\varepsilon &= \varepsilon (\partial_1^2 g^{(-3)})'_\varepsilon - \varepsilon (\partial_1^3 g^{(-3)})_\varepsilon, \end{aligned}$$

we find

$$\begin{aligned} g_\varepsilon &= \varepsilon (g^{(-1)})'_\varepsilon - \varepsilon^2 (\partial_1 g^{(-2)})'_\varepsilon \\ &\quad + \varepsilon^3 (\partial_1^2 g^{(-3)})'_\varepsilon - \varepsilon^3 (\partial_1^3 g^{(-3)})_\varepsilon. \end{aligned}$$

Convolving with K_3 and integrating by parts,

$$\begin{aligned} K_3 * g_\varepsilon &= \varepsilon K_3' * (g^{(-1)})_\varepsilon - \varepsilon^2 K_3' * (\partial_1 g^{(-2)})_\varepsilon \\ &\quad + \varepsilon^3 K_3' * (\partial_1^2 g^{(-3)})_\varepsilon - \varepsilon^3 K_3 * (\partial_1^3 g^{(-3)})_\varepsilon; \end{aligned}$$

the boundary terms vanish since K_3 has a bounded support. We already know the first and third terms are $O(\varepsilon^3)$. The second term reads

$$\begin{aligned} \varepsilon^2 K_3' * (\partial_1 g^{(-2)})_\varepsilon &= \varepsilon^3 K_3' * (\partial_1 g^{(-3)})'_\varepsilon - \varepsilon^3 K_3' * (\partial_1^2 g^{(-3)})_\varepsilon \\ &= \varepsilon^3 K_3'' * (\partial_1 g^{(-3)})_\varepsilon - \varepsilon^3 K_3' * (\partial_1^2 g^{(-3)})_\varepsilon \\ &= \varepsilon^3 K_1 * (K_2'' * (\partial_1 g^{(-3)})_\varepsilon) \\ &\quad - \varepsilon^3 K_2' * (K_1' * (\partial_1^2 g^{(-3)})_\varepsilon), \end{aligned}$$

and is also $O(\varepsilon^3)$ by using Lemma 3 twice. Finally, by Young's convolution inequality, the fourth term satisfies

$$\|\varepsilon^3 K_3 * \partial_1^3 g^{(-3)}\|_\infty \leq \varepsilon^3 \|K_3\|_1 \|\partial_1^3 g^{(-3)}\|_\infty,$$

hence is also $O(\varepsilon^3)$; notice $\partial_1^3 g^{(-3)}$ is bounded by assumption, and so is K_3 . The sum of the four terms is therefore also $O(\varepsilon^3)$, which concludes the proof. \square

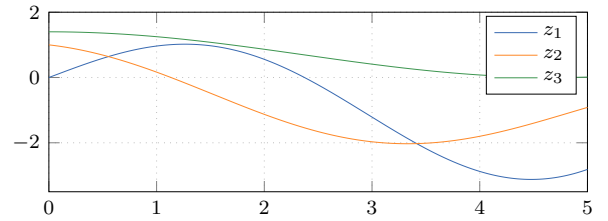


Figure 2. Encoded signals $z_1(t)$, $z_2(t)$, $z_3(t)$.

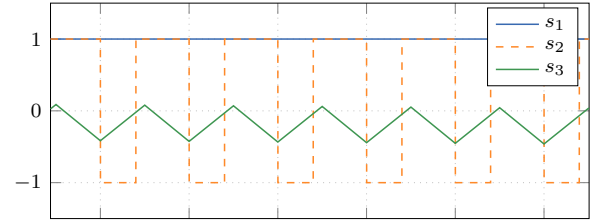


Figure 3. Carriers $s_1(t)$, $s_2(t)$, $s_3(t)$ (zoom).

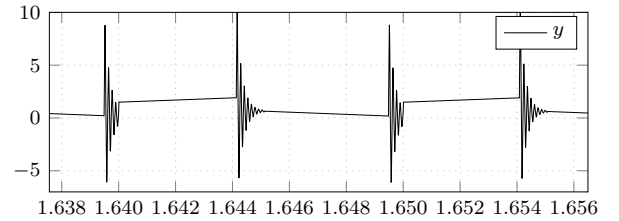


Figure 4. Composite signal $y(t)$ (zoom).

IV. NUMERICAL EXPERIMENTS

We illustrate the error analysis of Theorem 1 with numerical experiments for $k = 1, 2, 3$. Consider the composite signal y defined on $[0, 5]$ by

$$y(t) = z_1(t)s_1(t, \frac{t}{\varepsilon}) + z_2(t)s_2(t, \frac{t}{\varepsilon}) + z_3(t)s_3(t, \frac{t}{\varepsilon}) + d(t, \frac{t}{\varepsilon}),$$

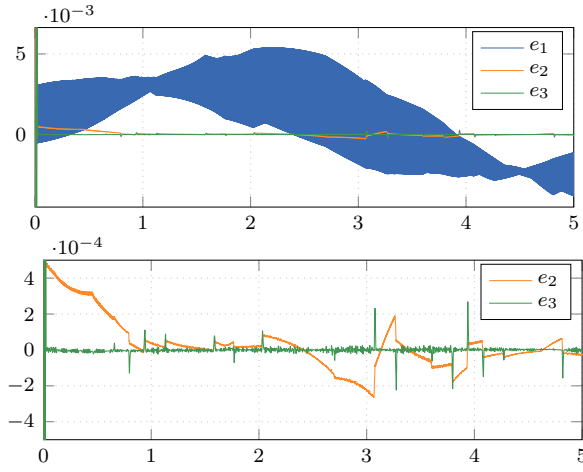


Figure 5. Errors $e_k(t)$ for $\varepsilon = 10^{-2}$: full view (top), zoom (bottom).

with encoded signals z_1, z_2, z_3 (see Fig. 2)

$$\begin{aligned} z_1(t) &:= 2 \sin(t) - 1.5 \sin\left(\frac{t}{2}\right) \\ z_2(t) &:= \cos(t) - 1.2 \sin\left(\frac{t}{4}\right) \\ z_3(t) &:= 1.4 \cos\left(\frac{t}{3}\right)^2; \end{aligned}$$

and carriers s_1, s_2, s_3 (see Fig. 3)

$$\begin{aligned} s_1(t, \sigma) &:= 1 \\ s_2(t, \sigma) &:= \text{sign}\left(\frac{t}{20} + \sigma - 0.5\right) \\ s_3(t, \sigma) &:= \begin{cases} \cos(t) + \sigma & \sigma \leq 0.5 \\ \cos(t) + 1 - \sigma & \sigma \geq 0.5, \end{cases} \end{aligned}$$

where $\sigma := \sigma \bmod 1$. The support of the disturbance d is

$$D_t := \left[f(t) - \frac{1}{20}, f(t) + \frac{1}{20}\right] \cup \left[g(t) - \frac{1}{20}, g(t) + \frac{1}{20}\right],$$

with $f(t) := \frac{1}{2}(1 + \sin(t))$ and $g(t) := \frac{1}{2}(1 + \cos(t))$; hence, on a window of length ε between 10 % (when the two intervals coincide) and 20 % (when the two intervals are disjoint) of the signal is corrupted. Fig 4 displays the resulting signal y , with the spikes caused by d clearly visible.

We select the simplest demodulating basis that is zero on D_t , namely $R(t, \sigma) := (1 - \mathbb{1}_{D_t}(\sigma))S(t, \sigma)$; tedious but routine computations show $\overline{SR^T} - SR^T$ is \mathcal{A}_k for $k = 1, 2, 3$. We check numerically that $\overline{SR^T}(t)$ is invertible by plotting its condition number κ , see Fig. 6: indeed, $\overline{SR^T}(t)$ is always well-conditioned, except during the filter initialization.

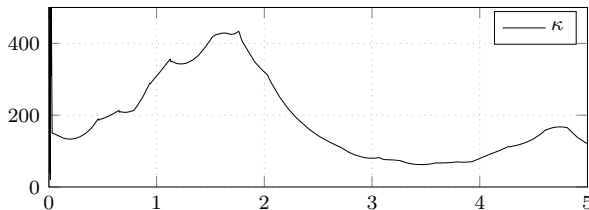


Figure 6. Condition number $\kappa(t)$ of matrix $\overline{SR^T}(t)$.

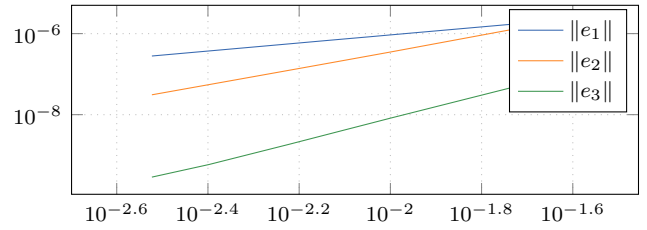


Figure 7. L^2 -error $\|e_k\|$ as a function of ε .

We focus on the recovery of z_2 , since it is modulated by the least regular carrier. We consider the error $e_k(t) := z_2(t) - P_k^2[y](t)$, where $P_k^2[y]$ denotes the second component of $P_k[y]$. For ε fixed, the error decreases as anticipated with k , see Fig 5. To study the asymptotic behavior as a function of ε , we consider the L_2 -error $\|e_k\| := (\int_1^5 (e_k(t))^2 dt)^{\frac{1}{2}}$; the first second of data is discarded to ensure the filters are well initialized. As anticipated, the plots in log scale are straight lines with slopes equal to the orders of the estimates, see Fig. 7.

V. CONCLUSION

We have proposed a demodulation procedure to recover analog signals encoded by multiple carriers with slowly-varying shapes. Though the procedure is not completely surprising at first sight, proving that the overall demodulation error is arbitrarily small is not obvious. Arguably, the framework is somewhat peculiar, which explains why no similar work seems to exist in the literature. Nevertheless, the result is exactly what we need for the application we have in mind, namely the “sensorless” control of AC electric motors at or near standstill. In this application, the composite signal y to be decoded is the (vector) current in the motor, the motor itself acting as a multicarrier modulator when fed by a PWM inverter; a suitable processing of the demodulated signal then yields the rotor angle, which is needed to accurately control the motor.

REFERENCES

- [1] D. Surroop, P. Combes, P. Martin, and P. Rouchon, “Adding virtual measurements by PWM-induced signal injection,” in *American Control Conference*, 2020, pp. 2692–2698.
- [2] —, “Sensorless rotor position estimation by PWM-induced signal injection,” in *Annual Conference of the IEEE Industrial Electronics Society (IECON)*, 2020, pp. 367–372.
- [3] B. Lathi and Z. Ding, *Modern Digital and Analog Communication Systems*, 4th ed. Oxford University Press, 2010.
- [4] Verdú, *Multuser Detection*. Cambridge University Press, 1998.
- [5] A. Goldsmith, *Wireless Communications*. Cambridge University Press, 2005.
- [6] L. Yang, *Multicarrier Communications*. Wiley, 2009.
- [7] S. Pavan, R. Schreier, and G. Temes, *Understanding Delta-Sigma Data Converters*. Wiley, 2017.
- [8] D. Surroop, P. Combes, P. Martin, and P. Rouchon, “A new demodulation procedure for a class of multiplexed signals,” in *Annual Conference of the IEEE Industrial Electronics Society (IECON)*, 2019, pp. 48–53.
- [9] J.-P. Aubin, *Applied functional analysis*, 2nd ed. Wiley-Interscience, 2000.
- [10] J. Ellison, A. W. Sáenz, and H. S. Dumas, “Improved Nth order averaging theory for periodic systems,” *Journal of Differential Equations*, vol. 84, no. 2, pp. 383–403, 1990.
- [11] S. M. Ruiz, “An algebraic identity leading to Wilson’s theorem,” *The Mathematical Gazette*, vol. 80, no. 489, p. 579–582, 1996.