Convex Clustering for Multistatic Active Sensing via Optimal Mass Transport

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***Abstract*—In multistatic active sensing, such as active sonar, exploiting the spatial diversity provided by the use of several transmitting and receiving units can lead to increased resolution and target localization performance. For multi-target scenarios, the success of the task depends on correctly identifying the subsets of measurements associated with each target, a problem of combinatorial nature. In this paper, we propose to address this problem using a convex relaxation. In particular, we propose a method inspired by the concept of optimal mass transport capable of jointly performing the measurement clustering and target localization. We show that this method may be interpreted as maximum likelihood estimation on a grid, allowing for local post-processing achieving statistical efficiency. The behavior of the proposed method is illustrated using simulated 2D examples.**

***Index Terms*—multistatic active sensing, data assignment, convex clustering, optimal mass transport**

I. Introduction

Target localization appears in a wide variety of signal processing applications, such as radar, sonar, and audio signal processing, serving as a basis for tracking [1], [2], target identification [3], and noise reduction [4]. In multistatic scenarios, several spatially distributed transmitting and/or receiving units are utilized, with the goal of achieving increased estimation performance, as well as robustness, by exploiting the extra information provided by the spatial diversity [5], [6]. Commonly, target locations are identified based on measurements or estimates of time-of-arrival (ToA), time-difference-of-arrival (TDoA), or direction-of-arrival (DoA) [7], [8]. As exact matching is often not possible due to the presence of measurement noise and model errors, target locations are commonly found as the minimizers of fitting functions such as, e.g., the negative log-likelihood if a statistical model for the measurements is available. This problem is in general non-convex [7], though convex relaxations have been proposed [9]. However, in multi-target scenarios, the obtained measurements form an unordered set, i.e., there are no labels indicating which measurements that are associated with which targets. Specifically, in order to find, e.g., the maximum likelihood estimates (MLEs) of the target locations, the measurements must first be grouped together into subsets, with each subset corresponding to a particular target [10]. Failure to find the correct grouping of measurements may lead to severely biased location estimates, especially if quadratic data fit functions are utilized [11]. Commonly, the grouping is estimated as the one leading to the minimal total cost. However, finding this minimizing association is a combinatorial problem, with the data fit function having to be minimized with respect to the target locations for each considered candidate grouping. The data association problem has been addressed by utilizing a generalized K-means approach [11], by considering Lagrangian relaxations [12], as well as by greedy methods [13] and multi-stage approaches [14]. These strategies are often non-convex, and the result thereby depends on the initialization of the clustering algorithm.

In this work, we propose a convex method for simultaneous measurement grouping and multi-target localization, based on measurements of ToA. In particular, we propose to formulate the task of clustering and localization as a problem of optimal mass transport (OMT). The Monge-Kantorovich problem of OMT [15] is concerned with finding the most efficient way of moving one mass distribution to another, where efficiency is dictated by a cost of transport on the product space of the domains of definition of the mass distributions. OMT has earlier been used for defining distances between stochastic processes [16]–[19], robust tracking of dynamic systems [20], [21], multi-microphone noise reduction [22], automatic control [23], as well as for fundamental frequency estimation for inharmonic signals [24], [25]. In the context of this work, OMT is utilized for constructing an association between the set of ToA measurements and a grid of candidate target locations, reminiscent of the convex clustering technique in [26]. As each triplet of transmitter, receiver, and ToA defines an ellipsoid of consistent target locations, the proposed method simultaneously clusters groups of ToA measurements and estimates target locations by minimizing the total cost of transport between the set of ellipsoids and the grid of candidate locations. If the cost of transport is selected as the Euclidean distance, the OMT problem models actual transport in $\mathbb{R}^d$, for $d = 2$ or $d = 3$, i.e., moving the candidate targets to the respective ellipsoids. If a statistical model for the ToA error is available, the cost of transport may instead be selected as the negative log-likelihood. In this case, the proposed method corresponds to an approximation of the MLE restricted to the candidate grid. As we show in this work, if the targets

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are not too closely spaced, this allows for finding statistically efficient estimates of the target locations. The behavior of the proposed method is illustrated in simulation studies in $d = 2$ dimensions.

II. Signal model

Consider a signal transmitter, located at $\mathbf{y}^{(t)} \in \mathbb{R}^d$, for $d = 2$ or 3, transmitting a signal that reflects on K targets and which is then recorded at R receivers. Let the targets be located at $\mathbf{x}_k \in \mathbb{R}^d$, $k = 1, \ldots, K$, and the receivers be located at $\mathbf{y}_m^{(r)} \in \mathbb{R}^d$, $m = 1, \ldots, R$. Letting the speed of signal propagation be $\rho \in \mathbb{R}_+$, the nominal delay time from the transmitter to receiver m via target k is

$$\tilde{\tau}_{m,k} = \frac{\left\|\mathbf{y}^{(t)} - \mathbf{x}_k\right\|_2 + \left\|\mathbf{y}_m^{(r)} - \mathbf{x}_k\right\|_2}{\rho}$$

Thus, the task of target localization is finding $\{\mathbf{x}_k\}_k$ that are consistent with the ToA measurements $\{\tilde{\tau}_{m,k}\}_{m,k}$. Consider the ToA corresponding to an arbitrary target location $\mathbf{x}$,

$$f_m(\mathbf{x}) = \frac{\left\|\mathbf{y}^{(t)} - \mathbf{x}\right\|_2 + \left\|\mathbf{y}_m^{(r)} - \mathbf{x}\right\|_2}{\rho}, \tag{1}$$

for $m = 1, \ldots, R$. Then, the set $\mathcal{E}_{m,k} = \{\mathbf{x} \mid f_m(\mathbf{x}) = \tilde{\tau}_{m,k}\}$ constitutes the uncertainty ellipsoid associated with a particular sensor m and a ToA $\tilde{\tau}_{m,k}$, i.e., all points in space that are consistent with $\tilde{\tau}_{m,k}$. Localizing a source at $\mathbf{x}_k$ associated with ToAs $\tilde{\tau}_{1,k}, \ldots, \tilde{\tau}_{R,k}$, corresponds to identifying the set

$$\Omega_k = \cap_{m=1}^{R} \mathcal{E}_{m,k}, \tag{2}$$

where the target is identifiable if Ω_k is a single point. However, in practice and in multi-target scenarios, two difficulties arise. Firstly, some of the sets Ω_k may be empty, i.e., there exists no points in space consistent with the measurements. This may be caused by, e.g., inexact knowledge of the transmitter and sensor locations, in which the functions f_m are only approximately known, due to synchronization errors, or due to inaccurate estimates of the ToAs. Secondly, if several targets are present, it is a non-trivial task to even construct the sets Ω_k, since each sensor m registers a set of unlabeled ToAs $\tilde{\tau}_{m,k}$, for $k = 1, \ldots, K$. Thus, in order to construct Ω_k one has to identify which set of ToAs that corresponds to each target, which is a combinatorial problem. As to reflect this, and to simplify notation, we relabel the ellipsoids according to $\mathcal{E}_q$, for $q = 1, \ldots, Q$, where $Q = KR$. Correspondingly, let the ToAs be $\tilde{\tau}_q$, $q = 1, \ldots, Q$. Furthermore, in order to take into account errors in the ToAs or, equivalently, the ellipsoids, we model τ_q as random variables according to

$$\tau_q \sim p(\cdot\,; \tilde{\tau}_q), \tag{3}$$

where p is a probability density function (pdf) parametrized by the nominal $\tilde{\tau}_q$, i.e., the ToA without error.[1] Accordingly, each ellipsoid $\mathcal{E}_q$ is a stochastic set in $\mathbb{R}^d$. Thus, the localization problem corresponds to identifying $\{\mathbf{x}_k\}_{k=1}^K$ from the set $\{\mathcal{E}_q\}_{q=1}^Q$. It may be noted that $Q = KR$ can be relaxed to $Q \leq KR$ as to cover scenarios in which some sensors may fail to detect some of the targets, e.g., if targets do not reflect the signal in all directions. Herein, we propose to view this as a clustering problem and to address it via a convex relaxation inspired by the concept of OMT.

[1] As τ_q is commonly estimated by matched filtering [27], subject to Gaussian measurement noise, p may for example be selected as corresponding to a Gaussian distribution.

III. Clustering via Optimal Mass Transport

Consider two non-negative vectors $\boldsymbol{\nu} \in \mathbb{R}_+^{N_1}$, $\boldsymbol{\mu} \in \mathbb{R}_+^{N_2}$ such that $\boldsymbol{\nu}^T \mathbf{1}_{N_1} = \boldsymbol{\mu}^T \mathbf{1}_{N_2}$, where $\mathbf{1}_N$ denotes a column vector of ones of length N. Then, for a cost matrix $\mathbf{C} \in \mathbb{R}^{N_1 \times N_2}$ the discrete Monge-Kantorovich problem of OMT [15], [28] is stated as

$$\begin{aligned} &\underset{\mathbf{M} \in \mathbb{R}_+^{N_1 \times N_2}}{\text{minimize}} \sum_{n_1=1}^{N_1} \sum_{n_2=1}^{N_2} \mathbf{C}_{n_1,n_2} \mathbf{M}_{n_1,n_2} = \operatorname{tr}\left(\mathbf{C}^T \mathbf{M}\right) \\ &\text{subject to } \mathbf{M}^T \mathbf{1}_{N_1} = \boldsymbol{\mu} \ , \ \mathbf{M} \mathbf{1}_{N_2} = \boldsymbol{\nu}, \end{aligned}$$

where $\operatorname{tr}(\cdot)$ denotes the trace operation. Here, $\mathbf{C}_{n_1,n_2} \in \mathbb{R}$ is the cost of transporting a unit mass from index n_1 of $\boldsymbol{\nu}$ to index n_2 of $\boldsymbol{\mu}$. The mass transported between these two indices is $\mathbf{M}_{n_1,n_2} \in \mathbb{R}_+$, where $\mathbf{M}$ is referred to as the transport plan. The constraints ensure that $\boldsymbol{\nu}$ and $\boldsymbol{\mu}$ are marginals of $\mathbf{M}$, i.e., the solution models actual transport between $\boldsymbol{\nu}$ and $\boldsymbol{\mu}$. The objective $\operatorname{tr}\left(\mathbf{C}^T \mathbf{M}\right)$ is the total cost of transport between $\boldsymbol{\nu}$ and $\boldsymbol{\mu}$. Herein, we propose to solve the localization problem by modeling transport between a set $\mathcal{X} = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$ of candidate target locations $\mathbf{x}_n \in \mathbb{R}^d$, $n = 1, \ldots, N$, for some $N \in \mathbb{N}$, and the set of uncertainty ellipsoids $\mathcal{E}_q$, $q = 1, \ldots, Q$. More specifically, we consider transport between mass distributions $\boldsymbol{\nu} \in \mathbb{R}_+^N$ and $\boldsymbol{\mu} \in \mathbb{R}_+^Q$ corresponding to the sets of candidate targets and ellipsoids, respectively. As we know that we have exactly Q ellipsoids, and all measurements have to be accounted for, $\boldsymbol{\mu} = \mathbf{1}_Q$. In contrast to the standard OMT problem, the distribution $\boldsymbol{\nu}$ is not known; in fact, this is the quantity to be estimated. However, some partial information of $\boldsymbol{\nu}$ is available; as K targets are sought, $\boldsymbol{\nu}^T \mathbf{1}_N = K$, and $\boldsymbol{\nu} \leq \mathbf{1}_N$, where the inequality is to be interpreted elementwise, as each position can correspond to at most one target. Letting c be a transport cost function such that $c(\mathbf{x}_n, \mathcal{E}_q)$ is the cost of associating $\mathbf{x}_n$ with $\mathcal{E}_q$, and setting $\mathbf{C}_{n,q} = c(\mathbf{x}_n, \mathcal{E}_q)$, an OMT problem for localization may be formulated as

$$\begin{aligned} \underset{\mathbf{M} \in \mathbb{R}_+^{N \times Q},\, \boldsymbol{\nu} \in \mathbb{R}_+^N}{\text{minimize}} \quad & \operatorname{tr}\left(\mathbf{C}^T \mathbf{M}\right) \\ \text{subject to} \quad & \mathbf{M}^T \mathbf{1}_N = \boldsymbol{\mu} \ , \ \mathbf{M} \mathbf{1}_Q \leq R\boldsymbol{\nu} \\ & \mathbf{M} \leq \boldsymbol{\nu} \mathbf{1}_Q^T \ , \ \boldsymbol{\nu}^T \mathbf{1}_N = K \ , \ \boldsymbol{\nu} \leq \mathbf{1}_N, \end{aligned} \tag{4}$$

where it may be noted that both the transport plan $\mathbf{M}$ and the mass distribution $\boldsymbol{\nu}$ for the candidate grid are problem variables. Strictly speaking, $\mathbf{M}$ models transport between $\boldsymbol{\mu}$ and $R\boldsymbol{\nu}$, as each target is expected to be associated with R ellipsoids, corresponding to the number of receivers. This is enforced through the constraint $\mathbf{M} \mathbf{1}_Q \leq R\boldsymbol{\nu}$, where the

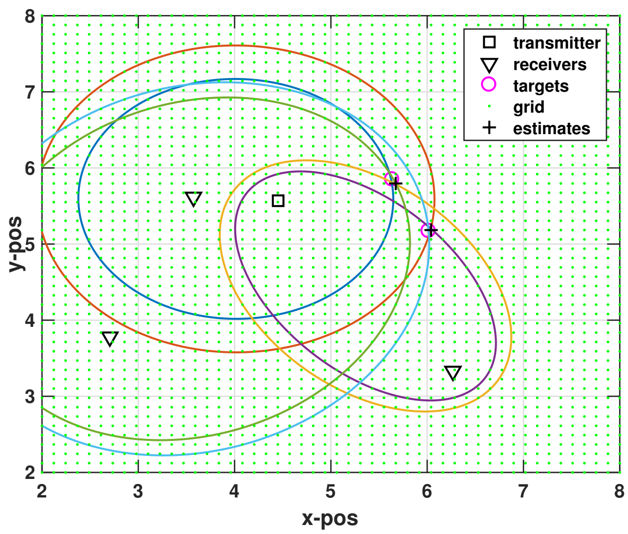

Fig. 1. Localization scenario with $K = 2$ targets and $R = 3$ receivers, together with estimates provided by (4).

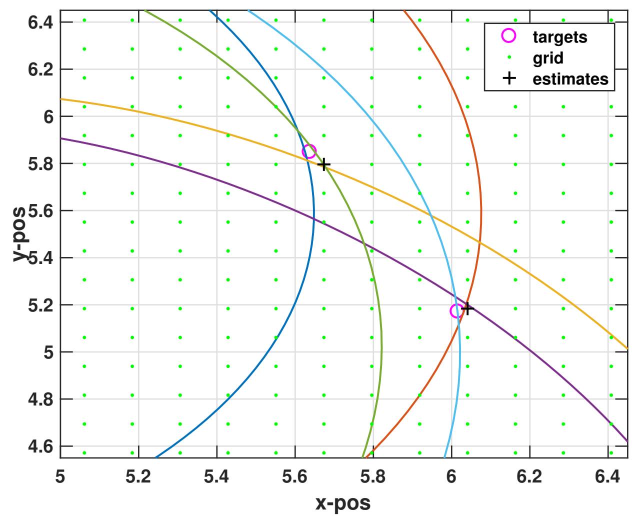

Fig. 2. Zoomed-in portion of Figure 1.

inequality allows for potentially missed detections or sensor failures; if $Q = RK$, the inequality can be replaced by equality. Furthermore, the constraint $\mathbf{M} \leq \boldsymbol{\nu}\mathbf{1}_Q^T$, i.e., $\mathbf{M}_{n,q} \leq \boldsymbol{\nu}_n$, for all n and q, ensures that mass is only transported to candidate targets $\mathbf{x}_n$ that are assigned mass by $\boldsymbol{\nu}$. It may be noted that (4) is a convex, in fact linear, program for any choice of cost function c. Furthermore, it can be seen as a convex relaxation of a corresponding combinatorial problem in which integer constraints are imposed on the elements of the mass distribution $\boldsymbol{\nu}$, i.e.,

$$\begin{aligned} \underset{\mathbf{M}\in\mathbb{R}_+^{N\times Q},\,\boldsymbol{\nu}\in\mathbb{R}_+^N}{\text{minimize}} \quad & \operatorname{tr}\left(\mathbf{C}^T\mathbf{M}\right) \\ \text{subject to} \quad & \mathbf{M}^T\mathbf{1}_N = \boldsymbol{\mu}\ ,\ \mathbf{M}\mathbf{1}_Q \leq R\boldsymbol{\nu} \\ & \mathbf{M}\leq\boldsymbol{\nu}\mathbf{1}_Q^T\ ,\ \boldsymbol{\nu}^T\mathbf{1}_N = K\ ,\ \boldsymbol{\nu}\in\{0,1\}^N. \end{aligned} \tag{5}$$

That is, the $\mathbf{M}$ solving (5) gives the optimal assignment of K candidate target locations to the uncertainty ellipsoids, where $\mathbf{M}_{n,q} = 1$ if target n is assigned to ellipsoid q, and 0 otherwise. It is then clear that if exactly K elements of $\boldsymbol{\nu}$ solving (4) are non-zero, then this is also a solution to (5). With this, it remains to select the cost function c defining the cost matrix $\mathbf{C}$. A natural choice would be

$$c^{\text{Euclid}}(\mathbf{x}, \mathcal{E}) = \min_{\tilde{\mathbf{x}}\in\mathcal{E}} \|\mathbf{x} - \tilde{\mathbf{x}}\|_2^2,$$

i.e., the squared distance to the closest, in the Euclidean sense, point on the ellipsoid from $\mathbf{x}$. With this choice, (5) finds the set of points that are spatially closest to being consistent with the measurements $\mathcal{E}_1, \ldots, \mathcal{E}_Q$. However, if a statistical description of the measurements, in the form of the pdf p, is available one could let the cost be the negative log-likelihood

$$c^p(\mathbf{x}, \mathcal{E}) = -\ln p(\tau; f(\mathbf{x})), \tag{6}$$

where τ and f is the ToA and delay function, as given in (1), defining the ellipsoid $\mathcal{E}$. Using c^p, and for $\mathbf{M}^\star$ solving (5), the objective $\operatorname{tr}\left(\mathbf{C}^T\mathbf{M}^\star\right)$ in (5) is then equal to the negative of the maximum log-likelihood achievable for a collection of K targets on the grid $\mathcal{X}$. Furthermore,

$$\operatorname{tr}\left(\mathbf{C}^T\mathbf{M}^\star\right) \leq \min_{\mathbf{x}_1,\ldots,\mathbf{x}_K\in\mathcal{X}} -\sum_{k=1}^{K}\sum_{m=1}^{R} \ln p(\tau_{m,k}; f_m(\mathbf{x}_k)),$$

where the right hand side is the negative of the maximum log-likelihood achievable on $\mathcal{X}$ when each target is associated with the correct set of uncertainty ellipsoids, i.e., when the Ω_k in (2) are known. For spatially well-separated targets, we have observed that the association determined by (5) corresponds to the ground truth with high probability, i.e., (5) yields the MLE restricted to $\mathcal{X}$. Here, the notion of target separation is related to the concentration of mass of the pdf p, i.e., to which extent p is flat. We summarize the connection between the convex program in (4) and the MLE as follows:

- Let the cost function c be as in (6), and assume that the targets are well-separated with respect to the density p. Let $(\mathbf{M}^\star, \boldsymbol{\nu}^\star)$ be the solution to (4). Then, if $\operatorname{card}(\boldsymbol{\nu}^\star) = K$, where $\operatorname{card}(\boldsymbol{\nu}^\star)$ is the number of non-zero elements of $\boldsymbol{\nu}^\star$, the corresponding set of estimated target positions is with high probability the MLE restricted to $\mathcal{X}$.

Remark 1: In some applications, the distribution for the ToA errors may come from a mix of distributions, and, in particular, with heavy-tailed components. In this case, the negative log-likelihood in (6) may be replaced by more robust alternatives such as, e.g., Huber's loss. It may be noted that this does not affect the convexity of (4).

Remark 2: As to not be restricted to the grid $\mathcal{X}$, the target locations from the solution of (4) may be refined by a local non-linear search. If $\mathbf{M}^\star$ yields the correct grouping of the uncertainty ellipsoids, this corresponds to the MLE of the target locations. The grid-based initial location estimates then serves as a good initial point for the optimization routine. It may be noted that the negative log-likelihood is (in general) non-convex in the locations.

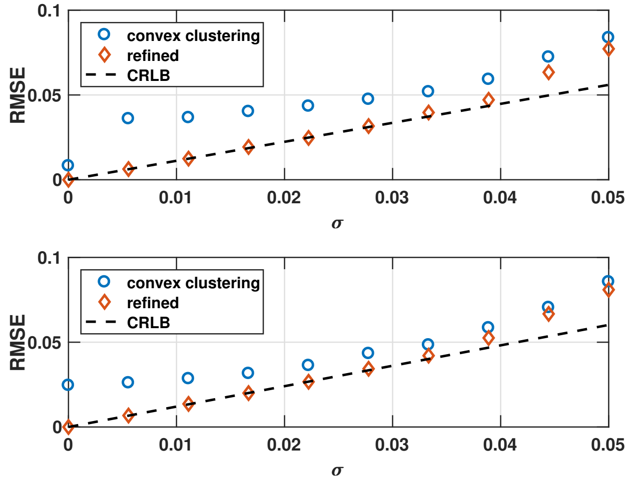

Fig. 3. Localization performance of (4), as well as refined estimates obtained using a local non-linear search, as a function of the standard deviation σ of the Gaussian timing errors. Top panel: first target. Bottom panel: second target.

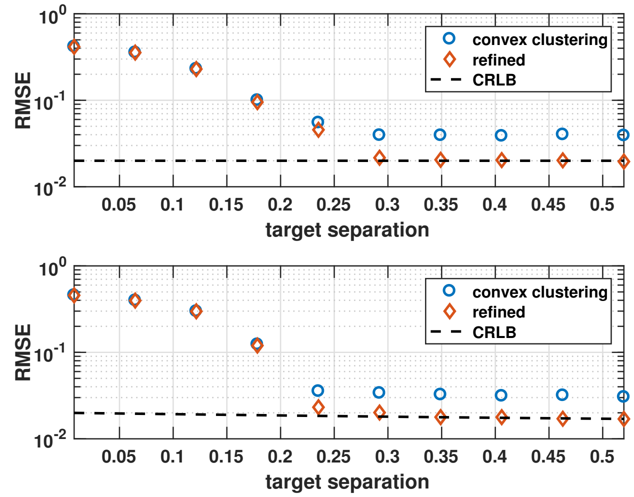

Fig. 4. Localization performance of (4), as well as refined estimates obtained using a local non-linear search, as a function of the target separation. Top panel: first target. Bottom panel: second target.

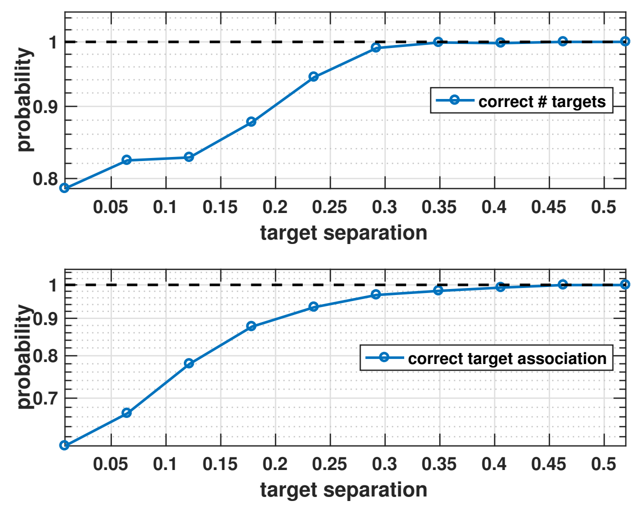

Fig. 5. Ability of (4) to identify the correct number of targets and two correctly cluster the uncertainty ellipses into two sets. Top panel: probability of identifying exactly two targets. Bottom panel: probability of correctly identifying the two sets of uncertainty ellipses.

A localization scenario in $d = 2$ dimensions is shown in Figure 1, displaying $K = 2$ reflecting targets, $R = 3$ receivers, and the corresponding six uncertainty ellipses. As may be noted, there are no points where any set of three ellipses intersect, due to stochastic timing errors. Also shown are the localization estimates obtained using (4), as well as the corresponding grid $\mathcal{X}$ utilized, with Figure 2 providing a zoomed-in view. As can be seen, the grid points closest to the actual targets are selected by (4). For solving (4), CVX [29] was used.

IV. Numerical Illustrations

As to illustrate the proposed clustering and localization method (4), consider two Monte Carlo simulations studies based on the scenario in Figure 1. Letting the ToA errors be Gaussian, we firstly consider varying the standard deviation, σ, of the error. Secondly, for a fixed $\sigma = 2 \times 10^{-2}$, we study the performance of (4) as a function of the target separation, moving the targets in Figure 1 closer and closer together. In the first scenario, the target separation is fixed to 0.78. For both scenarios, the square $[2, 8] \times [2, 8]$ is gridded as to yield a grid $\mathcal{X}$ with resolution 6×10^{-2} in each dimension. For each setting, i.e., for each σ and target separation, respectively, we perform 1000 Monte Carlo simulations. In each simulation, the nominal target positions are perturbed with a random vector uniform on a square corresponding to the grid resolution as to avoid biasing artefacts. Thus, the nominal target separations correspond to the expected values of the actual target separation in each simulation. The root mean squared error (RMSE) for the locations of the two targets are shown in Figures 3 and 4 for the scenarios varying σ and the target separation, respectively. In both figures, the top and bottom panels display the results for the first and second target, respectively. For reference, the corresponding (square root) Cramér-Rao lower bound (CRLB) is provided. The CRLB for the localization error for the two respective targets is here given by $\text{tr}\left(\mathbf{F}_1^{-1}\right)$ and $\text{tr}\left(\mathbf{F}_2^{-1}\right)$, respectively, where

$$\mathbf{F}_k = \sum_{m=1}^{R} \mathbb{E}\left[\nabla_{\mathbf{x}} \ln p(\tau_{m,k}; f_m(\mathbf{x})) \nabla_{\mathbf{x}} \ln p(\tau_{m,k}; f_m(\mathbf{x}))^T \big|_{\mathbf{x}=\mathbf{x}_k}\right],$$

for $k = 1, 2$, is the Fisher information matrix,

$$p(\tau; f_m(\mathbf{x})) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(\tau - f_m(\mathbf{x}))^2},$$

and where $\mathbb{E}$ denotes the expectation operator. Also, as to illustrate the connection between (4) and the MLE of the target locations, the RMSE obtained by refining the estimates from (4) using a local non-linear search implemented by Matlab's *fminsearch* is also shown. As may be noted from Figure 3, as the standard deviation of the ToA errors decreases, the grid-based estimates from (4) come close enough to the maximum of the log-likelihood for the refinement to correspond to the MLE. In this scenario, the uncertainty ellipses are, except in a few cases, associated to the correct targets. In contrast, when decreasing the distance between the targets, the probability of erroneous clustering of the ellipses increases. This is shown in Figure 5, displaying the probability of identifying exactly two targets and the probability of correctly clustering the ellipses to their corresponding targets, in the top and bottom panels, respectively. As may be noted from Figures 4 and 5, as the probability of correct association approaches 1, the RMSE also approaches the CRLB.

V. CONCLUSIONS

In this work, we have presented a convex method for simultaneous ToA measurement clustering and target localization in multi-target settings. By utilizing the concept of optimal mass transport, ToAs are clustered together as to minimize the total cost of transport to a set of candidate target locations. When the cost of transport is selected as the negative log-likelihood corresponding to the ToA error, the proposed method approximates the maximum likelihood estimator restricted to a discrete grid. We have seen that for not too closely spaced targets, this allows for finding statistically efficient estimates of the target locations, without prior knowledge of the ToA labels.

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