One-bit Error-Feedback Quantizer for Uniform Linear Antenna Array

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***Abstract*—One-bit error-feedback quantizers for uniform linear antenna arrays are developed to enable efficient spatial noise shaping. The SNR of the maximum ratio combiner for received signals quantized by one-bit error-feedback quantizers is evaluated. Then, the optimal error-feedback quantizer that maximizes the worst SNR for the signals whose arrival angles are in a fixed range of interest is designed. Numerical results are provided to show that our one-bit spatial noise shaping exhibits flat BER performance for the signals from the prescribed range.**

***Index Terms*—Error-feedback quantizer, Delta-Sigma modulator, antenna array, noise shaping**

I. Introduction

Massive MIMO that equips base stations with very large antenna arrays is a key technology for 5G wireless communication systems [1]. To enable reasonable massive MIMO, it is necessary to reduce the hardware cost and power consumption of radio-frequency (RF) font-ends. Especially at higher bandwidths and sampling rates, high-resolution analog-to-digital converters (ADCs) and digital-to-analog converters (DACs) are expensive and require a large amount of energy, which are prohibitable for small devices. This motivates the research on massive MIMO systems with low-resolution ADC and DAC (see, e.g., [2] and references therein).

Energy-efficient low-dynamic range power amplifiers can be utilized for low-resolution ADC and DAC. There exists a trade-off between numbers of bits for ADC and DAC and implementation/running costs. One-bit ADC and DAC are the simplest and the most power-saving, whereas they generate the largest quantization errors. Fortunately, it is often the case that a very large number of antennas can mitigate the performance loss due to quantization errors.

For example, in the uplink of massive MIMO systems, low-resolution ADCs can achieve satisfying spectral efficiency [3]; one-bit ADCs achieve sufficient channel estimation accuracy [4]. For the downlink, one-bit precoding for one-bit DACs has been developed in [5], [6].

The original $\Delta\Sigma$ modulator has been provided in [7] for code modulation, which has been used as efficient ADC and DAC [8]. The noise shaping of the original $\Delta\Sigma$ modulator can also be applied to multiple antenna array systems [9], [10], where the outputs of the multiple antenna array at one-time slot successively are fed into the $\Delta\Sigma$ modulator, which can be considered as a spatial $\Delta\Sigma$ modulation. For massive MIMO with one-bit $\Delta\Sigma$, the spectral efficiency has been studied in [11] and channel estimation has been considered in [12]. For the downlink, one-bit precoding methods have been presented using the original $\Delta\Sigma$ modulator [13] as well as the second order $\Delta\Sigma$ modulator [14]. However, most of them utilizes the original $\Delta\Sigma$ modulator, which exhibits poor performance compared with modern $\Delta\Sigma$ modulators. To obtain better noise shaping properties, the filters in the $\Delta\Sigma$ modulators should be well designed [8].

On the other hand, error-feedback quantizers have been presented to reduce quantization error in the coefficients of digital filters [15]–[17]. In an error-feedback quantizer, the quantization error of a uniform quantizer is filtered and fed back into the input of the uniform quantizer. As shown in [18], any stable $\Delta\Sigma$ modulators can be converted into error-feedback quantizer and optimal error-feedback filters can be designed based on linear matrix inequalities (LMIs).

The object of this paper is to develop one-bit error-feedback quantizers for massive MIMO systems with uniform linear antenna arrays. To illustrate the spatial noise shaping for uniform linear antenna arrays, we focus on the uplink of the single-user with one transmit antenna to a base station with a uniform linear antenna array.

First, we present an one-bit spatial noise shaping by an error-feedback quantizer for the received signal of an antenna array, which subsumes the spatial noise shaping by the original $\Delta\Sigma$ modulator as a special case. Then, we evaluate the SNR of the quantized received signal, which is a function of the amplitude response of the noise shaping filter (NSF) of the error-feedback quantizer at the frequency that corresponds to the arrival angle of the signal. To obtain flat performance for different arrival angles in a fixed range, we minimize the maximum of the amplitude response of the NSF at frequencies in the fixed range. The minimization is cast into a convex optimization, which can be solved numerically and efficiently by using a numerical solver like CVX [19]. Numerical results are provided to show that designed filters have flat low amplitude responses in the frequency range and that our one-bit spatial noise shaping exhibits flat BER performance for angles in the prescribed range.

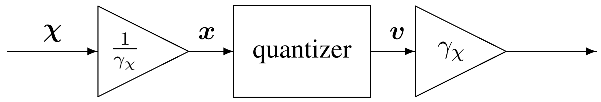

Fig. 1: Quantization and scaling.

II. Uniform linear antenna array and one-bit error- feedback quantization

Let us consider a single-cell system, where each user has one antenna and the base station equips a uniform linear antenna array with N antennas. For the simplicity of presentation, we deal with a single-user case with a single-path channel, using discrete-time signals. The extension from the single-user case to the multi-user case with multi-path channels is possible as in [11]. We confine our attention to the uplink with one-bit quantization. Similar results can be developed for the downlink with one-bit precoding as in [6] but are omitted here.

During one time-slot, the user sends an symbol s with zero mean and variance σ_s^2 over a single-path channel. Let us denote the arrival angle of the signal from the user to the base station and the antenna spacing of the uniform linear array as $\theta \in [-\pi/2, \pi/2]$ and d, respectively. The wavelength is assumed to be λ.

The received vector at the base station can be modeled as

$$\boldsymbol{y} = a(\theta)s + \boldsymbol{n} \tag{1}$$

where $a(\theta)$ is the steering vector given by

$$a(\theta) = \frac{1}{\sqrt{N}}\left[1, e^{-\mathrm{j}\frac{2\pi d}{\lambda}\sin(\theta)}, \dots, e^{-\mathrm{j}(N-1)\frac{2\pi d}{\lambda}\sin(\theta)}\right]^T \tag{2}$$

with $(\cdot)^T$ being the transpose of the vector and $\boldsymbol{n}$ is the additive noise vector at the receiver. The noise vector is assumed to be a circularly symmetric white Gaussian vector having zero mean and covariance matrix $\sigma_n^2\boldsymbol{I}$ with $\boldsymbol{I}$ being an $N \times N$ identity matrix.

We assume that the range of arrival angles is given by $[-\Theta, \Theta]$ where the angular spread $\Theta(> 0)$ is less than π. This implies that the range is symmetric with respect to the arrival angle 0.

We utilize the subscripts R and I to denote the real and the imaginary part of a scalar or a vector. For example, $\boldsymbol{y} = \boldsymbol{y}_R + j\boldsymbol{y}_I$ where $\boldsymbol{y}_R$ and $\boldsymbol{y}_I$ are respectively the real and the imaginary part of $\boldsymbol{y}$.

For the uniform linear array with a symmetric range, we independently quantize the real part and the imaginary part of the received signal. Let us denote the real-valued input vector of size N to our quantizer as $\boldsymbol{\chi}$ and express the l_∞ norm of $\boldsymbol{\chi}$ as γ_χ, which is the maximum of absolute values of entries of $\boldsymbol{\chi}$ denoted by $\|\boldsymbol{\chi}\|_\infty$.

A schematic diagram of our quantization is illustrated in Fig. 1. Before quantization, we normalize the input vector such as $\boldsymbol{\chi}/\gamma_\chi$ and denote the normalized input vector as $\boldsymbol{x}$. Let us express the nth entry of the vector $\boldsymbol{x}$ as x_n. Then, $|x_n|$ is bounded such as $|x_n| \leq 1$ for $n \in [1, N]$. The output vector of the quantizer is denoted as $\boldsymbol{v}$. After quantization, we multiply $\boldsymbol{v}$ by γ_χ to scale the output of the quantizer to the original range.

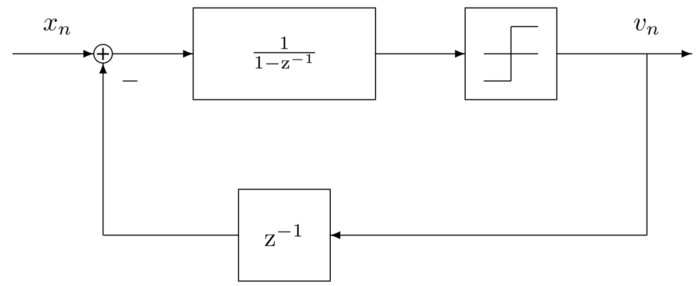

Fig. 2: $\Delta\Sigma$ modulator

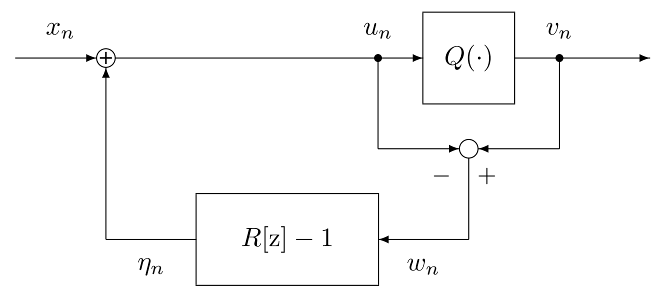

Fig. 3: Error-feedback quantizer.

Fig. 2 depicts the original $\Delta\Sigma$ modulator presented in [7], where x_n and v_n are the input and the output scalar sequence of the modulator, and z^{-1} is the unit-time delay operator. More efficient $\Delta\Sigma$ modulator can be obtained by replacing z^{-1} and $1/(1-\mathrm{z}^{-1})$ with appropriate filters.

On the other hand, Fig. 3 shows the error-feedback quantizer, where $Q(\cdot)$ is a uniform quantizer and $R[\mathrm{z}] - 1$ is an error-feedback filter which is stable and strictly proper. Error-feedback quantizers have been originally proposed to mitigate the quantization errors of uniform quantizers in digital filters [15]–[17]. In the error-feedback quantizer, the quantization error w_n of the uniform quantizer is filtered and fed back into the input of the quantizer.

Any stable $\Delta\Sigma$ modulator can be converted into its corresponding error-feedback quantizer [18]. For example, the original $\Delta\Sigma$ modulator is equivalent to the error-feedback quantizer with $R[\mathrm{z}] = 1 - \mathrm{z}^{-1}$ and

$$Q(u_n) = \mathrm{sgn}(u_n). \tag{3}$$

Thus, we adopt error-feedback quantizers for our quantization. It should be noted that with $R[\mathrm{z}] = 1$, the error-feedback quantizer boils down to the uniform quantizer.

Let us express the z-transform of a sequence denoted by a lowercase letter as its corresponding uppercase letter. The z-transform of the output v_n is related to the z-transform of the output x_n such as $V[\mathrm{z}] = X[\mathrm{z}] + R[\mathrm{z}]W[\mathrm{z}]$ [8]. Since $R[\mathrm{z}]$ shapes the noise spectrum, $R[\mathrm{z}]$ is called a noise shaping filter (NSF), an error shaping filter, or a noise transfer function. This paper considers FIR $R[\mathrm{z}]$ with real coefficients that is expressed by

$$R[\mathrm{z}] = \sum_{n=0}^{n_r} r_n \mathrm{z}^{-1} \tag{4}$$

with $r_0 = 1$. We assume that the order n_r of $R[\mathrm{z}]$ is much less than N.

The one-bit uniform quantizer is overloaded if $|u_n| > 2$. When an overloading occurs, the quantization error w_n may take a large value. Then, successive overloading may unstabilize the error-feedback quantizer. It is easy to see that if

$$\sum_{n=0}^{n_r} |r_n| \leq 1 \tag{5}$$

then there is no overloading and the quantization error of the uniform quantizer is bounded such as $|w_n| \leq 1$.

III. Design of one-bit error-feedback quantizer

The object of this paper is to design a stable error-feedback quantizer for a uniform linear antenna array receiver.

If a vector of inputs to the uniform quantizer and a noise vector of quantization errors of the uniform quantizer are defined as $\boldsymbol{u} = [u_1, u_2, \ldots, u_N]^T$ $\boldsymbol{w} = [w_1, w_2, \ldots, w_N]^T$, then the output vector is given by $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{w}$ (see Fig. 3). On the other hand, the vector $\boldsymbol{u}$ can be expressed as $\boldsymbol{u} = \boldsymbol{x} + (\boldsymbol{R} - \boldsymbol{I})\boldsymbol{w}$, where $\boldsymbol{R}$ is a lower triangular matrix whose diagonal entries and nth sub-diagonal entries are respectively r_0 and r_n, that is,

$$\boldsymbol{R} = \begin{bmatrix} 1 & 0 & & \\ r_1 & \ddots & \ddots & \\ r_2 & \ddots & 1 & 0 \\ & \ddots & r_1 & 1 \end{bmatrix}. \tag{6}$$

It follows that the vector consisting of the outputs of the error-feedback quantizer is given by $\boldsymbol{v} = \boldsymbol{x} + \boldsymbol{R}\boldsymbol{w}$.

The real part and the imaginary part of the received vector $\boldsymbol{y}$ are independently quantized. We denote the quantization error vectors for the real part and the imaginary part of $\boldsymbol{y}$ as $\boldsymbol{w}_R$ and $\boldsymbol{w}_I$, respectively. Then, after quantization and scaling, the quantized received vector $\tilde{\boldsymbol{y}}$ can be expressed as

$$\tilde{\boldsymbol{y}} = \boldsymbol{y} + \boldsymbol{e} \tag{7}$$

where $\boldsymbol{e}$ is the quantization error vector of error-feedback quantizers given by

$$\boldsymbol{e} = \boldsymbol{R}(\gamma_{y_R}\boldsymbol{w}_R + j\gamma_{y_I}\boldsymbol{w}_I) \tag{8}$$

with $\gamma_{y_R} = \|\boldsymbol{y}_R\|_\infty$ and $\gamma_{y_I} = \|\boldsymbol{y}_I\|_\infty$.

Now, let us consider the output of the maximum ratio combiner (MRC), which is given by $z = a^*(\theta)\tilde{\boldsymbol{y}}$, where $(\cdot)^*$ is the complex conjugate transpose operator.

For a sufficiently large N, the output of the MRC can be approximated as

$$z = s + a^*(\theta)\left[\boldsymbol{n} + R[e^{j\frac{2\pi d}{\lambda}\sin(\theta)}](\gamma_{y_R}\boldsymbol{w}_R + j\gamma_{y_I}\boldsymbol{w}_I)\right]. \tag{9}$$

The additive noise $\boldsymbol{n}$ is independent of $\boldsymbol{w}_R$ and $\boldsymbol{w}_I$. For our analysis and synthesis, we assume that $\boldsymbol{w}_R$ and $\boldsymbol{w}_I$ are uncorrelated with each other and that $a^*(\theta)\boldsymbol{w}_R$ and $a^*(\theta)\boldsymbol{w}_I$ have the same variance $\sigma_q^2 = E\{|a^*(\theta)\boldsymbol{w}_R|^2\} = E\{|a^*(\theta)\boldsymbol{w}_I|^2\}$ where $E\{\cdot\}$ stands for the expectation operator. Then, the received SNR of our system is given by

$$\frac{\sigma_s^2}{\sigma_n^2 + |R[e^{j\omega}]|^2(\gamma_{y_R}^2 + \gamma_{y_I}^2)\sigma_q^2} \tag{10}$$

where $\omega = \frac{2\pi d}{\lambda}\sin(\theta)$.

Since we are interested in the arrival angles in $[-\Theta, \Theta]$, to maximize the worst-case received SNR for $\theta \in [-\Theta, \Theta]$, we would like to minimize the maximum of $|R[e^{j\omega}]|$ for $\omega \in [-\Omega, \Omega]$ with $\Omega = \frac{2\pi d}{\lambda}\sin(\Theta)$ subject to the no-overloading condition. We can mathematically formulate our problem as:

$$\min_{r_1, r_2, \ldots, r_{n_r}} \max_{\omega \in [-\Omega, \Omega]} |R[e^{j\omega}]| \tag{11}$$

subject to the no-overloading condition given by (5). This problem is equivalent to: $\min_{r_1, r_2, \ldots, r_{n_r}, \mu} \mu$ subject to (5) and

$$|R[e^{j\omega}]|^2 < \mu \quad \text{for} \quad \omega \in [-\Omega, \Omega]. \tag{12}$$

Let us cast our constrained optimization into a convex optimization. First, we define state-space matrices $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D})$ of $R[\mathrm{z}]$ as

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{13}$$

$$\boldsymbol{C} = \begin{bmatrix} r_{n_r}, & r_{n_r-1}, & \cdots & r_1 \end{bmatrix}, \boldsymbol{D} = 1. \tag{14}$$

By using the generalized KYP lemma [20], the inequality (12) holds if and only if there exist symmetric matrices $\boldsymbol{Y} > 0$ and $\boldsymbol{X}$ such that [21]

$$\begin{bmatrix} \boldsymbol{M_1} & \boldsymbol{M_2} & \boldsymbol{C^T} \\ \boldsymbol{M_2^T} & M_3 & 1 \\ \boldsymbol{C} & 1 & -1 \end{bmatrix} < 0 \tag{15}$$

where

$$\boldsymbol{M_1} = \boldsymbol{A^T X A} + \boldsymbol{Y A} + \boldsymbol{A^T Y} - \boldsymbol{X} - 2\boldsymbol{Y}\cos\Omega \tag{16}$$

$$\boldsymbol{M_2} = \boldsymbol{A^T X B} + \boldsymbol{Y B} \tag{17}$$

$$M_3 = \boldsymbol{B^T X B} - \mu. \tag{18}$$

Then, (15) is a linear matrix inequality (LMI) in our design variables $r_1, r_2, \ldots, r_{n_r}$ and μ, which is convex.

On the other hand, introducing non-negative auxiliary variables $\bar{r}_n \geq 0$ for $n = 1, \ldots, n_r$ such that $\bar{r}_n = |r_n|$, we can express (5) as linear constraints [22]:

$$\sum_{n=1}^{n_r} \bar{r}_n \leq 1 \tag{19}$$

$$-\bar{r}_n \leq r_n \leq \bar{r}_n, \quad \bar{r}_n \geq 0 \quad \text{for} \quad n = 1, \ldots, n_r. \tag{20}$$

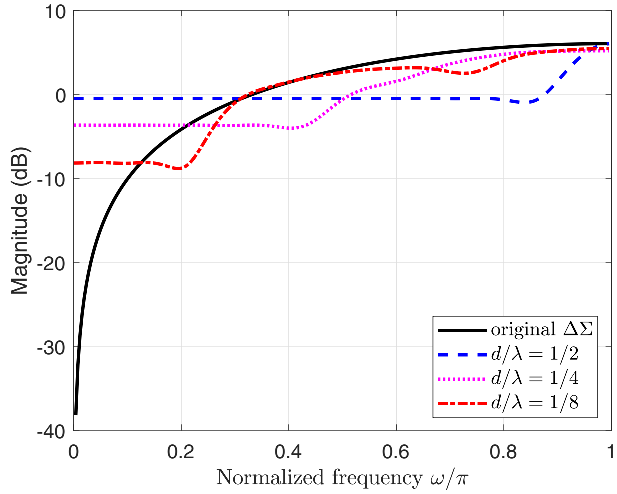

Fig. 4: Amplitude responses of designed $R[z]$ for $d/\lambda = 1/2, 1/4, 1/8$ and $R[z] = 1 - z^{-1}$ of the original $\Delta\Sigma$ modulator.

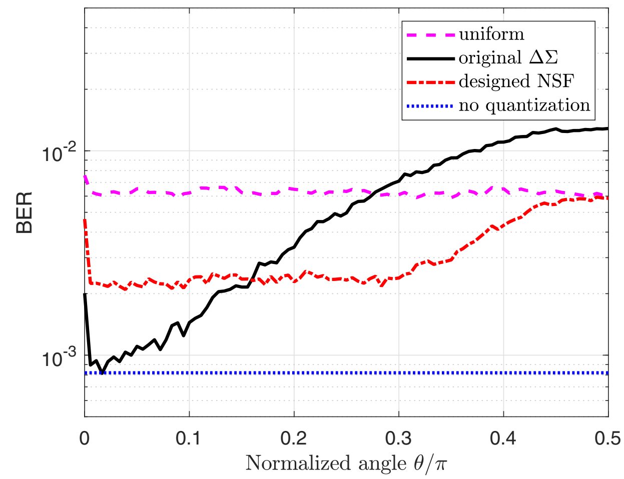

Fig. 5: BERs with designed quantizer, original $\Delta\Sigma$ modulator, and uniform quantizer compared with BER without quantization, when $N = 64$, $d/\lambda = 1/8$ and $\Theta = \pi/3$.

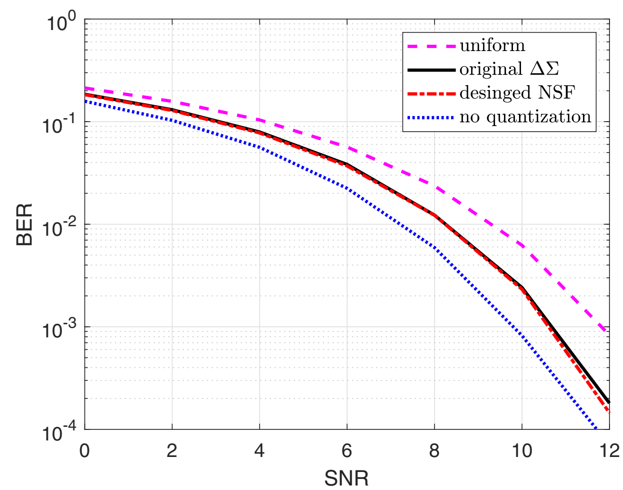

Fig. 6: BERs at $\theta = \pi/6$ with designed quantizer, original $\Delta\Sigma$ modulator, and uniform quantizer compared with BER without quantization, when $N = 64$, $d/\lambda = 1/8$ and $\Theta = \pi/3$.

Thus, our design problem is cast into the convex optimization: $\min_{r_1,\ldots,r_{n_r},\bar{r}_1,\ldots,\bar{r}_{n_r}} \mu$, subject to (15), (19), and (20). The problem can be solved numerically and efficiently by using a numerical solver like CVX [19].

IV. Simulation Results

First, we design our NSF $R[z]$ of order $n_r = 16$ for $d/\lambda = 1/2, 1/4$, and $1/8$, solving the convex optimization with CVX. The angular range is set to be $[-\pi/3, \pi/3]$.

Fig. 4 compares amplitude responses of designed $R[z]$ with the amplitude response of $R[z] = 1 - z^{-1}$ of the original $\Delta\Sigma$ modulator.

For $d/\lambda = 1/2$,$d/\lambda = 1/4$, and $d/\lambda = 1/8$, the maximum values Ω for ω are $1/2$, $1/4$ and $1/8$, respectively. Since we minimize the maximum of $|R[e^{j\omega}]|$ for $\omega \in [-\Omega, \Omega]$, the amplitude responses are flat for $\omega \in [-\Omega, \Omega]$. As d/λ decreases, the maximum values of $|R[e^{j\omega}]|$ for $\omega \in [-\Omega, \Omega]$ decreases and hence the received SNR improves.

For every d/λ, the amplitude response of the designed $R[z]$ is larger than the amplitude response of the original $\Delta\Sigma$ modulator from 0 up to almost a half of Ω, whereas the former is smaller than the latter from almost a half of Ω to Ω.

Next, we evaluate bit error rates (BERs) for MRC with our designed error-feedback quantizer, the original $\Delta\Sigma$ modulator, and the uniform quantizer, when $\Theta = \pi/3$, $N = 64$, $d/\lambda = 1/8$, and the SNR $\sigma_s^2/\sigma_n^2 = 10$ dB We generate 10^5 QPSK symbols and Gaussian noise vectors and compute BERs for arrival angles from 0 to $\pi/2$. It should be noted that the range of angular angles is $[-\pi/3, \pi/3]$ ($1/3 \approx 0.333$), in other words, normalized angular angles from 0.333 to 0.5 are out of our target.

Fig. 5 shows BERs for different angles, where the BER without quantization is also plotted, which is the limit of BER with quantization. It is remarked that theoretical BERs are even functions of the arrival angle.

From 0 to 0.167, the BER of our designed quantizer is worse than the BER of the original $\Delta\Sigma$ modulator, whereas from 0.167 to 0.333, the former is better than the latter. Our designed quantizer exhibits better BER than the uniform quantizer for the whole range of angles of interest, whereas the original $\Delta\Sigma$ modulator does not always.

Although the amplitude of the NSF of the original $\Delta\Sigma$ modulator is very small around 0, the BER of the original $\Delta\Sigma$ modulator is not that small. This is due to the fact that the quantization noise variance σ_q^2 in (10) is also a function of the angle and is large when the angle is around 0, which

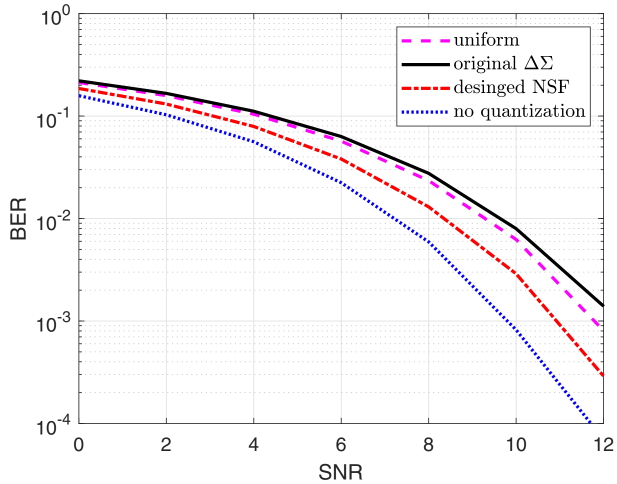

Fig. 7: BERs at $\theta = \pi/3$ with designed quantizer, original $\Delta\Sigma$ modulator, and uniform quantizer compared with BER without quantization, when $N = 64$, $d/\lambda = 1/8$ and $\Theta = \pi/3$.

has been pointed out in [23] and reported by [13].

Fig. 6 compares BERs for the arrival angle $\theta = \pi/6$, which is the half of the angular spread Θ. The BERs of our designed quantizer is almost identical with the BERs of the original $\Delta\Sigma$ modulator.

Fig. 7 depicts BERs for the arrival angle $\theta = \pi/3$, which is identical with the angular spread Θ. Our designed quantizer enjoys better performance than the uniform quantizer, whereas the original $\Delta\Sigma$ modulator does worse performance. This demonstrates the robustness of our designed quantizer to arrival angles.

V. Conclusion

We have developed one-bit error-feedback quantization for massive MIMO systems with uniform linear antenna arrays. We have designed the optimal error-feedback quantizer that maximizes the worst SNR for the signals whose arrival angles are in a fixed range of interest. Numerical results are provided to show that our one-bit spatial noise shaping exhibits flat BER performance for the prescribed range of angles.

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