On Maximum Likelihood DOA Estimation for Space Surveillance Radar

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Abstract—The detection and tracking of space targets moving in the low Earth orbit (LEO) region is an important emerging radar application. In this paper, we consider maximum likelihood direction of arrival (DOA) estimation for LEO targets using a phased array radar system. We validate a linear DOA motion model using a Kepler orbit assumption and demonstrate the effect of the chosen model for the estimation results in terms of DOA estimation bias, variance and SNR loss.

Index Terms—Radar signal processing, direction-of-arrival estimation, phased arrays, space surveillance radar

I. INTRODUCTION

Direction of arrival (DOA) estimation is an important task of a phased array radar system. Typically, the signal processing relies on applying the matched filter (MF). Locating the DOA that maximizes the MF output is equivalent to maximum likelihood (ML) estimation (considering a single target) [1], [2]. For a given array, the signal-to-noise ratio (SNR) needs to be increased for improving the estimation accuracy of ML. This can be achieved by integrating a number of pulses during a coherent processing interval (CPI). When estimating the parameters of a moving target, the MF signal model needs to include all of the target motion parameters that affect the phase of the received signal. Otherwise, the SNR and the motion estimation accuracy degrade.

An important emerging radar application is space surveillance: the detection and tracking of targets moving in the low Earth orbit (LEO) region. In this space surveillance scenario, a ground-based radar system is used to accurately determine the orbits of satellites and space debris particles to avoid possible collisions [3]. A number of advanced systems are currently operating or being developed for this purpose [4]–[7].

Due to the kinematics of LEO targets, their motion parameters (range, Doppler, and DOA) change significantly even for relatively short CPI lengths of less than a second. The range and Doppler motion models and MF formulation for LEO targets have been considered in [8]–[10]. To obtain the best possible SNR gain from the entire CPI, the DOA motion model also needs to be carefully analyzed. An inadequate model leads to degraded DOA estimation accuracy (increased bias and variance), which results in a poorer orbit estimation accuracy.

Previously, ML DOA estimation has been considered in the case of changing target amplitude in [11], [12]. However, these references only take into account the amplitude change due the antenna scan pattern, not the DOA change due the target motion. Estimating the DOA of moving targets has been previously considered e.g. in [13] in the context of a passive phased array. In this paper, we apply a similar approach for an active radar sensor and LEO targets.

The contribution of this paper is two-fold:

- 1) We validate a linear DOA motion model during the CPI for a LEO target using a Kepler orbit assumption, and
- 2) we demonstrate the effect of the chosen model for the estimation results in terms of DOA bias, variance and SNR loss.

The paper is structured as follows: After introducing the background of ML DOA estimation in Section II, we analyze the DOA kinematics of Kepler targets and formulate a linear DOA motion model in Section III. Section IV presents a numerical validation of the model and demonstrates the achieved estimation accuracy, while Section V concludes our findings.

II. THEORETICAL BACKGROUND

In this section, we describe the signal model and the conventional ML DOA estimator. We denote by θ the elevation angle of the target, and by ϕ the azimuth angle. The directional cosine vector is defined as $\boldsymbol{u} = [\boldsymbol{u} \ \boldsymbol{v}]^T = [\sin \theta \cos \phi \ \sin \theta \sin \phi]^T$.

A. Signal Model

We consider a 2D phased array radar system consisting of N elements. When N is large, the radar system typically employs a so-called beamspace transformation to reduce the data dimension. The reduced dimension leads to a reduced computational burden for the DOA estimation. We use the signal model described in [14], [15] for the beamspace data of the *m*th pulse (m = 0, ..., K - 1), i.e.

$$\boldsymbol{s}_m = \boldsymbol{B}^H \boldsymbol{a}_m(\boldsymbol{u}_m) \boldsymbol{d}(\boldsymbol{u}_m) + \boldsymbol{B}^H \boldsymbol{n}_m, \quad (1)$$

where $\boldsymbol{B} \in \mathbb{C}^{N \times L}$ is the beamformer, $\boldsymbol{d}(\boldsymbol{u}) \in \mathbb{C}^{N \times 1}$ is the array steering vector, $a_m(\boldsymbol{u}) \in \mathbb{C}$ is the complex amplitude, and $\boldsymbol{n}_m \in \mathbb{C}^{N \times 1}$ is complex white Gaussian noise. Furthermore, L is the beamspace dimension (i.e. the number of receive channels). We consider a CPI comprising K pulses, during which the DOA \boldsymbol{u}_m changes according to the motion



Fig. 1. A schematic illustration of using an inadequate DOA motion model in ML estimation. The various undesirable effects are marked in purple.

of the target. In addition, the amplitude a_m also changes due to the changing target range and location inside the transmit beam of the array.

B. Maximum Likelihood Estimation

Assuming the target signal is independent from pulse to pulse, the ML estimator maximizes the joint probability density function [16]

$$p(\boldsymbol{s}_0, \dots, \boldsymbol{s}_{K-1} | \boldsymbol{a}_0, \dots, \boldsymbol{a}_{K-1}, \boldsymbol{u}_0, \dots, \boldsymbol{u}_{K-1}, \boldsymbol{Q}) = \prod_{m=0}^{K-1} p(\boldsymbol{s}_m | \boldsymbol{a}_m, \boldsymbol{u}_m, \boldsymbol{Q}_m),$$
(2)

where the likelihood function of the mth pulse (assuming white Gaussian noise) is

$$p(\boldsymbol{s}_{m}|\boldsymbol{a}_{m},\boldsymbol{u}_{m},\boldsymbol{Q}_{m}) = \frac{1}{\pi^{N}|\boldsymbol{Q}_{m}|}e^{-(\boldsymbol{s}_{m}-\boldsymbol{a}_{m}\boldsymbol{V}(\boldsymbol{u}_{m}))^{H}\boldsymbol{Q}_{m}^{-1}(\boldsymbol{s}_{m}-\boldsymbol{a}_{m}\boldsymbol{V}(\boldsymbol{u}_{m}))}$$
(3)

and $V(\boldsymbol{u}_m) = \boldsymbol{B}^H \boldsymbol{d}(\boldsymbol{u}_m)$. For the case of identically and independently distributed noise samples, the $L \times L$ noise covariance matrix is $\boldsymbol{Q}_m = \boldsymbol{Q} = \boldsymbol{B}^H \boldsymbol{R} \boldsymbol{B}$, where $\boldsymbol{R} = E \{\boldsymbol{n} \boldsymbol{n}^H\}$.

The alternative approach assumes a constant DOA and amplitude for all the pulses within the CPI. Thus, maximizing (2) simplifies into [2]

$$\hat{\boldsymbol{u}} = \arg\max_{\boldsymbol{u}} \sum_{m=0}^{K-1} \left| \boldsymbol{w}^{H}(\boldsymbol{u}) \boldsymbol{s}_{m} \right|^{2}$$
(4)

with the weight vector

$$\boldsymbol{w}(\boldsymbol{u}) = \frac{\boldsymbol{Q}^{-1}\boldsymbol{V}(\boldsymbol{u})}{\sqrt{\boldsymbol{V}^{H}(\boldsymbol{u})\boldsymbol{Q}^{-1}\boldsymbol{V}(\boldsymbol{u})}}.$$
 (5)

The amplitude estimation of the mth pulse can be solved in closed form as [2]

$$\hat{a}_m(\hat{\boldsymbol{u}}) = \frac{\boldsymbol{V}^H(\hat{\boldsymbol{u}})\boldsymbol{Q}^{-1}\boldsymbol{s}_m}{\boldsymbol{V}^H(\hat{\boldsymbol{u}})\boldsymbol{Q}^{-1}\boldsymbol{V}(\hat{\boldsymbol{u}})},\tag{6}$$

and the estimated integrated power as

$$\hat{A} = \sum_{m=0}^{K-1} \left| \hat{a}_m(\hat{u}) \right|^2 = K \left| \hat{a}(\hat{u}) \right|^2.$$
(7)

For every \hat{u} under test, a detection is declared according to the well-known Neyman-Pearson test [16]: A target is detected if \hat{A} is bigger than a pre-defined threshold corresponding to a maximum allowed false alarm rate.

When the DOA changes linearly from pulse to pulse (i.e. $\Delta u_m = u_m - u_{m-1} = c$, where $c \in \mathbb{R}^{2 \times 1}$ is a constant vector) and the amplitude a_m is constant, the ML DOA estimation result from (4) is the average DOA during the CPI. The likelihood function in (4) is a sum of single pulse contributions, which amplify each other the most for the middle value $u_{\lfloor K/2 \rfloor}$. However, when the received signal amplitude changes with every pulse, the estimate is no longer the average. For example, if the signal amplitude monotonically decreases from pulse to pulse, there is an estimation bias towards the DOA at the beginning of the CPI.

From (6) we also see that a bias in \hat{u} decreases the magnitude of the estimated power – degrading the SNR. As a result, the probability of detection P_D is also decreased. Furthermore, a degraded SNR leads to an increased estimation variances σ_u^2 and σ_a^2 for both the target DOA and amplitude, respectively. This process is illustrated in Fig. 1. To avoid these undesirable effects, the constant DOA model must be replaced by a suitable time-dependent model in the ML estimation.

III. ML DOA ESTIMATION FOR LEO TARGETS

In this section, we consider the motion model required for accurate DOA estimation of a Keplerian target. First, we consider the exact DOA model during the CPI. Then, we formulate the ML problem with a time-dependent DOA model u_m . Without loss of generality, we analyze orbits with $\phi = 0$, i. e. a case where the target's orbit passes right above the radar station.

A. Kepler Orbit Kinematics

We base our formulations on [17] to model the DOA change as a function of time, during a single CPI. Several basic relationships help to find the true anomaly ϑ (angle from perigee, taken from Earth's center) as a function of time. The distance between the target and Earth'c center is given by

$$R_{\rm orb} = a \frac{1 - e_c^2}{1 + e_c \cos \vartheta},\tag{8}$$

where $a = (r_p + r_a)/2$, and r_p , r_a are the perigee and apogee radius, respectively. The orbit's eccentricity is e_c . The period of the orbit can be expressed as

$$\Gamma = 2\pi \sqrt{\frac{a^3}{\mu}} \tag{9}$$

where $\mu = 398600$ is the standard gravitational constant. The mean anomaly (auxiliary variable) is given by

$$M_e(t) = \frac{2\pi t}{T},\tag{10}$$



Fig. 2. The directional cosine u (top) and its change rate du/dt (bottom) of a Keplerian target for a single CPI. The observed behavior is linear to a very high degree of accuracy.

and the eccentric anomaly E_a is expressed in

$$M_e(t) = E_a(t) - \sin E_a(t).$$
 (11)

The angle ϑ can be calculated as a function of time as

$$\vartheta(t) = 2\tan^{-1}\left(\sqrt{\frac{1+e_c}{1-e_c}}\tan\frac{E_a(t)}{2}\right).$$
 (12)

Due to the (11), the angle $\vartheta(t)$ during the CPI needs to be evaluated numerically. This procedure is carried out as follows: Given the length of the CPI, denoted as $T_{\rm CPI}$, we can evaluate M_e from (10) as a function of time for $0 \le t \le T_{\rm CPI}$. The next step is solving the transcendental equation (11) using an iterative procedure as described in [17]. Finally, the values of E_a are used to compute $\vartheta(t)$ according to (12).

The final step converts the true anomaly ϑ into the radar's elevation angle θ . We use the cosine theorem to express r as

$$r = \sqrt{R_{\rm E}^2 + R_{\rm orb}^2 - 2R_{\rm E}R_{\rm orb}\cos\vartheta},$$
 (13)

and the sine theorem to calculate $\boldsymbol{\gamma}$ as

$$\sin \gamma = \frac{R_{\rm orb}}{r} \sin \vartheta. \tag{14}$$

We denote Earth's radius as $R_{\rm E}$. The radar elevation angle is $\theta = \pi - \gamma$. Plugging in r from (13) into (14) yields

$$\theta(t) = \pi - \arcsin\left(\frac{R_{\rm orb}\sin\vartheta(t)}{r}\right).$$
(15)

B. ML with Linear DOA Motion

In the case where the target DOA during the CPI can be approximated by a linear model, the DOA of each pulse can be expressed as

$$\boldsymbol{u}_m = \begin{bmatrix} u_0 + \alpha_u m\\ v_0 + \alpha_v m \end{bmatrix} \tag{16}$$

with initial position $u_0 = [u_0 \ v_0]^T$ and slope $\boldsymbol{\alpha} = [\alpha_u \ \alpha_v]^T$. In contrast to the constant model in (4), there are now two additional parameters to estimate. The ML estimation problem is now given by

$$(\hat{\boldsymbol{u}}_{0}, \hat{\boldsymbol{\alpha}}) = \operatorname*{arg\,max}_{\boldsymbol{u}_{0}, \boldsymbol{\alpha}} \sum_{m=0}^{K-1} \left| \boldsymbol{w}^{H} (\boldsymbol{u}_{0} + \boldsymbol{\alpha}m) \boldsymbol{s}_{m} \right|^{2}.$$
(17)

This was solved using an iterative 2D grid search. The estimated DOA of the mth pulse is now

$$\hat{\boldsymbol{u}}_m = \hat{\boldsymbol{u}}_0 + \hat{\boldsymbol{\alpha}}(m-1), \tag{18}$$

and the estimated average DOA is given by

$$\hat{\boldsymbol{u}} = E\left[\hat{\boldsymbol{u}}_m\right],\tag{19}$$

where $E[\cdot]$ denotes the mean. The estimated integrated power from (7) is modified to yield

$$\hat{A} = \sum_{m=0}^{K-1} |\hat{a}_m(\hat{u}_m)|^2 \,. \tag{20}$$

IV. NUMERICAL RESULTS

In this section, we demonstrate the DOA estimation of LEO targets with the GESTRA system [6], [9]. The radar parameters for the simulations are the following: The transmit beamwidth of the array $BW = 6^{\circ}$, the number of pulses during the CPI K = 24 with a PRF of 30 [Hz], resulting in $T_{\rm CPI} = 0.8$ s, the number of antenna elements N = 256. The beamformer **B** used transformed the data into L = 16 steered sumbeams with a rectangular pattern of beam positions, corresponding to the paving beamformer in [14].

To intercept signals outside of the 3 dB Tx BW, we consider for this scenario a spatial Rx coverage area of $BW_{Rx} = 12^{\circ}$, $300 \le r \le 3000$ km and eccentricity $0 \le e_c \le 0.25$. Since we set $\phi = 0^{\circ}$, we note that $\boldsymbol{u} = [\sin \theta \ 0]^T$ and treat the DOA as $u = \sin \theta$ in the remainder of the text.

A. Linear Model Validation

To validate that a linear model is accurate for a LEO Keplerian target during a single CPI, we consider the worst case scenario (maximum possible acceleration in DOA). If the linear model holds for this scenario, it applies for every other possible target scenario as well.

For this purpose, we chose an elliptical orbit, where the zenith of the radar and the orbit's perigee are pointing in the



Fig. 3. Comparison of location estimation biases between the constant (top) and linear model (center). SNR comparison of both models (bottom) shows negligible values.

same direction. The target motion starts from zenith ($\theta = 0^{\circ}$ and u = 0) with increasing angle values. We choose the values of r = 300 km as the orbit height at perigee, and target eccentricity of $e_c = 0.25$ to produce the highest DOA change during the CPI.

We plot the DOA and DOA change rate (first order derivative) in Fig. 2 (for various target ranges). We see that the maximum DOA change (corresponding to the fastest moving target) within a single CPI is u = 0.022, which corresponds to $\theta = 1.36^{\circ}$. More importantly, the DOA has a constant change rate for all the ranges, with zero acceleration (below $2 \cdot 10^{-5} \text{ } 1/\text{s}^2$). We also see that as the target range increases, the DOA change rate is decreasing. Therefore, we conclude that a linear DOA motion model can be applied for GESTRA in the space surveillance application.

For other radar systems (and LEO objects), the linear model validity depends on the CPI length. A useful method for determining the maximum CPI length for any system for LEO targets has been considered in [10]. This method, which is based on the target's radar cross section decorrelation properties, gives the maximum CPI length of about 2 s for a relatively small satellite (of dimension $5.2 \text{ m} \times 2.11 \text{ m} \times 2.12 \text{ m}$). By extending the calculations shown in Fig. 2, we observed that a very long CPI with $T_{\text{CPI}} \approx 8 \text{ s using } 10K$ pulses still maintained a linear DOA behavior to a very high degree of accuracy (less than 3 % of relative change in du/dt).

In practice, the maximum achievable CPI length is limited by BW, since the target must be detectable during the entire CPI. Another contributing factor is the ratio $\Delta\theta_{CPI}/BW$ (which was 0.23 in our example), where $\Delta\theta_{CPI} = \theta_{K-1} - \theta_0$. When this ratio is high, the amplitude change during the CPI increases significantly. When the ratio is small, the amplitude stays approximately constant (excluding target rotation effects), and a constant DOA model yields close to optimal results.

B. Amplitude Calculation

To calculate the changing amplitude of the received signal within a single CPI, we use the simulated DOA motion from the previous section. For each value θ_m the signal amplitude a_m is calculated assuming a Gaussian beamshape with a beamwidth of BW degrees centered around the angle θ_{st} for the transmitted signal. This corresponds to a target motion within the CPI that is not centered around the array's steering angle (since the target starts from $\theta_0 = 0^\circ \neq \theta_{st}$).

C. Estimation Accuracy Comparison

Next, we compare the estimation accuracy of the linear motion model with the constant model. We calculate the estimation bias for targets within $r^i \in [300, 3000]$ km. For each target range r^i , we simulate a single target moving from $u_0^i = 0$ and r_0^i to u_{K-1}^i and r_{K-1}^i , with an amplitude changing from a_0^i to a_{K-1}^i . The amplitude change is determined by the location inside the beam and the change in range. The received signal from each pulse and target range is simulated using (1).

After simulating the received signal, we use two different ways of estimation: the constant DOA model (using (4) and (6)), and the linear DOA model (using (17) and (19)). Both estimators were implemented using a grid-based method, with an iteratively decreasing grid spacing.

The performance metric we use is the localization bias denoted as b_p . We calculate the target's true 2D location during a single CPI as

$$p_r = E[r_m]e^{iE[\theta_m]}.$$
(21)

The estimated 2D target location is

$$p_e = \hat{r}e^{i\theta}.$$
 (22)

The bias measures the distance between the two points as

$$b_p = |p_r - p_e|$$
. (23)

To avoid the effect of the range estimation error we assume that $\hat{r} = E[r_m]$.

Fig. 3 shows the bias comparison of the two methods for different beam center angles θ_{st} normalized by BW. The constant model exhibits an estimation bias up to 1.4 km for $\theta_{st} = BW$. In general, the bias decreases exponentially as the target range grows. For the linear model, the bias shrinks by three orders of magnitude for any beam center angle, the values being less than six meters. From a practical point of view, the bias vanishes completely, because the range resolution and accuracy of most space surveillance radar systems is much lower.

To calculate the SNR estimation bias (i.e. SNR loss due to the DOA estimation bias), we derive the integrated signal power during a single CPI from (7). We denote the true value, the constant model estimation result and linear model estimation result as A_r , A_c and A_l , respectively. Fig. 3 (bottom) shows the SNR estimation biases calculated as

$$b_{sc} = A_r / \hat{A}_c$$
 and $b_{sl} = A_r / \hat{A}_l$, (24)

for the constant (b_{sc}) and linear (b_{sl}) models.

Clearly, such low values ($b_{sc} < 0.05 \text{ dB}$ and $b_{sl} \approx 0 \text{ dB}$) are insignificant. We therefore conclude that the SNR loss, degraded probability of detection P_D , increased estimation variances σ_u^2 and σ_a^2 do not play an important role in this scenario. Nevertheless, this result needs to be validated for each considered system (due to different system parameters such as BW and T_{CPI}).

V. CONCLUSIONS

In this paper, we gave an overview of ML DOA estimation in a radar space surveillance scenario. By analyzing the orbital motion of LEO targets, we demonstrated that their DOA during the CPI can be accurately modeled as a linear function. For the conventional ML approach with a constant DOA and amplitude, a large localization bias was observed. By introducing the linear model, the bias almost completely vanished.

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