Partially Relaxed Fourier Domain Direction of Arrival Estimation

David Schenck, Minh Trinh-Hoang, and Marius Pesavento
Communication Systems Group, Technische Universität Darmstadt, Germany

***Abstract*—In this paper, the recently introduced concept of Partial Relaxation in Direction-of-Arrival (DoA) estimation [1], [2] is extended to a new class of Partially Relaxed Fourier Domain DoA estimation methods. The periodic cost function of the Partially Relaxed Deterministic Maximum Likelihood and the Partially Relaxed Weighted Subspace Fitting DoA estimation method are approximated by means of a truncated Fourier series expansion. The truncated Fourier series is expressed as a polynomial and the source DoAs are estimated from its roots, similar to root-MUSIC. Simulation results show that the proposed Partially Relaxed Fourier Domain DoA estimation techniques provide improved estimation accuracy over their conventional counterparts especially in difficult scenarios with limited sample size and low Signal-to-Noise Ratio.**

I. Introduction

DoA estimation is among the most classical research topics in signal processing with a wide field of applications including radar, sonar, seismic exploration, electronic surveillance and mobile communication [3]–[7]. A variety of DoA estimators has been proposed in the literature such as high resolution algorithms like Multiple Signal Classification (MUSIC) [8], the minimum variance method of Capon [9] and Estimation of Signal Parameters via Rotational Invariance Technique (ESPRIT) [10]. These conventional spectral search based estimators treat multi-source scenarios as single source scenarios and thereby ignore the presence of interfering sources. Hence, the dependence between the sources is neglected which allows to derive "low-cost" DoA estimators. However, the DoA estimation accuracy of these "low-cost" estimators strongly degrades in scenarios with closely spaced sources or large number of sources where the interference power increases [11], [12].

More recently, the Partial Relaxation (PR) framework has been introduced in [1], [2], [13] to overcome the aforementioned drawbacks of the conventional spectral-based DoA estimation techniques. Although, the presence of multiple sources is considered in the PR approach the computational demand is kept low due to a relaxation of the manifold structure of the interfering signal part. The manifold structure of the desired signal component remains unchanged. A closed-form solution for the relaxed undesired signal part is computed and substituted back into the initial optimization problem which considers multiple sources. A concentrated cost function is obtained that considers the dependence between the sources and admits simple spectral-based grid search. A computational efficient implementation of the partially relaxed DoA estimators which has similar computational complexity than the MUSIC DoA method can be found in [1].

This work was supported by the DFG PRIDE Project PE 2080/2-1 and the German Academic Exchange Service (DAAD)

A common way to further improve the estimation accuracy of an existing DoA estimation method is to express its cost function as a polynomial that can be rooted to estimate the DoAs [14]. This technique was successfully used in the root-MUSIC method which was obtained from the convention MUSIC cost function using a Uniform Linear Array (ULA) [15], [16]. Motivated by the performance improvement of root-MUSIC over conventional MUSIC, a Fourier Domain (FD) DoA estimation technique is proposed that is based on the PR approach and allows to estimate the DoAs by rooting a polynomial equation, similar to root-MUSIC. However, in comparison to root-MUSIC which is limited to ULAs only, the proposed Partially Relaxed FD DoA estimation techniques can be applied to any array geometry. The concept of approximating periodic cost functions by means of a truncated Fourier series was initially introduced in [17], [18] and can also be applied to DoA estimators under the PR framework. Simulation results reveal that the proposed Partially Relaxed FD DoA estimators provide a substantial performance improvement over the conventional PR methods especially at low Signal-to-Noise Ratios (SNRs) and in scenarios with limited sample size.

This paper is organized as follows. The signal model is introduced in Section II. The concept of conventional DoA estimators is introduced in Section III followed by the PR framework in IV. The proposed partially relaxed FD DoA estimation methods are introduced in Section V and simulation results are provided in Section VI. Finally, Section VII concludes this paper.

II. Signal Model

Consider a scenario with an antenna array that is equipped with M sensors and N impinging narrowband signals where $M > N$. The number of sources N is assumed to be known and the DoAs are denoted by $\boldsymbol{\theta} = [\theta_1 \ldots, \theta_N]^{\mathrm{T}}$. The full-rank steering matrix $\boldsymbol{A}(\boldsymbol{\theta}) \in \mathbb{C}^{M \times N}$ is given by

$$\boldsymbol{A}(\boldsymbol{\theta}) = [\boldsymbol{a}(\theta_1), \ldots, \boldsymbol{a}(\theta_N)],$$

where $\boldsymbol{a}(\theta_i) \in \mathbb{C}^M$ denotes the sensor array response of the i-th impinging signal. The transmitted source signal at time instant t is denoted by $\boldsymbol{s}(t)=[s_1(t),\ldots,s_N(t)]^{\mathrm{T}}\in\mathbb{C}^N$ and the received baseband signal $\boldsymbol{x}(t)\in\mathbb{C}^M$ is given by

$$\boldsymbol{x}(t)=\boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{s}(t)+\boldsymbol{n}(t),\quad t=1,\ldots,T,$$

where T denotes the number of snapshots and $\boldsymbol{n}(t)\in\mathbb{C}^M$ the sensor noise. The transmitted source signals and the noise are assumed to be statistically independent zero-mean circularly Gaussian distributed and the covariance matrix of the received signal is given by

$$\boldsymbol{R}=\boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{R}_{\mathrm{s}}\boldsymbol{A}(\boldsymbol{\theta})^{\mathrm{H}}+\sigma^2\boldsymbol{I}_M, \tag{1}$$

where $\boldsymbol{R}_{\mathrm{s}}=\mathbb{E}[\boldsymbol{s}(t)\boldsymbol{s}(t)^{\mathrm{H}}]$ denotes the covariance of the transmitted signal and $\sigma^2\boldsymbol{I}_M$ the noise covariance matrix. The covariance matrix $\boldsymbol{R}$ in (1) is unavailable in practice. Hence, the sample covariance matrix

$$\hat{\boldsymbol{R}}=\frac{1}{T}\sum_{t=1}^{T}\boldsymbol{x}(t)\boldsymbol{x}(t)^{\mathrm{H}},$$

is used instead. The eigenvalue decomposition of the sample covariance matrix is given by

$$\hat{\boldsymbol{R}}=\hat{\boldsymbol{U}}\hat{\boldsymbol{\Lambda}}\hat{\boldsymbol{U}}^{\mathrm{H}}=\hat{\boldsymbol{U}}_{\mathrm{s}}\hat{\boldsymbol{\Lambda}}_{\mathrm{s}}\hat{\boldsymbol{U}}_{\mathrm{s}}^{\mathrm{H}}+\hat{\boldsymbol{U}}_{\mathrm{n}}\hat{\boldsymbol{\Lambda}}_{\mathrm{n}}\hat{\boldsymbol{U}}_{\mathrm{n}}^{\mathrm{H}}, \tag{2}$$

where $\hat{\boldsymbol{\Lambda}}_{\mathrm{s}}\in\mathbb{R}^{N\times N}$ is a diagonal matrix containing the N-largest eigenvalues $\{\hat{\lambda}_1,\ldots,\hat{\lambda}_N\}$ and $\hat{\boldsymbol{U}}_{\mathrm{s}}\in\mathbb{C}^{M\times N}$ contains the associated N-principal eigenvectors of the sample covariance matrix $\hat{\boldsymbol{R}}$. Accordingly, $\hat{\boldsymbol{\Lambda}}_{\mathrm{n}}\in\mathbb{R}^{(M-N)\times(M-N)}$ and $\hat{\boldsymbol{U}}_{\mathrm{n}}\in\mathbb{C}^{M\times(M-N)}$ contain the $(M-N)$-noise eigenvalues $\{\hat{\lambda}_{M-N},\ldots,\hat{\lambda}_M\}$ and the associated noise eigenvectors, respectively. In the following, the conventional DoA estimation framework is introduced.

III. Conventional DoA Estimation Methods

In the conventional Maximum Likelihood (ML) DoA estimation framework, the DoAs $\boldsymbol{\theta}$ of the N signals are estimated by searching for the steering matrix $\boldsymbol{A}$ within the highly structured non-convex sensor array manifold which is denoted by

$$\mathcal{A}_N=\{\boldsymbol{A}|\boldsymbol{A}=[\boldsymbol{a}(\vartheta_1),\ldots,\boldsymbol{a}(\vartheta_N)],\ \vartheta_1<\cdots<\vartheta_N\}. \tag{3}$$

Multi-source ML DoA estimation problems generally take the form

$$\left\{\hat{\boldsymbol{A}}\right\}=\underset{\boldsymbol{A}\in\mathcal{A}_N}{\arg\min}\ f(\boldsymbol{A}), \tag{4}$$

where $f(\boldsymbol{A})$ denotes the multi-dimensional, non-convex cost function, following e.g. the Deterministic Maximum Likelihood (DML) [4] or the Weighted Subspace Fitting (WSF) criteria [19]. Due to the existence of multiple local minima, multi-source ML DoA estimation problems require computationally expensive multi-dimensional gird search to estimate the DoAs [19]–[21].

A common way to reduce the computational complexity is to find a sub-optimal solution to (4) by applying the single source approximation method [22], [23]. Instead of searching for the steering matrix $\boldsymbol{A}$ within the highly structured array manifold $\mathcal{A}_N$ of (3) only one source is considered while the remaining "interfering" sources are neglected. Hence, the feasible set of solutions in (4) reduces from $\mathcal{A}_N$ to $\mathcal{A}_1$ and $\boldsymbol{A}=\boldsymbol{a}\in\mathcal{A}_1$. Correspondingly, the dependence between the sources is not considered in the single source approximation. In general single source approximation optimization problems take the form

$$\{\hat{\boldsymbol{a}}\}={}^{N}\underset{\boldsymbol{a}\in\mathcal{A}_1}{\arg\min}\ f(\boldsymbol{a}),$$

where ${}^{N}\arg\min f(\cdot)$ denotes the N arguments at which the function $f(\cdot)$ attains its N-deepest separated local minima. A computational efficient one-dimensional grid search over the Field of View (FoV) of the sensor has to be performed in order to estimate the DoAs. However, since only one source is considered at a time and the dependence between the sources is neglected the estimation accuracy of single source approximation methods is usually worse than that of multi-source ML DoA estimation methods (see, e.g. [6]). In the following the PR framework is introduced which provides a good compromise between computational efficiency and high estimation accuracy.

IV. Partial Relaxation (PR) Framework

In the PR framework not only the signal from the "desired" direction is considered but also the signals from other "interfering" directions, hence the dependence between the sources is considered [1], [2]. However, the structure of the "interfering" signals is relaxed and the computational complexity is greatly reduced. Instead of searching for the steering matrix in the highly structured sensor array manifold in (3) the sensor array manifold is partially relaxed [1], [2]

$$\bar{\mathcal{A}}_N=\left\{\boldsymbol{A}|\boldsymbol{A}=[\boldsymbol{a}(\vartheta),\boldsymbol{B}],\ \boldsymbol{a}(\vartheta)\in\mathcal{A}_1,\ \boldsymbol{B}\in\mathbb{C}^{M\times(N-1)}\right\}. \tag{5}$$

Only the manifold structure of the first column of the steering matrix $\boldsymbol{A}$ is maintained which corresponds to the "desired" direction whereas the structure of the "interfering" signals is relaxed to an arbitrary matrix $\boldsymbol{B}$. The DoA estimators under the PR framework are obtained by replacing the highly structure array manifold $\mathcal{A}_N$ in (4) by the partially relaxed manifold $\bar{\mathcal{A}}_N$ in (5)

$$\{\hat{\boldsymbol{a}}_{\mathrm{PR}}\}={}^{N}\underset{\boldsymbol{a}\in\mathcal{A}_1}{\arg\min}\ \min_{\boldsymbol{B}}\ f([\boldsymbol{a},\boldsymbol{B}]). \tag{6}$$

The inner optimization problem with respect to $\boldsymbol{B}$ in (6) is solved in closed-form and substituted back into the cost function. Afterwards, a computationally efficient one-dimensional spectral search on $\boldsymbol{a}(\vartheta)\in\mathcal{A}_1$ is applied to search for the N-deepest local minima of the concentrated cost function. Next, the Partially Relaxed Deterministic Maximum Likelihood (PR-DML) and the Partially Relaxed Weighted Subspace Fitting (PR-WSF) DoA estimation methods are introduced.

A. Partially Relaxed Deterministic Maximum Likelihood (PR-DML)

The PR-DML DoA estimation technique is derived by applying the concept shown in (6) to the conventional DML cost function [4]

$$\{\hat{\boldsymbol{a}}_{\text{PR-DML}}\}={}^{N}\underset{\boldsymbol{a}\in\mathcal{A}_1}{\arg\min}\ \min_{\boldsymbol{B}}\ \operatorname{tr}\left[\boldsymbol{P}^{\perp}_{[\boldsymbol{a}(\vartheta),\boldsymbol{B}]}\hat{\boldsymbol{R}}\right], \tag{7}$$

where $\boldsymbol{P}_{\boldsymbol{A}} = \boldsymbol{A}(\boldsymbol{A}^{\mathrm{H}}\boldsymbol{A})^{-1}\boldsymbol{A}^{\mathrm{H}}$ denotes the projection matrix onto the subspace spanned by the columns of $\boldsymbol{A}$ and $\boldsymbol{P}_{\boldsymbol{A}}^{\perp} = \boldsymbol{I}_M - \boldsymbol{P}_{\boldsymbol{A}}$ denotes the corresponding orthogonal projection matrix. Solving the inner optimization problem in (7) with respect to $\boldsymbol{B}$ and substituting the optimal solution for $\boldsymbol{B}$ into the cost function we obtain the concentrated cost function [1], [2]

$$\begin{aligned} f_{\text{PR-DML}}(\vartheta) =& \min_{\boldsymbol{B}} \ \operatorname{tr}\left[\boldsymbol{P}_{[\boldsymbol{a}(\vartheta),\boldsymbol{B}]}^{\perp}\hat{\boldsymbol{R}}\right] \\ =& \sum_{k=N}^{M} \lambda_k\left(\boldsymbol{P}_{\boldsymbol{a}(\vartheta)}^{\perp}\hat{\boldsymbol{R}}\right), \end{aligned} \tag{8}$$

where $\lambda_k(\cdot)$ denotes the k-th largest eigenvalue of the matrix in the argument.

B. Partially Relaxed Weighted Subspace Fitting (PR-WSF)

Applying the PR framework to the conventional ML WSF DoA method [19] yields

$$\{\hat{\boldsymbol{a}}_{\text{PR-WSF}}\} = {}^{N}\arg\min_{\boldsymbol{a}\in\mathcal{A}_1} \ \min_{\boldsymbol{B}} \ \operatorname{tr}\left[\boldsymbol{P}_{[\boldsymbol{a},\boldsymbol{B}]}^{\perp}\hat{\boldsymbol{U}}_{\mathrm{s}}\boldsymbol{W}\hat{\boldsymbol{U}}_{\mathrm{s}}^{\mathrm{H}}\right], \tag{9}$$

where $\boldsymbol{W} \in \mathbb{C}^{N\times N}$ denotes a positive semidefinite weighting matrix. It was shown in [19] that the estimation error of the WSF DoA method asymptotically achieves the Cramer-Rao Bound (CRB) as the number of snapshots T tends to infinity if the weighting matrix is chosen as

$$\boldsymbol{W} = \left(\hat{\boldsymbol{\Lambda}}_{\mathrm{s}} - \hat{\sigma}^2\boldsymbol{I}_N\right)^2\hat{\boldsymbol{\Lambda}}_{\mathrm{s}}^{-1},$$

where $\hat{\sigma}^2 = \frac{1}{M-N}\operatorname{tr}[\hat{\boldsymbol{\Lambda}}_{\mathrm{n}}]$. The concentrated PR-WSF cost function is obtained by solving the inner optimization problem in (9) with respect to $\boldsymbol{B}$ and is given by [1], [2]

$$\begin{aligned} f_{\text{PR-WSF}}(\vartheta) =& \min_{\boldsymbol{B}} \ \operatorname{tr}\left[\boldsymbol{P}_{[\boldsymbol{a},\boldsymbol{B}]}^{\perp}\hat{\boldsymbol{U}}_{\mathrm{s}}\boldsymbol{W}\hat{\boldsymbol{U}}_{\mathrm{s}}^{\mathrm{H}}\right] \\ =& \sum_{k=N}^{M} \lambda_k\left(\boldsymbol{P}_{\boldsymbol{a}(\vartheta)}^{\perp}\hat{\boldsymbol{U}}_{\mathrm{s}}\boldsymbol{W}\hat{\boldsymbol{U}}_{\mathrm{s}}^{\mathrm{H}}\right). \end{aligned} \tag{10}$$

Next, the proposed Partially Relaxed FD DoA estimation technique is introduced.

V. Partially Relaxed Fourier Domain DoA Estimation Methods

In this Section, we combine the previously introduced PR framework [1] and the FD DoA estimation technique that was introduced in [17]. Two novel Partially Relaxed FD DoA estimators are introduced that allow to estimate the DoAs by rooting a polynomial equation, similar to root-MUSIC [15]. It can be observed that the PR-DML and the PR-WSF cost function in (8) and (10) are periodic in ϑ with the period 2π. The 2π-periodicity allows to equivalently express both cost function using a Fourier series expansion

$$f(\vartheta) = \sum_{m=-\infty}^{\infty} F_m e^{\mathrm{j}m\vartheta}, \tag{11}$$

where $f(\vartheta)$ denotes a generic 2π-periodic cost function and the Fourier series coefficients are given by

$$F_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\vartheta)e^{-\mathrm{j}m\vartheta}\mathrm{d}\vartheta. \tag{12}$$

The generic cost function $f(\vartheta)$ can be approximated by truncating the Fourier series in (11) to $2N_{\mathrm{D}} - 1$ points [17], [18] according to

$$f(\vartheta) \simeq \sum_{m=-(N_{\mathrm{D}}-1)}^{N_{\mathrm{D}}-1} F_m e^{\mathrm{j}m\vartheta} \triangleq \tilde{f}(\vartheta). \tag{13}$$

Using $z = e^{\mathrm{j}\vartheta}$ the truncated Fourier series in (13) can be expressed as a polynomial in z

$$\tilde{f}(\vartheta) = \sum_{m=-(N_{\mathrm{D}}-1)}^{N_{\mathrm{D}}-1} F_m z^m \triangleq \tilde{f}(z).$$

The Fourier series coefficients F_m for $m = -(N_{\mathrm{D}} - 1), \ldots, (N_{\mathrm{D}} - 1)$ in (12) can be approximated using the Discrete Fourier Transform (DFT) [17], [18]

$$F_m \simeq \frac{1}{2\pi}\sum_{l=-(N_{\mathrm{D}}-1)}^{N_{\mathrm{D}}-1} f(l\Delta\vartheta)e^{-\mathrm{j}ml\Delta\vartheta}\Delta\vartheta \triangleq \hat{F}_m, \tag{14}$$

where $\Delta\vartheta = 2\pi/(2N_{\mathrm{D}} - 1)$. The use of the DFT in (14) allows to compute the Fourier coefficients in a computationally efficient way. However, due to aliasing effects that are introduced by sampling the cost function $f(l\Delta\vartheta)$ for $l = -(N_{\mathrm{D}}-1), \ldots, (N_{\mathrm{D}}-1)$ in (14), the obtained coefficients $\hat{F}_m$ will be different from those in (12). With this we can approximated the 2π-periodic generic cost function $f(\vartheta)$ using a FD polynomial of degree $2N_{\mathrm{D}} - 2$ [17], [18]

$$\tilde{f}(z) \simeq \sum_{m=-(N_{\mathrm{D}}-1)}^{N_{\mathrm{D}}-1} \hat{F}_m z^m \triangleq \hat{f}(z) \tag{15}$$

$$= \sum_{m=-(N_{\mathrm{D}}-1)}^{N_{\mathrm{D}}-1} \hat{F}_m e^{\mathrm{j}m\vartheta} \triangleq \hat{f}(\vartheta). \tag{16}$$

In order to estimate the DoAs by rooting the polynomial equation $\hat{f}(z)$ in (15) the cost function $f(\vartheta)$ has to satisfy $\lim_{T\to\infty} f(\vartheta) \geq 0$ where equality holds iff $\vartheta \in \{\theta_1, \ldots, \theta_N\}$. This condition is a necessary condition for the unbiasedness of the FD estimator as the number of snapshots T tends to infinity. Whereas the PR-WSF cost function in (10) fulfills this condition by default, the PR-DML cost function in (8) converges to $\lim_{T\to\infty} f_{\text{PR-DML}}(\vartheta) = (M - N)\sigma^2$ for $\vartheta \in \{\theta_1, \ldots, \theta_N\}$. Therefore, we replace the PR-DML cost function in (8) with

$$g_{\text{PR-DML}}(\vartheta) = f_{\text{PR-DML}}(\vartheta) - \operatorname{tr}\left[\hat{\boldsymbol{\Lambda}}_{\mathrm{n}}\right], \tag{17}$$

such that $\lim_{T\to\infty} g_{\text{PR-DML}}(\vartheta) = 0$ for $\vartheta \in \{\theta_1, \ldots, \theta_N\}$. We remark that the local minima of the cost function $g_{\text{PR-DML}}(\vartheta)$ in (17) are identical to the ones of the cost function $f_{\text{PR-DML}}(\vartheta)$ in (8). However, due to the aforementioned unbiasedness condition, the FD method applied on the corrected cost function $g_{\text{PR-DML}}(\vartheta)$ in (17) is unbiased. On the other hand, direct application of the FD method on the cost function $f_{\text{PR-DML}}(\vartheta)$ in (8) results in a biased-estimator.

The roots of the FD polynomial in (15) appear in two different groups of root pairs. The first group of root pairs contains the root pairs that lie exactly on the unit circle and are caused by two sign changes of $\hat{f}(\vartheta)$ in (16). Although the modified PR-DML cost function $g_{\text{PR-DML}}(\vartheta)$ in (17) and the PR-WSF cost function $f_{\text{PR-WSF}}(\vartheta)$ in (10) are both non-negative functions by definition their corresponding FD approximation in (16) may take values that are slightly below zero in some of its minima [17]. This is due to the approximation error made by truncating the Fourier series and the aliasing effects that are introduced by using the DFT to compute the Fourier coefficients. As a direct consequence, each pair of roots of the polynomial $\hat{f}(z)$ in (15) that corresponds to a point where $\hat{f}(\vartheta)$ in (16) takes values smaller than zero (changes its sign twice) will lie exactly on the unit circle.

The second group of root pairs contains the roots that appear in conjugate reciprocal pairs and do not lie on the unit circle. It was shown in [17] that the polynomial $\hat{f}(z)$ in (15) satisfies the so-called conjugate reciprocity property which states the following. Assuming that z_0 is a root of $\hat{f}(z)$ in (15) that does not lie exactly on the unit circle, then $1/z_0^*$ is a root of the polynomial as well.

The procedure that is used to estimate the DoAs from the roots of the polynomial equation $\hat{f}(z)$ in (15) is as follows [17]:

- **Step 1**: Take the root that is closest to the unit circle.
- **Step 2**: Assign the root to one of the groups by verifying if its conjugate reciprocal value is a root as well.
- **Step 3**: If the root belongs to the first group, the corresponding DoA is estimated by taking the average of this root and its closest neighbor. Drop both roots and go to step 5.
- **Step 4**: If the root belongs to the second group, then use it to estimate the DoA. Drop this root and its conjugate reciprocal.
- **Step 5**: If fewer than N DoAs have been estimate, then go to step 1. Otherwise, stop.

In the following simulation results are provided.

VI. Simulation Results

In this Section, simulation results of the Root-Mean-Squared-Error (RMSE) performance of different DoA estimators are compared to the stochastic CRB [24]. All simulations are conducted for $N_{\text{R}} = 4000$ independent Monte Carlo trials. The RMSE is used as performance indicator and computed as

$$\text{RMSE} = \sqrt{\frac{1}{N_{\text{R}} N} \sum_{r=1}^{N_{\text{R}}} \sum_{n=1}^{N} \left(\hat{\theta}_n^{(r)} - \theta_n \right)^2},$$

where both the estimated DoAs $\hat{\boldsymbol{\theta}}^{(r)} = [\hat{\theta}_1^{(r)}, \ldots, \hat{\theta}_N^{(r)}]^{\text{T}}$ and the true DoAs $\boldsymbol{\theta} = [\theta_1, \ldots, \theta_N]^{\text{T}}$ are sorted in ascending order. A ULA with $M = 10$ antennas is considered. The ULA describes a special case as the PR cost functions in (17) and (10) are periodic in ϑ with period π. Hence, only the roots that lie within $-\pi/2$ and $\pi/2$ are considered. Furthermore, we consider $N = 2$ uncorrelated sources at $\boldsymbol{\theta} = [40^\circ, 50^\circ]^{\text{T}}$. The transmitted signals are statistically independent with zero-mean and unit power and the SNR is given by $\text{SNR} = 1/\sigma^2$.

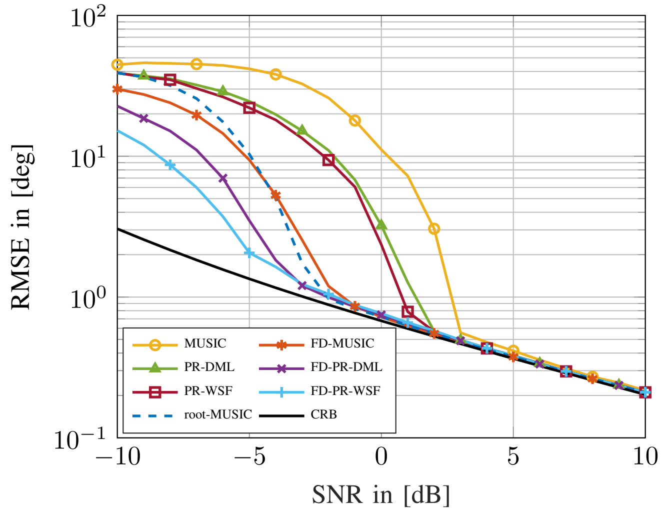

Fig. 1. Uncorrelated sources, number of snapshots $T = 40$

The cost functions of the PR-DML, the PR-WSF and the conventional MUSIC method are approximated by a polynomial of order $2N_{\text{D}} - 2$ where $N_{\text{D}} = 70$. We remark that due to the π-periodicity of the PR cost functions for ULAs, the polynomial order can be reduced to half of its size if the polynomial is used to approximate only one period of the cost function instead of two.

In Figure 1 the RMSE performance is investigated for different SNRs and $T = 40$ snapshots. It can be seen that both proposed methods FD-PR-DML and FD-PR-WSF outperform their spectral search based counterparts in terms of RMSE performance. Furthermore, both proposed methods provide better RMSE performance than conventional MUSIC, FD-MUSIC and root-MUSIC. Note that the threshold of the FD-PR-DML and the FD-PR-WSF method occurs at an even lower SNR than the one of root-MUSIC.

In Figure 2 the same scenario is considered with only $T = 15$ snapshots. Again, both proposed PR FD DoA estimators show improved RMSE performance and outperform their spectral search based counterparts. The proposed DoA estimators show enhanced threshold performance in comparison to the FD-MUSIC and the root-MUSIC method.

In Figure 3 the RMSE is investigated for a fixed SNR of -6dB and different numbers of snapshots. It can be observed that both proposed FD PR DoA estimation methods are superior to their spectral search based counterparts PR-DML and PR-WSF as they provide better RMSE performance especially in the smaller sample size region.

VII. Conclusion

In this paper, we have extended the class of PR DoA estimation methods by introducing two novel Partially Relaxed FD DoA estimation methods that allow to estimate the DoAs by rooting a polynomial equation. Both proposed FD DoA methods, the FD-PR-DML and the FD-PR-WSF exploit the 2π-periodicity of the corresponding spectral-based DoA

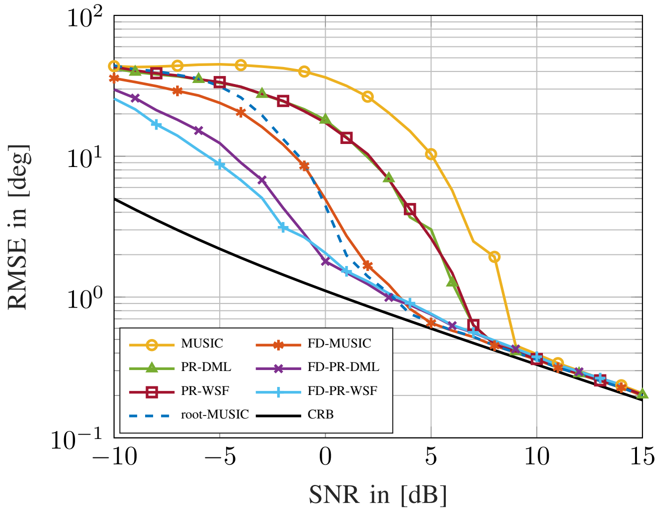

Fig. 2. Uncorrelated sources, number of snapshots $T = 15$

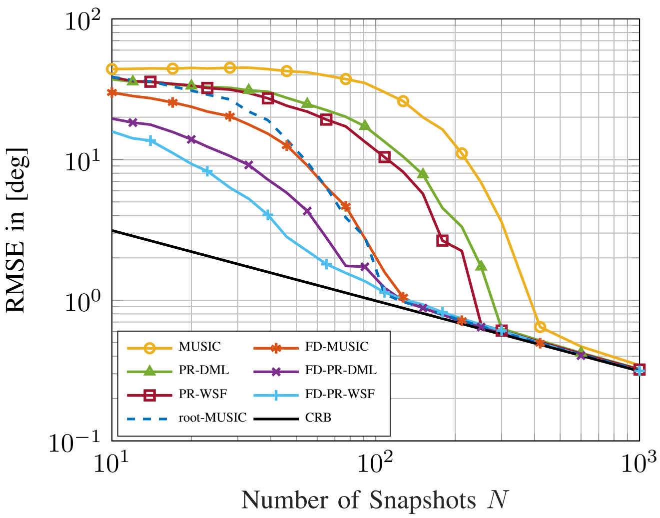

Fig. 3. Uncorrelated sources, SNR $= -6$dB

estimators. A truncated Fourier series expansion was used to approximate the PR-DML and the PR-WSF cost function and was reformulated as polynomial. Simulations have shown that the proposed Partially Relaxed FD DoA estimators provide RMSE performance that is superior to their spectral search based counterparts and root-MUSIC. Furthermore, the proposed Partially Relaxed FD methods are applicable to any type of array geometry whereas root-MUSIC is only applicable in case a ULA is used.

References

[1] M. Trinh-Hoang, M. Viberg, and M. Pesavento, "Partial Relaxation Approach: An Eigenvalue-Based DOA Estimator Framework," *IEEE Transactions on Signal Processing*, vol. 66, no. 23, pp. 6190–6203, Dec. 2018.

[2] ——, "An Improved DoA Estimator Based on Partial Relaxation Approach," *IEEE International Conference on Acoustics, Speech, and Signal Processing*, pp. 3246–3250, Apr. 2018.

[3] A. M. Rembovsky, A. V. Ashikhmin, V. A. Kozmin, and S. Smolskiy, *Radio Monitoring: Problems, Methods and Equipment*, ser. Lecture Notes in Electrical Engineering. Springer US, 2009.

[4] H. V. Trees, *Optimum Array Processing: Detection, Estimation, and Modulation Theory*, ser. Detection, Estimation, and Modulation Theory. Wiley, 2004.

[5] H. Krim and M. Viberg, "Two Decades of Array Signal Processing Research: The Parametric Approach," *IEEE Signal Processing Magazine*, vol. 13, no. 4, pp. 67–94, Jul. 1996.

[6] P.-J. Chung, M. Viberg, and J. Yu, *DOA Estimation Methods and Algorithms*, ser. Academic Press Library in Signal Processing. Elsevier, 2014, vol. 3.

[7] L. C. Godara, "Application of Antenna Arrays to Mobile Communications, Part II: Beam-Forming and Direction-of-Arrival Considerations," *Proceedings of the IEEE*, vol. 85, no. 8, pp. 1195–1245, Aug. 1997.

[8] R. Schmidt, "Multiple Emitter Location and Signal Parameter Estimation," *IEEE Transactions on Antennas and Propagation*, vol. 34, no. 3, pp. 276–280, Mar. 1986.

[9] J. Capon, "High-Resolution Frequency-Wavenumber Spectrum Analysis," *Proceedings of the IEEE*, vol. 57, no. 8, pp. 1408–1418, Aug. 1969.

[10] R. Roy and T. Kailath, "ESPRIT-Estimation of Signal Parameters Via Rotational Invariance Techniques," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. 37, no. 7, pp. 984–995, Jul. 1989.

[11] P. Stoica and A. Nehorai, "MUSIC, Maximum Likelihood and Cramér-Rao Bound: Further Results and Comparisons," *International Conference on Acoustics, Speech, and Signal Processing*, pp. 2605–2608 vol.4, May 1989.

[12] C. Vaidyanathan and K. M. Buckley, "Performance Analysis of the MVDR Spatial Spectrum Estimator," *IEEE Transactions on Signal Processing*, vol. 43, no. 6, pp. 1427–1437, Jun. 1995.

[13] D. Schenck, M. Trinh, H. X. Mestre, M. Viberg, and M. Pesavento, "Full Covariance Fitting DoA Estimation Using Partial Relaxation Framework," *European Signal Processing Conference (EUSIPCO)*, pp. 1–5, 2019.

[14] A. B. Gershman, M. Rübsamen, and M. Pesavento, "One- and Two-Dimensional Direction-of-Arrival Estimation: An Overview of Search-Free Techniques," *Signal Processing*, vol. 90, no. 5, pp. 1338–1349, 2010, special Section on Statistical Signal & Array Processing.

[15] A. Barabell, "Improving the Resolution Performance of Eigenstructure-Based Direction-Finding Algorithms," *IEEE International Conference on Acoustics, Speech, and Signal Processing*, vol. 8, pp. 336–339, Apr. 1983.

[16] B. D. Rao and K. V. S. Hari, "Performance Analysis of Root-Music," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. 37, no. 12, pp. 1939–1949, 1989.

[17] M. Rübsamen and A. B. Gershman, "Direction-of-Arrival Estimation for Nonuniform Sensor Arrays: From Manifold Separation to Fourier Domain MUSIC Methods," *IEEE Transactions on Signal Processing*, vol. 57, no. 2, pp. 588–599, 2009.

[18] ——, "Root-MUSIC Based Direction-of-Arrival Estimation Methods for Arbitrary Non-Uniform Arrays," *IEEE International Conference on Acoustics, Speech and Signal Processing*, pp. 2317–2320, 2008.

[19] M. Viberg and B. Ottersten, "Sensor Array Processing Based on Subspace Fitting," *IEEE Transactions on Signal Processing*, vol. 39, no. 5, pp. 1110–1121, May 1991.

[20] I. Ziskind and M. Wax, "Maximum Likelihood Localization of Multiple Sources by Alternating Projection," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. 36, no. 10, pp. 1553–1560, 1988.

[21] B. Ottersten, M. Viberg, P. Stoica, and A. Nehorai, *Exact and Large Sample Maximum Likelihood Techniques for Parameter Estimation and Detection in Array Processing*. Springer Berlin Heidelberg, 1993, pp. 99–151.

[22] A. Paulraj, B. Ottersten, R. Roy, A. Swindlehurst, G. Xu, and T. Kailath, *Subspace Methods for Directions-of-Arrival Estimation*, ser. Handbook of Statistics. Elsevier, 1993, vol. 10.

[23] P. Stoica and R. L. Moses, "Spectral Analysis of Signals," *Prentice-Hall*, 2005.

[24] P. Stoica, E. G. Larsson, and A. B. Gershman, "The Stochastic CRB for Array Processing: A Textbook Derivation," *IEEE Signal Processing Letters*, vol. 8, no. 5, pp. 148–150, 2001.