Comparison of Robust Hypothesis Tests for Fixed Sample Size and Sequential Observations

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***Abstract*—Fixed sample size and sequential performance of the asymptotically minimax robust hypothesis test is evaluated over a signal processing example and the results are compared to other well known robust hypothesis testing schemes and the nominal test. As a fixed sample size test around fifty samples are sufficient to observe the minimax properties of the asymptotically minimax robust test. Comparisons indicate that Huber's minimax robust test and the nominal test degrade their performances drastically when imposed to uncertainties due to modeling errors. Similarly, Dabak and Johnson's asymptotically robust test degrades its performance, hence it is not minimax robust. This indicates that choosing the right uncertainty model and the corresponding minimax test is crucial in applications. For the sequential test a new definition of minimax robustness is made. The simulations indicate that the new definition is satisfied by the asymptotically minimax robust test asymptotically.**

***Index Terms*—Hypothesis testing, detection, robustness, least favorable distributions, spectrum sensing**

I. Introduction

The detection of the presence or absence of an event is paramount to various applications such as cognitive radio, radar, sonar or communications, and can be realized by a binary hypothesis test [1]. The classical hypothesis testing is not robust in the sense that small deviations from the nominal model, due to for example non-Gaussian noise, outliers present in the collected data or modeling errors made prior to detection, can lead to large performance degradation [2]. In order to be able to deal with such cases usual approach is to extend the nominal model to a much wider model, which accepts a set of distributions under each hypothesis. A minimax robust test is then designed over these sets to minimize the worst case error probability. Hence, the designed test provides the best (in terms of minimum error probability) guaranteeable detection performance despite uncertainties imposed by the uncertainty model [3].

One of the earliest works in robust detection was presented by Huber in 1965, where he showed that the minimax robust test for the ϵ-contamination neighborhood was the clipped likelihood ratio test (CLRT) [4]. Moreover, he also showed that the sequential version of the same test was asymptotically minimax robust for the false alarm and miss detection probabilities. A more in dept analysis of sequential robustness of Huber's test can be found in [5]. Later Huber and Strassen showed that the CLRT was the minimax robust test for larger classes of distributions [6], [7]. Huber's ϵ-contamination neighborhood is a proper model to deal with outliers. However, in order to deal with modeling errors, according to Dabak et al. [8], it is more appropriate to consider distance based uncertainty classes since modeling errors do not result in abrupt changes on the distribution functions. While Dabak at el. considers the KL-divergence for large sample sizes (asymptotically), Levy [9] and Gül, [10], [11] consider the KL-divergence and the α-divergence for a single sample over randomized decision rules. The resulting test of Dabak is still a nominal likelihood ratio test but with a modified threshold, whereas the minimax robust test for the whole α-divergence neighborhood (including the KL-divergence as a special case) is a censored likelihood ratio test. Recently, asymptotically minimax hypothesis testing was derived both in Neyman-Person as well as in Bayesian sense [12]. Furthermore, it was theoretically shown that Dabak's test was not asymptotically minimax robust. However, the fixed sample size as well as sequential versions of the asymptotically minimax robust test have never been analyzed before.

In this paper fixed sample size and sequential performances of the asymptotically minimax robust hypothesis test have been evaluated considering a signal processing example from spectrum sensing. For comparison three other robust hypothesis testing schemes are considered [4], [8], [10]. Although the theory of asymptotically minimax robustness is well established, it is not known in which extend the word asymptotical can be used. Moreover, it is also not known whether asymptotically minimax robust test can remain minimax for sequential setup. In order to evaluate the latter, a new definition of sequential minimax robustness is made. This definition makes sense because for engineering applications, a guaranteed power of detection performance is sought for a fixed expected number of samples. It was shown that this new definition was satisfied by the asymptotically minimax robust hypothesis test asymptotically for the considered signal processing example.

The rest of this paper is organized as follows. In Section II, minimax robust hypothesis testing and its extension to fixed sample size and sequential tests are introduced. Furthermore, a new definition of sequential minimax robustness is made. In Section III, four different robust hypothesis testing schemes are presented. In Section IV, numerical results are provided in order to evaluate the performance of the asymptotically minimax robust hypothesis test in comparison to other robust tests. Finally in Section V, the paper is concluded.

II. Minimax Robust Hypothesis Testing

Let $Y_1, \dots, Y_n$ be a sequence of independent and identically distributed random variables with a common distribution G defined on a measurable space $(\Omega, \mathscr{A})$. Let furthermore F_0 and F_1 be the nominal, and G_0 and G_1 be the actual probability distributions having the density functions f_0, f_1, g_0 and g_1 with respect to a measure μ, respectively. Consider the following hypothesis testing problem

$$\begin{aligned} \mathcal{H}_0 &: G = G_0, \quad G_0 \in \mathscr{G}_0, \\ \mathcal{H}_1 &: G = G_1, \quad G_1 \in \mathscr{G}_1, \end{aligned}$$

where the distribution G equals G_0 or G_1, which belongs to disjoint uncertainty classes $\mathscr{G}_0$ and $\mathscr{G}_1$, if the hypothesis $\mathcal{H}_0$ or $\mathcal{H}_1$ is true. Most common way of defining the uncertainty classes is either based on a distance or on a model. Here we consider (in Section III A) the uncertainty classes based on the $\epsilon-$contamination model

$$\mathscr{G}_j = \{G_j | G_j = (1-\epsilon_j)F_j + \epsilon_j H_j, H_j \in \Xi, \}, \quad j \in \{0,1\},$$

where $0 \le \epsilon_j < 1$ and Ξ is the set of all probability measures on $(\Omega, \mathscr{A})$, as well as (in Section III B-D) the uncertainty classes

$$\mathscr{G}_j = \{G_j : D(G_j, F_j) \le \epsilon_j\}, \quad j \in \{0,1\}, \tag{1}$$

which are induced by the KL-divergence

$$D(G_j, F_j) = \int_\Omega \log\left(g_j / f_j\right) g_j \mathrm{d}\mu.$$

A. Fixed Sample Size Tests

Given the uncertainty classes, fixed sample size likelihood ratio test (the decision rule) is defined as

$$\delta = \begin{cases} 0, & l_n < t \\ \kappa & l_n = t \\ 1, & l_n > t \end{cases} \tag{2}$$

where

$$l_n = \log \prod_{k=1}^n \frac{dG_1(Y_k)}{dG_0(Y_k)} = \log \prod_{k=1}^n \frac{g_1(Y_k)}{g_0(Y_k)} = \sum_{k=1}^n \log \frac{g_1(Y_k)}{g_0(Y_k)} \tag{3}$$

is the log-likelihood ratio, $t \in \mathbb{R}$ is a threshold and $\kappa \in \{0,1\}$ is a random variable. Let $P_0 = P(\mathcal{H}_0)$ and $P_1 = P(\mathcal{H}_1)$ be the a priori probabilities of the hypotheses. Furthermore, let $P_F(\delta, G_0) = G_0[\delta = 1]$ and $P_M(\delta, G_1) = G_1[\delta = 0]$ define the false alarm and the miss detection probabilities, respectively, and

$$P_E(\delta, G_0, G_1) = P_0 P_F(\delta, G_0) + P_1 P_M(\delta, G_1)$$

define the overall error probability. Then, the objective of minimax decision making is to find a solution to

$$\begin{aligned} &\sup_{(G_0,G_1)\in\mathscr{G}_0\times\mathscr{G}_1} \min_{\delta\in\Delta} P_E(\delta, G_0, G_1) \\ &= \min_{\delta\in\Delta} \sup_{(G_0,G_1)\in\mathscr{G}_0\times\mathscr{G}_1} P_E(\delta, G_0, G_1) \end{aligned} \tag{4}$$

which implies a saddle value [13],

$$P_E(\hat{\delta}, G_0, G_1) \le P_E(\hat{\delta}, \hat{G}_0, \hat{G}_1) \le P_E(\delta, \hat{G}_0, \hat{G}_1), \tag{5}$$

where $\hat{\delta}$ is the robust decision rule, and $\hat{G}_0 \in \mathscr{G}_0$ and $\hat{G}_1 \in \mathscr{G}_1$ are the least favorable distributions (LFDs). Inequalities in 5 indicate that the minimax decision making guarantees a certain detection performance irrespective of the uncertainty imposed by $\mathscr{G}_0$ and $\mathscr{G}_1$ (left inequality) and this guaranteed performance is the best achievable among all other decision rules (right inequality).

B. Sequential Tests

Sequential probability ratio test (SPRT) was proposed by Wald as an alternative to classical hypothesis testing [14]. Let $l_0 = 0$ and at each iteration $k > 0$ let

$$l_k = l_{k-1} + \log \frac{g_1(Y_k)}{g_0(Y_k)}. \tag{6}$$

Then,

$$\delta = \begin{cases} 0, & l_n < t_l \\ \text{continue monitoring}, & t_l < l_n < t_u \\ 1, & l_n \ge t_u \end{cases} \tag{7}$$

defines the stopping rule for the stochastic process $(l_k)_{k>0}$ and

$$\tau = \min\{n \ge 1 : l_n \ge t_u \text{ or } l_n \le t_l\}$$

defines the stopping time, where $t_l \in \mathbb{R}^-$ and $t_u \in \mathbb{R}^+$ are the lower and upper thresholds of the SPRT. According to Huber an SPRT is minimax robust if

$$\begin{aligned} P_F(\delta(t_l,t_u), \hat{G}_0) &\ge P_F(\delta(t_l,t_u), G_0), \\ P_M(\delta(t_l,t_u), \hat{G}_1) &\ge P_M(\delta(t_l,t_u), G_1), \end{aligned} \tag{8}$$

and

$$\begin{aligned} \mathbb{E}_{\hat{G}_0}[\tau(t_l,t_u)] &\ge \mathbb{E}_{G_0}[\tau(t_l,t_u)], \\ \mathbb{E}_{\hat{G}_1}[\tau(t_l,t_u)] &\ge \mathbb{E}_{G_1}[\tau(t_l,t_u)], \end{aligned} \tag{9}$$

for all (t_l, t_u) and for all $(G_0, G_1) \in \mathscr{G}_0 \times \mathscr{G}_1$, where $\mathbb{E}_{G_j}$ denotes the expected value with respect to the distribution G_j. Notice that for minimax robustness Wald's approximations cannot be considered since the computations must be exact, which can be found e.g. in [10].

C. A New Definition of Sequentially Minimax Robustness

It is known that existing minimax robust hypothesis tests do not satisfy both (8) and (9) [3]. Moreover, it is not clear why both of these conditions must hold, because the related objective functions are not compatible, i.e. a test satisfying one does not imply satisfying the other. Here, we make another definition of sequential minimax robustness which is more practically oriented.

Definition II.1. Let $t_u = -ct_l$, $c > 0$, and

$$\begin{aligned} h_{G_0} &: \mathbb{E}_{G_0}[\tau(t_l,t_u)] \overset{n}{\mapsto} P_F(\delta(t_l,t_u), G_0), \\ h_{G_1} &: \mathbb{E}_{G_1}[\tau(t_l,t_u)] \overset{n}{\mapsto} P_M(\delta(t_l,t_u), G_1). \end{aligned}$$

Then, a sequential probability ratio test is minimax robust if

$$\begin{aligned} h_{\hat{G}_0}(n) &\geq h_{G_0}(n), \quad \forall n, \forall G_0 \in \mathscr{G}_0 \\ h_{\hat{G}_1}(n) &\geq h_{G_1}(n), \quad \forall n, \forall G_1 \in \mathscr{G}_1 \end{aligned} \tag{10}$$

where n is a common notation for $\mathbb{E}_{G_0}$ and $\mathbb{E}_{G_1}$. The test is called asymptotically minimax robust if (10) holds as $n \to \infty$.

Although the set of t_l and the corresponding t_u can be defined differently, the main idea remains the same; a sequential test satisfying Definition II.1 guarantees a certain level of detection performance, despite the uncertainty imposed by $\mathscr{G}_j$, for any given set of expected number of samples n.

III. Robust Hypothesis Testing Schemes

In this section four different robust hypothesis testing schemes are introduced. The following test is based on the $\epsilon-$contamination model whereas the latter three are derived elsewhere by considering the KL-divergence neighborhood.

A. Huber's Clipped Likelihood Ratio Test

The $\epsilon-$contamination classes of distributions are first considered by Huber in order to deal with outliers often present in the collected data [4]. The resulting minimax robust test, i.e. a solution to (4) is known to be the clipped likelihood ratio test with the robust likelihood ratio function

$$\hat{l}(y) = \frac{\hat{g}_1(y)}{\hat{g}_0(y)} = \begin{cases} bt_l, & l(y) \leq t_l \\ bl(y), & t_l < l(y) < t_u \\ bt_u, & l(y) \geq t_u \end{cases} \tag{11}$$

where $l = f_1/f_0$ is the nominal likelihood ratio function (LRF), y is the single observation, $b = (1-\epsilon_1)/(1-\epsilon_0)$ and $0 < t_l < t_u < \infty$. Huber's clipped likelihood ratio test is denoted as the (h)-test in the rest of the paper.

B. Dabak's Asymptotically Robust Hypothesis Test

It was first observed by Dabak and Johnson that the $\epsilon-$contamination model is not suitable for modeling errors and a smooth distance based uncertainty classes would be more appropriate. Considering the KL-divergence and large sample sizes, an asymptotically robust hypothesis testing scheme was derived. Surprisingly the test for n samples [2], [10]

$$\frac{1}{n}\sum_{k=1}^{n} \log l(y_k) \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} t^{'} = \frac{\log\left(\frac{s(1-v)}{s(u)} t^{1/n}\right)}{1-(u+v)} \tag{12}$$

is still a likelihood ratio test, but with a modified threshold $t^{'}$, where

$$s(u) = \int_{\mathbb{R}} f_1(y)^u f_0(y)^{1-u} \mathrm{d}y \tag{13}$$

and u and v are the variables to be determined based on the robustness parameters ϵ_0 and ϵ_1, [8], and t is the threshold of the nominal test e.g. $t = 1$. Dabak's asymptotically robust test is denoted as the $(a)^*$-test in the rest of the paper.

TABLE I
Robust Tests Used in Simulations for Comparison

Acronym	Description
(a)-test	Asymptotically minimax robust test [12]
(a*)-test	Dabak's asymptotically robust test [8]
(h)-test	Huber's clipped likelihood ratio test [4]
(m)-test	Minimax robust test for modeling errors [10]
(n)-test	Nominal test

C. Minimax Robust Hypothesis Test for Modeling Errors

The KL-divergence does not allow a minimax robust test to exist over deterministic decision rules [15]. This is actually the main motivation for Dabak et al. to go for asymptotics. However, a minimax robust test for a single sample (possibly multivariate) exists for the KL-divergence neighborhood if randomized decision rules are considered [9], [10]. For the most general case, the robust decision rule is given by [10]

$$\hat{\delta}(y) = \begin{cases} 0, & l(y) < t_l \\ \frac{\log(l(y)/t_l)}{\log(l_u/l_l)} & t_l \leq l(y) \leq t_u \\ 1, & l(y) > t_u \end{cases} \tag{14}$$

with the robust likelihood ratio function

$$\hat{l}(y) = t_l^{\hat{\delta}(y)-1} t_u^{-\hat{\delta}(y)} l(y). \tag{15}$$

The parameters t_l and t_u are again found based on ϵ_0 and ϵ_1. Notice that this test may not be extended to fixed sample size or sequential tests without loosing the minimax robustness property. This is due to the loss of randomization information provided by $\hat{\delta}$. The minimax robust test for modeling errors is denoted as the (m)-test in the rest of the paper.

D. Asymptotically Minimax Robust Hypothesis Test

It was shown theoretically in [12] that Dabak's test is not asymptotically minimax robust and the least favorable distributions of the asymptotically minimax robust test can be obtained by solving

$$\min_{u\in(0,1)} \max_{(G_0,G_1)\in\mathscr{G}_0\times\mathscr{G}_1} \int_{\Omega} {g_1}^u {g_0}^{1-u} \mathrm{d}\mu.$$

It was further derived that the LFDs for the KL-divergence neighborhood can be given as

$$\hat{g}_0 = \exp\left[\frac{-\lambda_0-\mu_0}{\lambda_0}\right] \exp\left[\frac{(1-u)\hat{l}^u}{\lambda_0}\right] f_0,$$

$$\hat{g}_1 = \exp\left[\frac{-\lambda_1-\mu_1}{\lambda_1}\right] \exp\left[\frac{u\hat{l}^{u-1}}{\lambda_1}\right] f_1,$$

and the robust likelihood ratio function $\hat{l}$ is obtained by solving

$$\hat{l} = \exp\left[\frac{u\hat{l}^{u-1}-\mu_1}{\lambda_1} + \frac{(u-1)\hat{l}^u+\mu_0}{\lambda_0}\right] l, \tag{16}$$

where λ_0, λ_1, μ_0, μ_1 are the Lagrangian parameters, which are also determined by solving a set of equations. For details see [12]. In the rest of the paper the asymptotically minimax robust test will be denoted as the (a)-test and the nominal likelihood ratio test will be denoted as the (n)-test.

IV. Numerical Results

In this section, accuracy and robustness of the (a)-test will be evaluated in comparison to the other robust tests which are listed in Table I. For solving all systems of equations damped Newton's method [16] is used. The notation $|_a^b$ stands for testing with the (a)-test while the data samples are obtained from the LFDs of the (b)-test.

Let us consider spectrum sensing used in cognitive radio to allow unlicensed or secondary users to use spectrum holes that are not occupied by licensed or primary users [17]. Presence or absence of a signal is formulated by a binary hypothesis test

$$\mathcal{H}_0 : y[n] = w[n], \quad n \in \{1, \ldots, N\}$$
$$\mathcal{H}_1 : y[n] = \theta x[n] + w[n], \quad n \in \{1, \ldots, N\}$$

where $w[n]$ are noise samples, $x[n]$ are unattenuated samples of the primary signal, $\theta > 0$ is the unknown channel gain and $y[n]$ are the received signal samples. Both the primary signal samples $x[n]$ and the noise samples $w[n]$, which are independent of $x[n]$, are i.i.d. standard Gaussian. Under each hypothesis, it is assumed that the distribution of Y may deviate from its nominal distribution by a factor of $\epsilon_0 = \epsilon_1 = 0.02$ with respect to the KL-divergence. Furthermore, the channel gain is assumed to be perfectly estimated as $\theta = \sqrt{3}$. The LFDs corresponding to the robust tests listed in Table I are found by solving the related equations provided in [4], [8], [10], [12]. The LFDs of the (h)-test are determined from the ϵ-contamination neighborhood such that $D(\hat{G}_0, F_0) = \epsilon_0$ and $D(\hat{G}_1, F_1) = \epsilon_1$. In Figure 1 the ratio of the robust LRF to the nominal LRF for four different robust hypothesis testing schemes is depicted. Both similarities and differences can be observed, and in particular $\hat{l}/l$ is not integrable for the (m)-test. For the aforementioned scenario, the goal is to evaluate the performance of the (a)-test, (a^*)-test, (h)-test and the (m)-test under fixed sample size and sequential concepts for various statistics of the data samples.

A. Fixed Sample Size Test

For a fixed number of samples $n \in \{1, \ldots, 100\}$, the performance of the robust tests are evaluated with Monte-Carlo simulations for 10^6 samples. The threshold of the fixed sample size test is set to $t = 0$. False alarm and miss detection probabilities of the (a)-test in comparison to that of the (a^*)-test are illustrated in Figures 2 and 3, when the tested data samples are obtained from the LFDs of the robust tests listed in Table I. In Figures 4 and 5 similar experiments are repeated for the (h)-test in comparison to the (n)-test. The following conclusions can be made from these experiments.

1) The (a)-test does not degrade its performance as the theory suggests.
2) The (a*)-test degrades its performance for the data samples obtained from the LFDs of the (a)- and (h)-tests, see Figure 2.

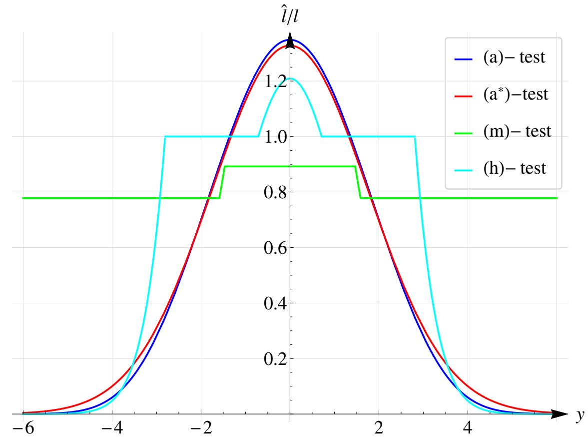

Fig. 1. The ratio of the robust likelihood ratio function to the nominal likelihood ratio function for various tests.

3) The data samples obtained from the LFDs do not always yield worse results than that of the nominal test, cf. $P_{M_a^n}$ with $P_{M_a^m}$ in Figures 3 and 5.
4) The performance of the (h)-test degrades significantly if indeed the uncertainties can be well modeled by the KL-divergence neighborhood, see Figure 4.

B. Sequential Test

The performance of the sequential version of the (a)-test can similarly be evaluated. Of particular interest is whether the new definition of the minimax robust sequential test in Section II-C holds. The value of $c = 1$ is chosen. The SPRT is run for every $t_u \in \{0.01, 0.02, \ldots, 4\}$ assuming the same experimental setup used for fixed sample size tests. Figure 6 illustrates the false alarm and miss detection probabilities resulting from the sequential (a)-test as a function of n. According to results, the sequential (a)-test does not degrade its performance as n gets larger for any input data that is considered. It was also verified that the (a)-test is not asymptotically minimax robust for Huber's definitions given by (8) and (9).

V. Conclusion

Minimax robustness of the asymptotically minimax robust hypothesis test was evaluated for fixed sample size and sequential tests considering a signal processing example. According to numerical results, fifty samples were sufficient to observe the minimax robustness property of the asymptotically minimax robust test, which implies the best guaranteeable detection performance regardless of the uncertainties. Unlike the (a)-test, Huber's minimax robust test as well as the nominal test experienced significant performance degradation. It was also verified experimentally that Dabak's asymptotically designed test also experienced performance degradation, hence it was not minimax robust. Since Huber's definition of minimax robustness is more mathematically oriented, a new definition of minimax robusness for sequential tests was made. It was shown that the (a)-test satisfies this new definition asymptotically. More general theoretical results are currently not available but can be a future work.

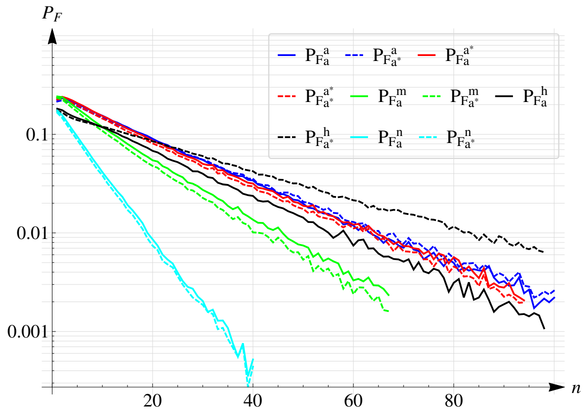

Fig. 2. False alarm probability as a function of the total number of samples for the asymptotically minimax robust test ((a)-test) in comparison to Dabak's test ((a^*)-test).

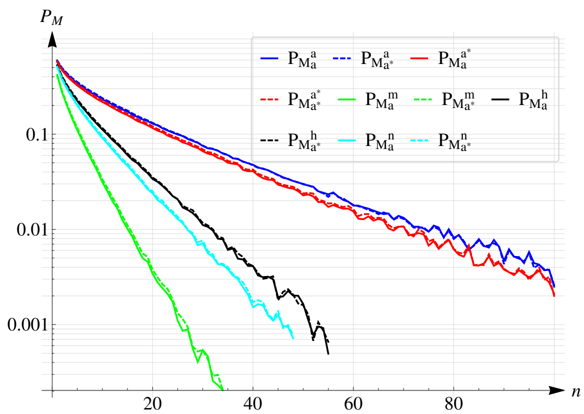

Fig. 3. Miss detection probability as a function of the total number of samples for the asymptotically minimax robust test ((a)-test) in comparison to Dabak's test ((a^*)-test).

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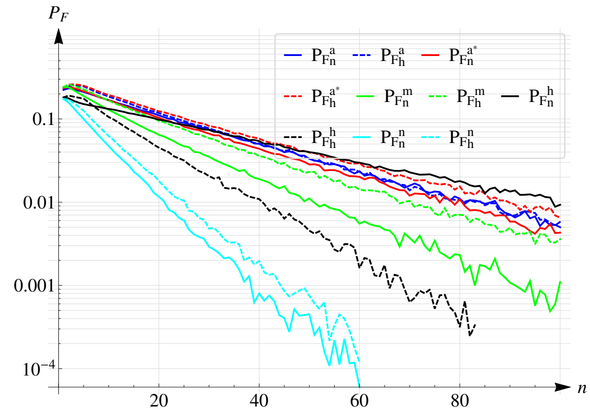

Fig. 4. False alarm probability as a function of the total number of samples for the nominal test ((n)-test) in comparison to Huber's test ((h)-test).

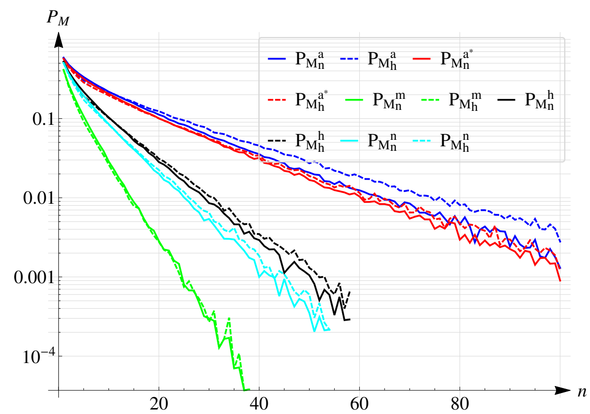

Fig. 5. Miss detection probability as a function of the total number of samples for the nominal test ((n)-test) in comparison to Huber's test ((h)-test).

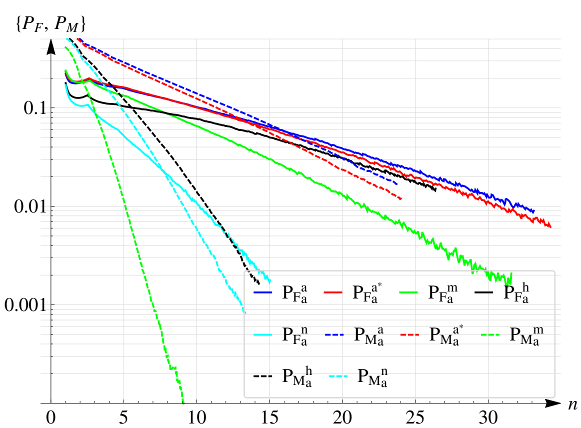

Fig. 6. False alarm and miss detection probabilities resulting from the sequential (a)-test as a function of n for the given signal processing example.

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