Evaluation of Performance Bounds in Distributed Detection

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***Abstract*—A generalized version of the majority voting rule is proposed to speed up the calculation of global error probabilities in parallel sensor networks, which otherwise requires exponentially many computations in general. The motivation behind proposing such a fusion rule is to inform the other researchers that the recently available performance bounds are not tight as claimed and can only be used for very limited decentralized detection problems. The proposed fusion rule can be computed in quasi-linear time for identically and quasi-quadratic time for non-identically distributed independent sensor observations. Effectiveness of the proposed fusion rule is illustrated with comparative experiments for identical and non-identical sensors over varying size of networks and various quantization levels.**

***Index Terms*—Distributed detection, quantization, performance bounds, sensor networks, error analysis**

I. Introduction

Detecting events of interest through distributed sensor networks has some advantages, such as reliability, survivability and much reduced usage of bandwidth compared to centralized networks. Despite all these advantages, optimization of distributed sensor networks may be a very challenging problem due to high computational complexity. For parallel sensor networks an optimum design is an NP-complete problem in general [1], [2]. Assuming that such a design has already been performed, i.e., all local sensor thresholds and the global fusion rule are determined by an algorithm, just evaluating the global error probability has an exponentially increasing time complexity in the total number of sensors [3, p. 314]. It is therefore of high interest, especially for the designers, to have sub-optimal yet efficient methods/bounds in order to calculate the global error probability.

There are several approaches to performance evaluation which circumvent direct computation of the global error probabilities. In [4] an approximation was made by determining the asymptotic error exponents of the distributed sensor network as the number of sensor nodes tends to infinity. In [5], Aldosari and Moura have presented an application of the saddlepoint method to determine the global error probabilities. However, the expressions obtained require the numerical solution of a saddle-point equation. In a more recent work [6] computationally simple upper-bounds were presented by considering probability inequality introduced by Hoeffding [7] and employing a multiplicative form factor following a technique developed by Talagrand [8].

In this paper the upper-bounds proposed by [6] are shown in fact not to be tight. Noting that any sub-optimal fusion rule is a valid upper-bound on the minimum error probability, a generalization of the majority voting rule to multilevel decisions is proposed in order to calculate approximate global error probabilities in parallel sensor networks. The proposed fusion rule is scalable since it can be computed in quasi-linear (or quasi-quadratic) time for identically (or non-identically) distributed sensors, respectively cf. [9], [10]. Numerical results indicate that the time complexities below that of the proposed fusion rule may experience serious problems especially when the number of quantization intervals is large and/or when the sensors are not identically distributed.

The rest of this paper is organized as follows. In Section II, the decentralized detection problem is introduced for parallel sensor networks. In Sections III and IV, majority fusion rule is introduced and its computational complexity is discussed in comparison to [6]. In Section V, the performance of the proposed fusion rule, as an approximation to the minimum error probability, is evaluated over identically and non-identically distributed independent sensor observations. Finally in Section VI, the paper is concluded.

II. Decentralized Detection

Consider a distributed detection network with K decision makers $\phi_1, \ldots, \phi_K$ and a fusion center γ as illustrated by Figure 1. Each sensor ϕ_k makes an observation $y_k \in \Omega_k$ from a certain phenomenon, where Ω_k is an interval, and gives a multilevel decision $u_k \in \{0, \ldots, N_k - 1\}$. The phenomenon is modeled by a binary hypothesis testing problem

$$\begin{aligned} \mathcal{H}_0 &: Y_k \sim F_0^k, \\ \mathcal{H}_1 &: Y_k \sim F_1^k, \end{aligned} \tag{1}$$

where the random variables Y_k corresponding to the observations y_k are mutually independent and follow the probability distribution function F_0^k or F_1^k, conditioned on the hypothesis $\mathcal{H}_0$ or $\mathcal{H}_1$. The fusion center receives multilevel decisions from all sensors and gives a binary decision u_0.

A. Local sensors

Optimum quantization, which minimizes the error probability of the fusion center is known to be the monotone likelihood ratio test [3]. Let f_0^k and f_1^k be the density functions corresponding to F_0^k and F_1^k, respectively and $l_k = f_1^k/f_0^k$

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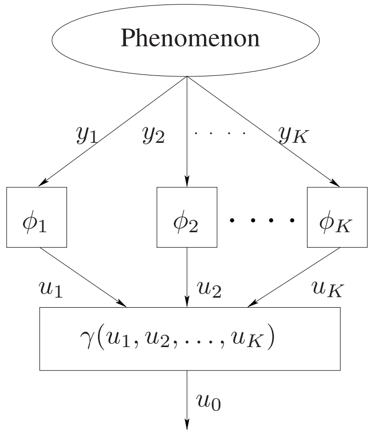

Fig. 1. Distributed detection network with K decision makers, each represented by the decision rule ϕ, and a fusion center associated with the fusion rule γ.

denote the likelihood ratio function. Then, the decisions can be obtained by

$$\phi_k(y_k) = u_k^{i_k} \quad \text{if} \quad \lambda_k^{i_k-1} \leq l_k(y_k) < \lambda_k^{i_k}, \tag{2}$$

where $\lambda_k^{i_k}$ denotes the thresholds, $k \in \{1, \ldots, K\}$ denotes the indices of sensors and $i_k \in \{1, \ldots, N_k\}$ denotes the indices of the multilevel decision u_k for the kth sensor. The upper and lower thresholds are given by $\lambda_k^0 := \inf \Omega_k$ and $\lambda_k^{N_k} := \sup \Omega_k$, leaving $N_k - 1$ unknown thresholds to be determined per sensor. From (1) and (2) the probability mass functions (p.m.f.s) of the decisions conditioned on the hypothesis $\mathcal{H}_j$ can be found by

$$p_j^k(u_k^{i_k}) = F_j[\lambda_k^{i_k-1} \leq l_k(Y_k) < \lambda_k^{i_k}], \quad j \in \{0, 1\}. \tag{3}$$

B. Fusion center

Let p_0 and p_1 denote the joint probability mass functions of the random variables U_k, corresponding to the multilevel decisions u_k, conditioned on the hypotheses $\mathcal{H}_0$ and $\mathcal{H}_1$, respectively. Furthermore, let the transmitted decisions u_k be reformed optimally by the fusion center as

$$u_k := \log \frac{p_1^k(u_k)}{p_0^k(u_k)}. \tag{4}$$

Then, the optimum test at the fusion center can be obtained by [11, p. 39]

$$\log \frac{p_1(u_1, \ldots, u_K)}{p_0(u_1, \ldots, u_K)} = \sum_{k=1}^{K} \log \frac{p_1^k(u_k)}{p_0^k(u_k)} = \sum_{k=1}^{K} u_k \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \log \frac{\pi_0}{\pi_1}, \tag{5}$$

where π_0 and π_1 are the a-priori probabilities of the hypothesis $\mathcal{H}_0$ and $\mathcal{H}_1$, respectively.

III. Majority Voting Rule

Since the test statistic in (5) corresponds to the summation of K random variables U_k, the probability mass function of the sum can be obtained by K-fold convolution of the marginal mass functions as

$$g_j(z) = \sum_{i_1=1}^{N_1} \cdots \sum_{i_K=1}^{N_K} p_j^1(u_1^{i_1}) \cdots p_j^K(u_K^{i_K}) \delta\Big(\sum_{k=1}^{K} u_k^{i_k} - z\Big), \tag{6}$$

where δ is the dirac delta function. Then, for $M = \dim(g_m)$ the minimum error probability can be found by

$$P_E = \sum_{n=1}^{M} \min\left(\pi_0 g_0(n), \pi_1 g_1(n)\right), \tag{7}$$

In general, evaluating error probabilities using (6) is of exponential complexity [12]. Here, we simplify (6) by omitting (4), hence keeping the original domain $u_k \in \{0, \ldots, N_k - 1\}$ and considering the test

$$\sum_{k=1}^{K} u_k \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \frac{1}{2} \sum_{k=1}^{K} N_k - \frac{K}{2}. \tag{8}$$

This approximation, which is the majority voting rule extended to multilevel quantization, simplifies the p.m.f. of the sum of the random variables U_k to

$$g_j = p_j^1 * p_j^2 * \ldots * p_j^K, \tag{9}$$

where $(*)$ stands for the ordinary discrete convolution,

$$(p_j^1 * p_j^2)[n] = \sum_{m=0}^{N_1+N_2-2} p_j^1(m) p_j^2(n-m). \tag{10}$$

Hence, given a set of quantization thresholds for K sensors as in [6], the performance can be evaluated approximately but much faster than the true calculation of the global error probability.

IV. Complexity Analysis

Computational complexity of the proposed fusion rule can be obtained by using the convolution theorem. Suppose that $N = N_k \forall k$. Then, the output of the K-fold convolution given by (9) has $NK - K + 1$ values. Therefore, the discrete Fourier transform of all p_js should include zero padding up to $NK - K + 1$ samples. Taking the fast Fourier transform (FFT) has a complexity of $\mathcal{O}(KN \log KN)$, where $\mathcal{O}$ is the standard Landau notation. For identically distributed p.m.f.s, i.e., $p_j^1 = p_j^k \forall k$, after the FFT, raising each sample to the Kth power requires $\mathcal{O}(KN)$ and taking the inverse FFT (IFFT) requires again $\mathcal{O}(KN \log KN)$. Hence, the overall complexity for the identical p.m.f.s is $\mathcal{O}(KN \log KN)$. If the p.m.f.s are not identical, then FFT should be taken for all zero padded p_js having the overall complexity of $\mathcal{O}(K^2 N \log KN)$. Multiplying them together requires $\mathcal{O}\left(K^2 N\right)$ and the IFFT needs $\mathcal{O}(K^2 N \log KN)$. Hence, the overall complexity for non-identical p.m.f.s is $\mathcal{O}(K^2 N \log KN)$. The complexity of the proposed scheme is slightly worse than that of the bound proposed in [6], which has $\mathcal{O}(KN)$ complexity. As it will be shown in the next section, it is not plausible in general to have a linear complexity in the number of sensors K and obtain tight bounds on an exponentially complex problem.

V. Numerical Results

In this section, the performance of the proposed approximation to the true error probability is evaluated and compared to [6] for both identically as well as non-identically distributed sensor observations. In all cases the sensors are assumed to be mutually independent.

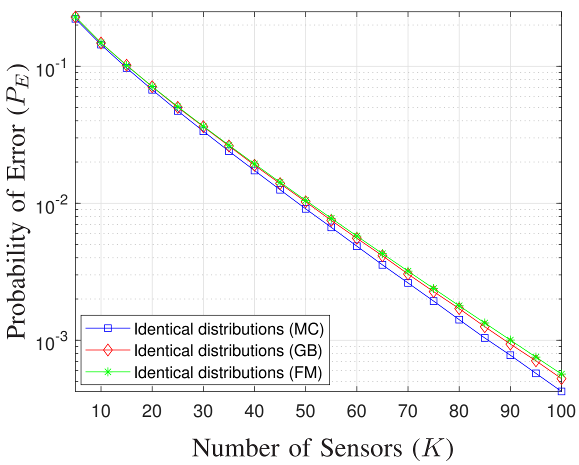

Fig. 2. Minimum error probability versus the total number of sensors over identical p.m.f.s with $N = 4$.

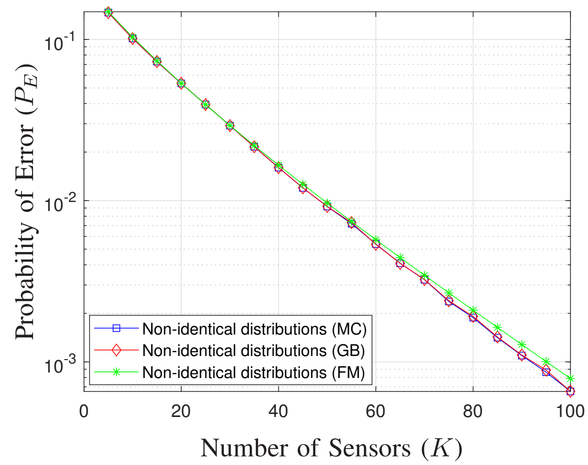

Fig. 3. Minimum error probability versus the total number of sensors over non-identical p.m.f.s with $N = 2$.

A. Identically distributed observations

Consider a decentralized detection network with $K = 5$ to $K = 100$ number of sensors and $N = 4$ levels of quantization. Suppose that the a-priori probabilities of the hypotheses are equal $\pi_0 = \pi_1 = 1/2$ and the probability mass functions of each sensor are represented by the following probabilities,

$$\begin{aligned} &p_0^k(0) = 0.3,\ p_0^k(1) = 0.3,\ p_0^k(2) = 0.2,\ p_0^k(3) = 0.2, \\ &p_1^k(0) = 0.1,\ p_1^k(1) = 0.2,\ p_1^k(2) = 0.3,\ p_1^k(3) = 0.4, \end{aligned} \quad (11)$$

for all $k \in \{1, \ldots, K\}$, as given by [6]. Figure 2 illustrates the numerical results regarding this problem. The names of the methods are abbreviated by the initials of the authors, i.e., (GB) stands for the proposed approximation, (FM) denotes [6], whereas (MC) stands for extensive Monte-Carlo simulations with up to 10^9 samples per sensor. The results indicate that the proposed scheme is only slightly better than that of [6], whereas both methods are slightly worse than the minimum global error probability.

B. Non-identically distributed observations

1) Bernoulli distributed observations: For the same sensor network as before assume now that each sensor makes a binary quantization, i.e., $N = 2$ and the a-priori probabilities of the hypotheses are $\pi_0 = 0.8$ and $\pi_1 = 0.2$. The conditional probability mass functions corresponding to each sensor are characterized by the following local false alarm and miss detection probabilities

$$\begin{aligned} &p_0^k(1) = 0.2 + 0.002(k-1), \\ &p_1^k(0) = 0.5 - 0.002k,\ k \in \{1, \ldots, K\}. \end{aligned} \quad (12)$$

For this problem, which is again adopted from [6], Figure 3 illustrates the error probabilities of the proposed scheme in comparison to those obtained by [6]. Although the results of the proposed scheme is almost the same with the true error probabilities obtained by extensive Monte-Carlo simulations, the error probabilities obtained by [6] is slightly off and the related gap seems to be increasing with the number of sensors.

2) χ^2-distributed observations: Consider a signal detection problem, where each sensor is an energy detector over a static channel model facing a presumably different signal-to-noise ratio (SNR). The details of this problem can be found in [11, p. 42]. The probability mass functions of the quantized observations are given by their dependence on the quantization thresholds $\lambda_k^{i_k}$,

$$\begin{aligned} p_0^k(u_k^{i_k}) &= \frac{\Gamma_u\left(\frac{W}{2}, \frac{\lambda_k^{i_k-1}}{2}\right) - \Gamma_u\left(\frac{W}{2}, \frac{\lambda_k^{i_k}}{2}\right)}{\Gamma\left(\frac{W}{2}\right)}, \\ p_1^k(u_k^{i_k}) &= \frac{\Gamma_u\left(\frac{W}{2}, \frac{\lambda_k^{i_k-1}}{2(\psi_k+1)}\right) - \Gamma_u\left(\frac{W}{2}, \frac{\lambda_k^{i_k}}{2(\psi_k+1)}\right)}{\Gamma\left(\frac{W}{2}\right)}, \end{aligned} \quad (13)$$

where W is the number of samples collected by each sensor, ψ_k is the SNR of the kth sensor, Γ is the gamma function and Γ_u is the upper incomplete gamma function. It is assumed that each sensor collects $W = 10$ samples, as in [11], and ψ_k are found by dividing the SNR range of $[-3, 2]$dB uniformly to the total number of sensors in the network as in [13]. The thresholds $\lambda_k^{i_k}$ are either found by the Gaussian approximation method [9] or by the Chernoff information based method [13]. We are interested in, first, whether the bound proposed by [13] is tight, i.e. whether one can obtain similar error probabilities to the true error probability, and second, whether the evaluated approximations to the true error probability can distinguish between the performances of Gaussian approximation and Chernoff information based quantization methods. Figures 4, 5 and 6 illustrate the error probabilities P_E obtained by two different schemes for sensor networks consisting of $5, \cdots, 50$ sensors, where each sensor transmits 1-, 2- and 3 bits, respectively. The proposed scheme (GB) provides consistently close results to the true error probabilities, whereas the bound proposed by [6] (FM) is off by a considerable margin and this margin increases both with the number of sensors as well

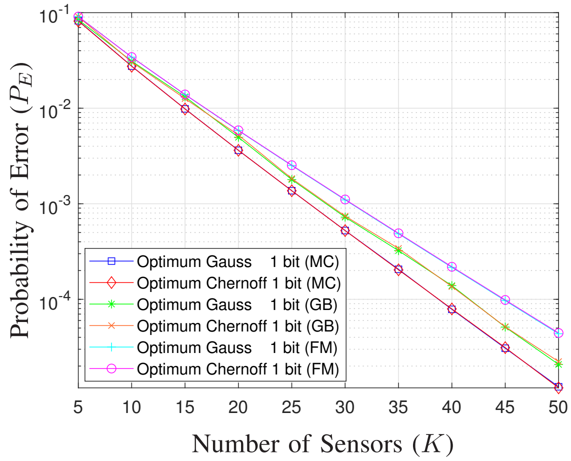

Fig. 4. Minimum error probability versus the total number of sensors over non-identical distributions for 1-bit quantization.

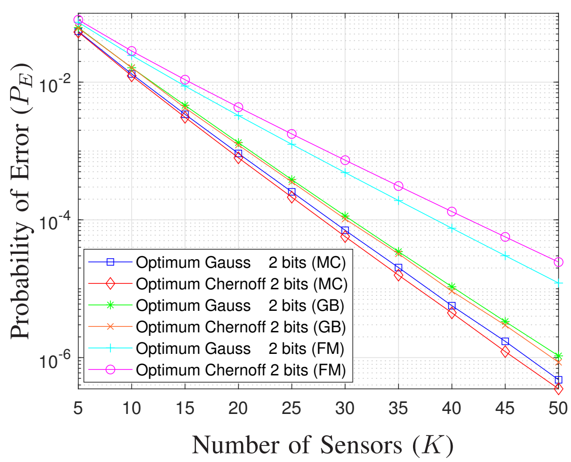

Fig. 5. Minimum error probability versus the total number of sensors over non-identical distributions for 2-bit quantization.

as with the number of transmitted bits. Moreover, the bounds proposed by [6] is not capable of distinguishing the better performing quantization method, i.e., in all figures it wrongly suggests that the Gaussian approximation method is better although this is not true. Another serious inaccuracy is that it shows that around $K = 50$ sensors, the error probabilities obtained by 3-bit quantization is almost the same with that of 1-bit quantization.

VI. Conclusion

The purpose of this paper was to show that performance bounds running in linear time may not be tight for distributed detection problems and in some cases may totally be useless. In order to illustrate this, a generalized version of the majority voting rule was proposed as an alternative approximation to calculate the true error probabilities and its performance was compared to the one available in the literature. Computational complexity of the proposed scheme is quasi-linear for identically distributed and quasi-quadratic for non-identically distributed independent sensors. Comparative simulations have

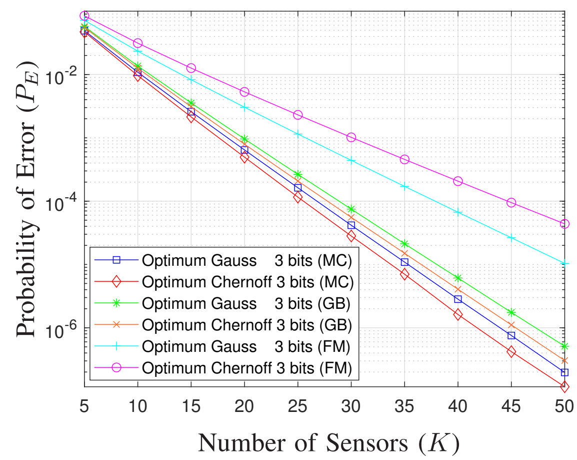

Fig. 6. Minimum error probability the total number of sensors over non-identical distributions for 3-bit quantization.

revealed that the proposed scheme is a much better approximation, although imposed computational complexities can be slightly more, yet can efficiently be computed with ordinary personal computers. The proposed scheme is supposed to be useful, where multiple evaluations of the performance measures of interest are necessary.

References

[1] Christos H. Papadimitriou and John Tsitsiklis, "Intractable problems in control theory," *SIAM Journal on Control and Optimization*, vol. 24, no. 4, pp. 639–654, 1986.

[2] J. Tsitsiklis and M. Athans, "On the complexity of decentralized decision making and detection problems," *IEEE Transactions on Automatic Control*, vol. 30, no. 5, pp. 440–446, May 1985.

[3] John N. Tsitsiklis, "Decentralized detection," in *In Advances in Statistical Signal Processing*. 1993, pp. 297–344, JAI Press.

[4] J. Chamberland and V. V. Veeravalli, "Decentralized detection in sensor networks," *IEEE Transactions on Signal Processing*, vol. 51, no. 2, pp. 407–416, Feb 2003.

[5] Saeed Aldosari and Jose Moura, "Saddlepoint approximation for sensor network optimization," in *ICASSP, IEEE International Conference on Acoustics, Speech and Signal Processing - Proceedings*, 04 2005, vol. 4, pp. 741–744.

[6] Gernot Fabeck and Rudolf Mathar, "Tight performance bounds for distributed detection," in *International Conference on Acoustics, Speech, and Signal Processing*, 05 2007, vol. 3, pp. 1049–1052.

[7] Wassily Hoeffding, "Probability inequalities for sums of bounded random variables," *Journal of the American Statistical Association*, vol. 58, no. 301, pp. 13–30, March 1963.

[8] Michel Talagrand, "The missing factor in hoeffding's inequalities," *Annales de l'I.H.P. Probabilités et statistiques*, vol. 31, no. 4, pp. 689–702, 1995.

[9] G. Gül and M. Baßler, "Fast multilevel quantization for distributed detection based on Gaussian approximation," in *28th European Signal Processing Conference (EUSIPCO)*, Jan 2021, pp. 2433–2437.

[10] G. Gül and M. Baßler, "Scalable multilevel quantization for distributed detection," in *IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, 2021, pp. 5200–5204.

[11] C. Altay and H. Delic, "Optimal quantization intervals in distributed detection," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 52, no. 1, pp. 38–48, February 2016.

[12] John Tsitsiklis, "Decentralized detection by a large number of sensors," *Mathematics of Control, Signals, and Systems (MCSS)*, vol. 1, pp. 167–182, 06 1988.

[13] G. Fabeck and R. Mathar, "Chernoff information-based optimization of sensor networks for distributed detection," in *2009 IEEE International Symposium on Signal Processing and Information Technology (ISSPIT)*, Dec 2009, pp. 606–611.