A Super Resolution Phase Retrieval Method for Sparse Signals with Arbitrary Scattering Function

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***Abstract*—The phase retrieval problem studies the recovery of the original signal from its phaseless Fourier intensity measurement. Unlike traditional phase retrieval algorithms that only recover the discrete approximation of the original signal, the recently proposed super resolution phase retrieval theories first realize continuous-domain phase retrieval of sparse signals. However, these current methods maintain too strict restriction on the scattering function and there is unnecessary redundancy in the parameter estimation models. This paper proposes a novel super resolution sparse phase retrieval method suitable for arbitrary scattering function and can reduce nearly half of the redundant parameters. First, after a recursive data processing procedure, we use Prony's method to calculate the support intervals. Then, the support of the original signal can be restored through a reordering algorithm. Finally, under the premise of known support, recovering the amplitude is equivalent to solving a series of nonlinear equations, which can be solved by Chebyshev's method. The simulation results verify the effectiveness of the proposed method.**

***Index Terms*—super resolution phase retrieval; sparse signal; scattering function.**

I. Introduction

Phase retrieval is a widespread inverse problem recovering a signal from the magnitude of its Fourier transform [1]. It is concerned by various fields, including crystallography [2], coherent diffraction imaging [3], radar waveform optimization [4], astronomical imaging [5], and more [1].

Over the past decade, many novel phase retrieval algorithms have been developed, such as PhaseLift [6], GESPAR [7], DOLPHIn [8], Wirtinger Flow [9], CoRK [10], PRIME [11], etc. However, these approaches usually recover the original signal in the discrete domain, that is, model the original signal as a discrete vector, which is a discrete approximation of original signal. In recent years, some explorations of continuous-domain phase retrieval have been published [12], [13]. In [12], Beinert *et al.* model the original signal as a parameterized sparse signal in the continuous domain, and restore these parameters through Prony's method. However, the method in [12] only provides the recovery method of stream of Dirac and B-spline function, which is not suitable for sparse signals under other bases. In [13], Baechler *et al.* use a compact structure that considers the original signal is the convolution of the Dirac stream and the fixed scattering function. Nevertheless, the method in [13] relies on the assumption that the frequency spectrum of the scattering function is a box function, which limits the applicable range of this method. In addition, the estimation models in [12] and [13] contain unnecessary redundant parameters, which will be interpreted in Section III.

This paper considers the continuous model in [12] and [13], and proposes a novel phase retrieval framework that is applicable to sparse parameterized signals with arbitrary scattering function. Our method is divided into three steps. First, use Prony's method to calculate the support interval set of the original signal after applying a recursive data processing procedure. Second, obtain the original signal support set from the support interval set through a reordering algorithm. Finally, use Chebyshev's method to recover the amplitude parameters. The main contribution of this paper is in the first step. We point out that if the scattering function is known, the requirement that the frequency spectrum of the scattering function is a box function is unnecessary. Besides, the number of unknown parameters in the location estimation is only half of that in the model of [12] and [13]. In this paper, we only discuss the one-dimensional case.

The rest of this paper is organized as follows: Section II states the problem; Section III presents our research motivation; Section IV introduces the proposed method; Section V provides simulation results; Section VI concludes this paper.

II. Problem statement

Consider a parameterized sparse signal of the form

$$x(t) = \sum_{i=1}^{K} c_i \phi(t - t_i) = (\sum_{i=1}^{K} c_i \delta(t - t_i)) * \phi(t), \quad (1)$$

This work was supported by the National Natural Science Foundation of China (NSFC, No. 62071149 and No. 61671177).

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which can be determined by $2K$ parameters $\{c_i\}_{i=1}^K$, $\{t_i\}_{i=1}^K$, and $\phi(t)$ is the scattering function known beforehand. To avoid a heavier notation, we assume t_i, c_i, and the range of $\phi(t)$ belong to $\mathbb{R}$, which is easily extended to $\mathbb{C}$.

In phase retrieval problem, the measurement of $x(t)$ is the square of its Fourier intensity $|\mathcal{F}x|$, which is given by

$$|\mathcal{F}x(\omega)|^2 := \Big|\int_{-\infty}^{\infty} x(t)\cdot e^{-jwt}dt\Big|^2. \tag{2}$$

Setting Ω as the sampling frequency, the underlying problem in this paper is to recover original signal $x(t)$ from the measurement samples $\{|\mathcal{F}x(n\Omega)|^2\}_{n=0}^{N-1}$, which is equivalent to recover $2K$ parameters $\{c_i\}_{i=1}^K$ and $\{t_i\}_{i=1}^K$ from $\{|\mathcal{F}x(n\Omega)|^2\}_{n=0}^{N-1}$.

III. Motivation

In this section, we first briefly introduce the methods in [12] and [13], and then introduce our research motivation.

An essential method to realize continuous-domain phase retrieval is Prony's method, which can be summarized as the following Lemma 1.

Lemma 1: [14] If a sequence $\{s_n|n\in\mathcal{K}\}$ can be expressed as the form

$$s_n = \sum_{i=1}^{K}\alpha_i u_i^n, \tag{3}$$

where $\mathcal{K}\subseteq\mathbb{Z}$, $\alpha_i, u_i\in\mathbb{C}$, $\alpha_i\neq 0$, and $u_i\neq u_j$ for $i\neq j$, then unknown variables $\{\alpha_i\}_{i=1}^K$ and $\{u_i\}_{i=1}^K$ can be estimated from $2K$ continuous non-zero measurements s_n.

Proof: see, e.g., [14], [15], [16].

Defining the auto-correlation function (ACF) of $x(t)$ is $A_x(t)$ given by

$$A_x(t) = x(t) * x(-t), \tag{4}$$

the methods in [12] and [13] both rely on the key relationship that the Fourier transform of $A_x(t)$ is the measurement $|\mathcal{F}x|^2$, i.e.,

$$A_x(t) = \mathcal{F}^{-1}\big[|\mathcal{F}x|^2\big], \tag{5}$$

which is also known as Winner-Khintchine formula. Consider (1) and (4), we have

$$A_x(t) = \sum_{k=1}^{K}\sum_{l=1}^{K} c_k c_l \psi(t-(t_k-t_l)) \tag{6}$$

$$= \Big[\sum_{k=1}^{K}\sum_{l=1}^{K} c_k c_l \delta(t-(t_k-t_l))\Big] * \psi(t), \tag{7}$$

where $\psi(t)$ is the ACF of $\phi(t)$. Define $\alpha_m = c_k c_l$, $\beta_m = t_k - t_l$, $u_m = e^{-j\Omega\beta_m}$, $m = 1,2,\ldots,M$, then according to (5) we have

$$|\mathcal{F}x(n\Omega)|^2 = \sum_{m=1}^{M}\alpha_m u_m^n |\Phi(n\Omega)|^2, \tag{8}$$

where $M = K^2 - K + 1$, $\Phi(\omega)$ is the Fourier transform of $\phi(t)$. If we assume that $|\Phi(\omega)|^2$ is a constant for some neighborhood of ω around zero (i.e. the scattering function is sinc function or Dirac function), then parameters $\{\alpha_m\}_{m=1}^M$ and $\{u_m\}_{m=1}^M$ can be properly estimated by Prony's method according to Lemma 1. The subsequent steps in [13] are able to recover the original parameters $\{c_i\}_{i=1}^K$ and $\{t_i\}_{i=1}^K$ from $\{\alpha_m\}_{m=1}^M$ and $\{u_m\}_{m=1}^M$. However, we point out:

- When the scattering function is known, it is unnecessary to require its frequency spectrum to be a box function.
- Considering (7), it is easy to see that there is redundancy in the parameters to be estimated in (8). For example, $t_k - t_l$ and $t_l - t_k$ will be estimated as two parameters, but they carry exactly the same information. Redundant parameters will increase the length of the constructed annihilating filter[1], which will reduce its efficiency and noise resilience.

To remove the restriction that the frequency spectrum of the scattering function must be a box function and eliminate the redundancy in the estimation model, we propose a novel continuous-domain sparse phase retrieval method. The method proposed in this paper can be applied when the scattering function is known, regardless of its form (not necessarily sinc function or Dirac function). At the same time, compared with the method in [13], the number of parameters to be estimated is greatly reduced.

IV. The Proposed Method

A. Support Intervals Recovery

Consider (1), the Fourier transform of $x(t)$ is

$$\mathcal{F}x(\omega) = \Phi(\omega)\sum_{i=1}^{K} c_i e^{-j\omega t_i}. \tag{9}$$

Thus the measurement can be written as

$$|\mathcal{F}x(\omega)|^2 = |\Phi(\omega)\sum_{i=1}^{K} c_i e^{-j\omega t_i}|^2 \tag{10}$$

$$= |\Phi(\omega)|^2\cdot\Big|\sum_{i=1}^{K} c_i e^{-j\omega t_i}\Big|^2. \tag{11}$$

Set $\mathbf{c} = [c_1, c_2, \ldots, c_K]^\mathsf{T}$, $\boldsymbol{\gamma}_\omega = [e^{-j\omega t_1}, e^{-j\omega t_2}, \ldots, e^{-j\omega t_K}]^\mathsf{T}$, then (11) can be rewritten as

$$\frac{|\mathcal{F}x(\omega)|^2}{|\Phi(\omega)|^2} = \mathbf{c}^\mathsf{T}\boldsymbol{\gamma}_\omega\cdot(\mathbf{c}^\mathsf{T}\boldsymbol{\gamma}_\omega)^\mathsf{H} \tag{12}$$

$$= \mathbf{c}^\mathsf{T}\cdot\underbrace{(\boldsymbol{\gamma}_\omega\boldsymbol{\gamma}_\omega^\mathsf{H})}_{\boldsymbol{\Gamma}_\omega}\cdot\bar{\mathbf{c}}. \tag{13}$$

Considering $\boldsymbol{\Gamma}_\omega$ is a Hermitian matrix, thus (13) is a quadratic form. Then we have

$$\frac{|\mathcal{F}x(\omega)|^2}{|\Phi(\omega)|^2} = \sum_{i=1}^{K}\sum_{j=1}^{K}\gamma_{\omega(ij)}c_i c_j, \tag{14}$$

where $\gamma_{\omega(ij)}$ is the entry in the ith row and jth column of the matrix $\boldsymbol{\Gamma}_\omega$. Since $\gamma_{\omega(ij)} = e^{j\omega(t_j-t_i)}$, it is easy to see the relationship

$$\gamma_{\omega(ij)}c_i c_j + \gamma_{\omega(ji)}c_j c_i = 2c_i c_j\cos\big(\omega(t_i-t_j)\big). \tag{15}$$

[1]A core step of Prony's method [14].

Therefore, we can rewrite (14) as

$$\frac{|\mathcal{F}x(\omega)|^2}{|\Phi(\omega)|^2} = \|\mathbf{c}\|_2^2 + 2\Big[c_1c_2\cos\big(\omega(t_1-t_2)\big)+ c_1c_3\cos\big(\omega(t_1-t_3)\big)+...+ c_{K-1}c_K\cos\big(\omega(t_{K-1}-t_K)\big)\Big]. \quad (16)$$

Note here $\|\mathbf{c}\|_2^2$ is a known quantity. If we assume that the scattering function is of finite length, and the scattering functions at different support positions in the original signal do not overlap, then we can infer that

$$\|\mathbf{c}\|_2^2 = \frac{\int_{-\infty}^{\infty}|\mathcal{F}x(\omega)|^2 d\omega}{\int_{-\infty}^{\infty}|\phi(t)|^2 dt}, \quad (17)$$

according to Parseval's theorem. Defining $f(\omega) = \frac{1}{2}\Big(\frac{|\mathcal{F}x(\omega)|^2}{|\Phi(\omega)|^2} - \|\mathbf{c}\|_2^2\Big)$ and $\alpha'_m = c_ic_j$, $\beta'_m = t_i - t_j$, $i<j$, we have

$$f(\omega) = \sum_{m=1}^{M}\alpha'_m\cos(\omega\beta'_m), \quad (18)$$

where $M = K(K-1)/2$. The sampled form of (18) is

$$f(n\Omega) = \sum_{m=1}^{M}\alpha'_m\cos(n\Omega\beta'_m),\ n=0,1,\ldots,N-1. \quad (19)$$

So far, we can see (19) is somewhat similar to (3). Applying a relationship in [17] that

$$\cos(n\theta) = \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\Big(\tbinom{n-k}{k} + \tbinom{n-1-k}{k-1}\Big)(-1)^k 2^{n-1-2k}\cos^{n-2k}(\theta), \quad (20)$$

we can obtain form (3) from (19).

Before giving a general conclusion, we first show a few specific steps to reveal the law. Suppose $\{\xi[n]\}_{n=0}^{N-1}$ is a sequence that can be expresses as the form of (3), and we define

$$\begin{cases} \xi[0] &:= f(0\Omega) = \sum_{m=1}^{M}\alpha'_m\big[\cos(\Omega\beta'_m)\big]^0, \\ \xi[1] &:= f(1\Omega) = \sum_{m=1}^{M}\alpha'_m\big[\cos(\Omega\beta'_m)\big]^1. \end{cases} \quad (21)$$

From (3) we can infer that

$$\xi[2] = \sum_{m=1}^{M}\alpha'_m\big[\cos(\Omega\beta'_m)\big]^2. \quad (22)$$

But unlike (21), the question now is how can we get the value of $\xi[2]$. Consider (19) and (20), we have

$$\begin{aligned} f(2\Omega) &= \sum_{m=1}^{M}\alpha'_m\cos(2\Omega\beta'_m) \\ &= \sum_{m=1}^{M}\alpha'_m\Big[2\big[\cos(\Omega\beta'_m)\big]^2 - 1\Big] \\ &= 2\cdot\underbrace{\sum_{m=1}^{M}\alpha'_m\big[\cos(\Omega\beta'_m)\big]^2}_{\xi[2]} - \underbrace{\sum_{m=1}^{M}\alpha'_m}_{\xi[0]}. \end{aligned} \quad (23)$$

Thus we have

$$\xi[2] = \frac{1}{2}\Big(f(2\Omega) + \xi[0]\Big). \quad (24)$$

Similar to (23), according to (19) and (20), it is easy to verify that

$$\begin{aligned} f(3\Omega) &= \sum_{m=1}^{M}\alpha'_m\cos(3\Omega\beta'_m) \\ &= \sum_{m=1}^{M}\alpha'_m\Big[4\big[\cos(\Omega\beta'_m)\big]^3 - 3\cos(\Omega\beta)\Big] \\ &= 4\cdot\xi[3] - 3\cdot\xi[1], \end{aligned}$$

and

$$\xi[3] = \frac{1}{4}\Big(f(3\Omega) + 3\xi[1]\Big). \quad (25)$$

Therefore, we can conclude a general expression that

$$\xi[n] := 2^{1-N}\Big(f(n\Omega) - G_n\Big), \quad (26)$$

where

$$G_n = \sum_{k=1}^{\lfloor\frac{n}{2}\rfloor}\Big(\tbinom{n-k}{k} + \tbinom{n-1-k}{k-1}\Big)(-1)^k 2^{n-1-2k}\xi[n-2k]. \quad (27)$$

Since (26) is a recursive formula, as long as $\xi[0]$ and $\xi[1]$ are determined, the value of $\{\xi[n]\}_{n=2}^{N-1}$ can be deduced by (26) and (27).

To be clear, the above steps allow us to obtain $\{\xi[n]\}_{n=0}^{N-1}$ from $\{f(n\Omega)\}_{n=0}^{N-1}$, and $\xi[n]$ can be expanded as

$$\xi[n] = \sum_{m=1}^{M}\alpha'_m\big[\cos(\Omega\beta'_m)\big]^n. \quad (28)$$

Then according to Lemma 1, the unknown parameters $\{\alpha'_m\}_{m=1}^{M}$ and $\{\cos(\Omega\beta'_m)\}_{m=1}^{M}$ can be estimated accurately by Prony's method. Since our goal in this subsection is to recover support intervals, we only need to calculate $\{\cos(\Omega\beta'_m)\}_{m=1}^{M}$. If we choose sampling frequency properly, satisfying

$$\Omega < \frac{\pi}{\max(|\beta'_m|)}, \quad (29)$$

then $\{\beta'_m\}_{m=1}^{M}$ can be uniquely determined from $\{\cos(\Omega\beta'_m)\}_{m=1}^{M}$ by applying a simple inverse cosine function.

B. Support Recovery

As we have obtain the support interval set $\mathcal{D} := \{\beta'_m | \beta'_m = t_i - t_j, i<j, m=1,2,\ldots,M\}$. Note these differences are unlabeled that we do not know the order of the elements in $\mathcal{D}$. To recover support set $\{t_i\}_{i=1}^{K}$ from $\mathcal{D}$, we can directly use the algorithm in [13], which is shown as Algorithm 1.

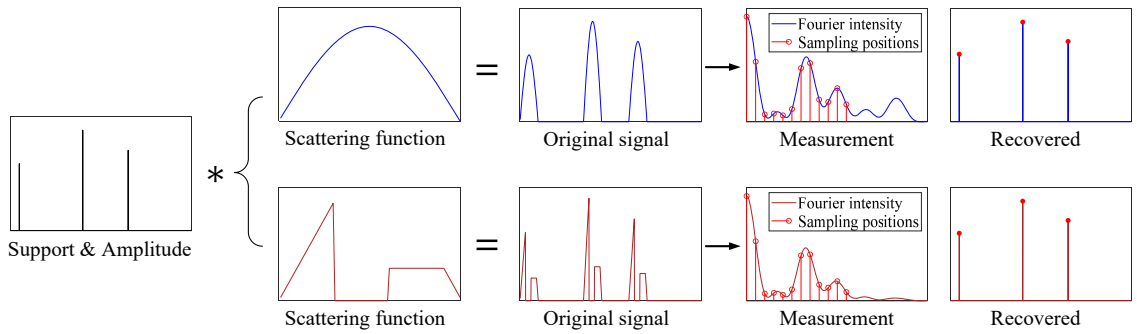

Fig. 1. Demonstrations of super resolution sparse phase retrieval under two different scattering functions.

Algorithm 1 Support Recovery [13]

Input: A set of $K(K-1)/2$ differences $\mathcal{D} = \{d_i\}_{i=1}^{M}$ ordered by their absolute value.

Output: A support set of K points $\hat{\mathcal{X}}$ such that their pairwise differences generate support interval set $\mathcal{D}$.

Initialize $\hat{\mathcal{X}}_2 = \{0, d_M\}$, $\mathcal{P}_2 = \mathcal{D} \setminus \hat{\mathcal{X}}_2$.
for $K = 2, ..., k-1$ **do**
  $\hat{x}_{k+1} = \arg\min_{p \in \mathcal{P}_k} \sum_{\hat{x} \in \hat{\mathcal{X}}_k} \min_{d \in \mathcal{D}} |p - \hat{x} - d|^2$.
  $\hat{\mathcal{X}}_{k+1} = \hat{\mathcal{X}}_k \cup \hat{x}_{k+1}$, $\mathcal{P}_{k+1} = \mathcal{P}_k \setminus \hat{x}_{k+1}$.
end for
return $\hat{\mathcal{X}}_K$.

C. Amplitude Recovery

Recover the amplitude $\{c_i\}_{i=1}^{K}$ when the support set $\{t_i\}_{i=1}^{K}$ is known is equivalent to solving a problem of a system of nonlinear equations. Again, we set $\mathbf{c} = [c_1, c_2, \ldots, c_K]^{\mathsf{T}}$. Defining $s_{kl} := \cos\big(\Omega(t_k - t_l)\big)$, the nonlinear equations we need to solve is $\{g_n(\mathbf{c}) = 0\}_0^{N-1}$, where

$$g_n(\mathbf{c}) := \sum_{k=1}^{K} \sum_{l>k}^{K} c_k c_l s_{kl}^n - \xi[n], \tag{30}$$

To solve the nonlinear equations $\{g_n(\mathbf{c}) = 0\}_0^{N-1}$, a variant of Chebyshev's method in [18] can be used. Defining vector $\mathbf{g} = [g_0(\mathbf{c}), g_1(\mathbf{c}), \ldots, g_{N-1}(\mathbf{c})]^{\mathsf{T}}$.This iterative method can de summarized as

$$\begin{cases} \mathbf{c}_0 \text{ and } p \in (0,1] \text{ given}, \\ \mathbf{g}'(\mathbf{c}_t)\boldsymbol{\delta}_t = -\mathbf{g}(\mathbf{c}_t), \ t > 0, \\ \mathbf{z}_t = \mathbf{c}_t + p\boldsymbol{\delta}_t, \\ \mathbf{g}'(\mathbf{c}_t)\mathbf{q}_t = -\frac{1}{p^2}((p-1)\mathbf{g}(\mathbf{c}_t) + \mathbf{g}(\mathbf{z}_t)), \\ \mathbf{c}_{t+1} = \mathbf{c}_t + \boldsymbol{\delta}_t + \mathbf{q}_t. \end{cases} \tag{31}$$

D. Analysis & Comparison

From (18), we know the number of parameters to be estimated in the proposed method is $M = \binom{K}{2} = K(K-1)/2$, where the model in [13] contains $K^2 - K + 1$ parameters to be estimated as can be seen in (8). This shows that our method reduces the number of parameters to be estimated by nearly half.

TABLE I
PERFORMANCE COMPARISON

	The method in [13]	**Proposed method**
Number of parameters to be estimated	$K^2 - K + 1$	$\frac{K(K-1)}{2}$
Complexity	$\mathcal{O}(K^6)$	$\mathcal{O}(K^6)$
Scattering function	sinc function or Dirac function	arbitrary function

The computational complexity of the method in [13] is $\mathcal{O}(K^6)$, where K is the sparsity of the original signal. The proposed method adds a recursive procedure as described in (26) and (27) to calculate $\{\xi[n]\}_{n=0}^{N-1}$. From (26) and (27), we can see calculating $\xi[n]$ requires $\lfloor \frac{n}{2} \rfloor$ additions and $\lfloor \frac{n}{2} \rfloor + 1$ multiplications. Thus the computational complexity of calculating $\{\xi[n]\}_{n=0}^{N-1}$ is $\mathcal{O}(N^2)$. Because the number of parameters to be estimated in our model is $K(K-1)/2$, according to Lemma 1, we know that $N = K(K-1)$ and thus the computational complexity of the recursive procedure is $\mathcal{O}(K^4)$. Therefore, the total computational complexity of proposed framework is $\mathcal{O}(K^4) + \mathcal{O}(K^6) = \mathcal{O}(K^6)$, which is the same as the method in [13].

Besides, another advantage of the proposed method is that it is suitable for arbitrary known scattering function given some mild conditions (such as the finite-length condition we assumed before (17)). The performance comparison between the proposed method and the method in [13] is summarized in Table I.

V. SIMULATION RESULTS

To verify the method proposed in this paper, we selected two general finite-length scattering functions to form the original signal under the same support and amplitude. We sampled the phaseless Fourier intensity distribution of the two original signals, and then used the proposed method to recover the original signal. The sampling and reconstruction results are shown in Fig. 1. It can be seen that for general scattering function without special characteristics in the frequency domain, the method in this paper can accurately reconstruct the support and amplitude of the original signal.

To further compare the recovery accuracy between the proposed method and the existing method in [13], we test the recovery errors of the two methods under different sparsity and different noise levels. Selecting sparsity as $K = 3, 4, 5$, we add white Gaussian noise with SNR ranging from 120 dB to -60 dB, and calculate the Normalized Square Error (NSE) of the reconstructed signal and the original signal as

$$\mathrm{NSE}(\mathbf{x}_r, \mathbf{x}_o) = \frac{\|\mathbf{x}_r - \mathbf{x}_o\|_2^2}{\|\mathbf{x}_o\|_2^2}, \tag{32}$$

where $\mathbf{x}_r$ is the reconstructed signal and $\mathbf{x}_o$ is the original signal. The Normalized Mean Square Error (NMSE) is averaged over 100 Monte Carlo simulations for every (K, SNR). The simulation results are shown in Fig. 2. The results show that both methods can not reconstruct the signal accurately in the low SNR region, and the NMSE decreases with the increase of SNR. When the sparsity is small, the recovery errors of the two methods are similar. But when the sparsity increases, the NMSE of the proposed method is steadily lower than that of [13] in the high SNR region.

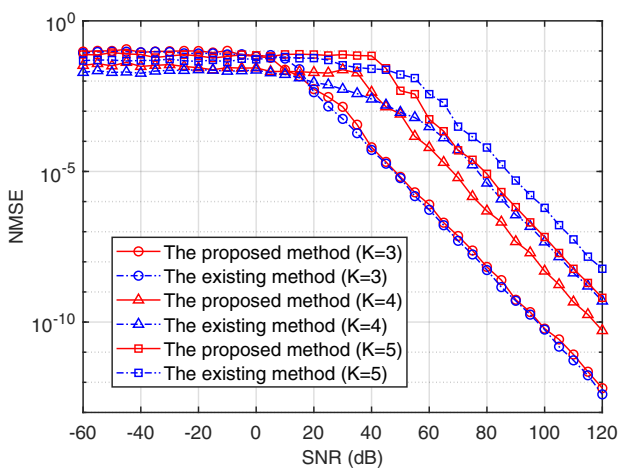

Fig. 2. Recovery error vs. noise SNR under sparsity $K = 3, 4, 5$.

VI. Conclusion

The existing super resolution sparse phase retrieval methods have too high limitation on the scattering function and contains unnecessary redundancy in estimation model. This paper provided a strategy to achieve super resolution sparse phase retrieval under arbitrary scattering function. In addition, by designing a recursive data processing procedure, the proposed method can reduce the number of parameters to be estimated by nearly half without increasing the computational complexity. Simulation results show that the proposed method can effectively reduce the recovery error.

References

[1] Y. Shechtman, Y. C. Eldar, O. Cohen, H. N. Chapman, J. Miao, and M. Segev, "Phase Retrieval with Application to Optical Imaging: A contemporary overview," *IEEE Signal Processing Magazine*, vol. 32, no. 3, pp. 87–109, 2015.

[2] J. Miao, P. Charalambous, J. Kirz, and D. Sayre, "Extending the methodology of X-ray crystallography to allow imaging of micrometre-sized non-crystalline specimens," *Nature*, vol. 400, no. 6742, pp. 342–344, 1999.

[3] P. Sidorenko, O. Kfir, Y. Shechtman, A. Fleischer, Y. C. Eldar, M. Segev, and O. Cohen, "Sparsity-based super-resolved coherent diffraction imaging of one-dimensional objects," *Nature Communications*, vol. 6, pp. 1–8, 2015.

[4] L. K. Patton and B. D. Rigling, "Phase retrieval for radar waveform optimization," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 48, no. 4, pp. 3287–3302, 2012.

[5] K. Jaganathan, S. Oymak, and B. Hassibi, "Sparse phase retrieval: Uniqueness guarantees and recovery algorithms," *IEEE Transactions on Signal Processing*, vol. 65, no. 9, pp. 2402–2410, 2017.

[6] E. J. Candès, T. Strohmer, and V. Voroninski, "PhaseLift: Exact and Stable Signal Recovery from Magnitude Measurements via Convex Programming," *Communications on Pure and Applied Mathematics*, vol. 66, no. 8, pp. 1241–1274, 2013.

[7] Y. Shechtman, A. Beck, and Y. C. Eldar, "GESPAR: Efficient phase retrieval of sparse signals," *IEEE Transactions on Signal Processing*, vol. 62, no. 4, pp. 928–938, 2014.

[8] A. M. Tillmann, Y. C. Eldar, and J. Mairal, "DOLPHIn - Dictionary Learning for Phase Retrieval," *IEEE Transactions on Signal Processing*, vol. 64, no. 24, pp. 6485–6500, 2016.

[9] E. J. Candès, X. Li, and M. Soltanolkotabi, "Phase retrieval via wirtinger flow: Theory and algorithms," *IEEE Transactions on Information Theory*, vol. 61, no. 4, pp. 1985–2007, 2015.

[10] K. Huang, Y. C. Eldar, and N. D. Sidiropoulos, "Phase Retrieval from 1D Fourier Measurements: Convexity, Uniqueness, and Algorithms," *IEEE Transactions on Signal Processing*, vol. 64, no. 23, pp. 6105–6117, 2016.

[11] T. Qiu, P. Babu, and D. P. Palomar, "PRIME: Phase Retrieval via Majorization-Minimization," *IEEE Transactions on Signal Processing*, vol. 64, no. 19, pp. 5174–5186, 2016.

[12] R. Beinert and G. Plonka, "Sparse Phase Retrieval of One-Dimensional Signals by Prony's Method," *Frontiers in Applied Mathematics and Statistics*, vol. 3, no. January, p. Art. no. 5, 2017.

[13] G. Baechler, M. Krekovic, J. Ranieri, A. Chebira, Y. M. Lu, and M. Vetterli, "Super Resolution Phase Retrieval for Sparse Signals," *IEEE Transactions on Signal Processing*, vol. 67, no. 18, pp. 4839–4854, 2019.

[14] M. Vetterli, P. Marziliano, and T. Blu, "Sampling signals with finite rate of innovation," *IEEE Transactions on Signal Processing*, vol. 50, no. 6, pp. 1417–1428, 2002.

[15] P. L. Dragotti and Y. M. Lu, "On sparse representation in fourier and local bases," *IEEE Transactions on Information Theory*, vol. 60, no. 12, pp. 7888–7899, 2014.

[16] B. B. Haro and M. Vetterli, "Sampling continuous-time sparse signals: A frequency-domain perspective," *IEEE Transactions on Signal Processing*, vol. 66, no. 6, pp. 1410–1424, 2018.

[17] Y.-h. LU, "Multiple angles formula of sine and cosine and its application," *College Mathematics*, vol. 3, 2013.

[18] J. Ezquerro and M. Hernández, "An optimization of chebyshev's method," *Journal of Complexity*, vol. 25, no. 4, pp. 343 – 361, 2009.