Identifiability Results for Nonuniform Linear and Rectangular Sensor Arrays

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***Abstract*—Sensor arrays with simple geometries play an important role in solving direction of arrival estimation and source separation problems. To reduce the number of sensors used, nonuniform linear and rectangular array geometries have been proposed. Unlike the case of uniform linear and rectangular arrays, identifiability conditions for direction of arrival estimation and source separation using nonuniform linear and rectangular arrays are not well-studied. Based on rank properties of Fourier matrices and tools from algebraic geometry, we present generic identifiability conditions for direction of arrival estimation and source separation problems when nonuniform linear and rectangular array geometries are used. Furthermore, based on properties of bilinear factorizations subject to polynomial/monomial equality constraints, we also briefly discuss how to obtain deterministic identifiability conditions.**

***Index Terms*—Array processing, nonuniform linear array, direction of arrival estimation, source separation, nonuniform rectangular array, harmonic retrieval, tensor, canonical polyadic decomposition, identifiability, missing sensors.**

I. Introduction

In this paper we revisit the classical array processing model in which R signals impinging on an array composed of I sensors located on a plane such that the output of the ith sensor at the kth observation is

$$x_{ik} = \sum_{r=1}^{R} s_r\left(k - \tau_{ri}\right), \tag{1}$$

where τ_{ri} denotes the delay between the ith sensor and the rth source. Assume that the sources are located in the far-field and that the narrowband assumption holds. Under these assumptions the array response vector associated with the rth source can be expressed as

$$\mathbf{a}_r = [e^{-\mathrm{i}\omega_c \mathbf{b}_r^T \mathbf{p}_1/c},\ \ldots\ , e^{-\mathrm{i}\omega_c \mathbf{b}_r^T \mathbf{p}_I/c}]^T \in \mathbb{C}^I, \tag{2}$$

where $\mathrm{i} = \sqrt{-1}$, ω_c is the carrier frequency, $\mathbf{p}_i \in \mathbb{R}^2$ is position of the ith sensor (in Cartesian coordinates), $\mathbf{b}_r = [\sin(\phi_r)\cos(\theta_r),\ \sin(\phi_r)\sin(\theta_r)]^T$ is the bearing vector in which θ_r and ϕ_r denote the azimuth and elevation angle, respectively, c is the speed of propagation, and the product $\mathbf{b}_r^T\mathbf{p}_i/c$ corresponds to the propagation delay in (1) associated with the ith sensor and the rth source, so that $\tau_{ri} = \mathbf{b}_r^T\mathbf{p}_i/c$. Assume that K snapshots are available such that $\mathbf{s}_r \in \mathbb{C}^K$ denotes the signal vector associated with the rth source. Then the observed data matrix admits the factorization

$$\mathbf{X} = \mathbf{A}\mathbf{S}^T \in \mathbb{C}^{I\times K},\ \mathbf{A} = \left[\mathbf{a}_1, \ldots, \mathbf{a}_R\right], \mathbf{S} = \left[\mathbf{s}_1, \ldots, \mathbf{s}_R\right]. \tag{3}$$

Depending on the application, the goal is to estimate the azimuth and elevation angles $\{\theta_r, \phi_r\}$ via $\mathbf{A}$ or to estimate the signal matrix $\mathbf{S}$. The former problem is known as Direction-Of-Arrival (DOA) estimation while the latter problem is known as source separation. To achieve these goals, either arrays with simple geometries and/or signals with certain known properties are used in practice. We mention the minimum redundancy arrays [7], nonredundant arrays [17], nested arrays [8] and co-prime arrays [15], where Nonuniform Linear Array (NLA) geometries are used and the signal matrix $\mathbf{S}$ is assumed to be columnwise orthogonal. Assuming that the temporal source signals are random, mutually uncorrelated, zero-mean, and the autocorrelation function of each source decays sufficiently fast, columnwise orthogonality of $\mathbf{S}$ is attained asymptotically as $K \to \infty$. In this paper we will study the identifiability properties of NLAs and Nonuniform Rectangular Arrays (NRAs) when the impinging signals are non-orthogonal. More precisely, we will only assume that $\mathbf{S}$ has full column rank.

The paper is organised as follows. Section II briefly reviews the NLA and NRA models while Section III discuss their connections to a type of decomposition that we will refer to as Generalized Vandermonde Matrix Factorization (GMVF). Based on the link between NLA/NRA and GVMF, we will in Section IV present and discuss both generic and deterministic identifiability conditions for the NLA and NRA models when the signal matrix $\mathbf{S}$ has full column rank. Section V concludes the paper.

II. Nonuniform Linear and Rectangular Arrays

Consider a Uniform Linear Array (ULA) [16], in which the sensors are equispaced on a line. Typically “half wavelength spacing” is used, meaning that $d_x = \lambda/2$ is the unit measure along the sensor axis (“x-axis”), where $\lambda = \frac{2\pi c}{\omega_c}$ denotes the signal wavelength. We will use this convention throughout the paper. To reduce the number of sensors used, NLAs have

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been considered (e.g., [7], [8], [15], [17]), in which the array response vectors $\mathbf{a}_r$ in (3) are of the form

$$\begin{aligned}\mathbf{a}_r &= \left[1 \; e^{-\mathrm{i}\omega_c d_x m_2 \cos(\theta_r)/c} \; \dots \; e^{-\mathrm{i}\omega_c d_x m_I \cos(\theta_r)/c}\right]^T \\ &= \left[1 \; e^{-\mathrm{i}\pi m_2 \cos(\theta_r)} \; \dots \; e^{-\mathrm{i}\pi m_I \cos(\theta_r)}\right]^T \\ &= [1 \; x_r^{m_2} \; \dots \; x_r^{m_I}]^T \in \mathbb{C}^I, \end{aligned} \tag{4}$$

where $m_i d_x$ is the distance between the reference sensor ($m_1 = 0$) and the ith sensor in the NLA in which $m_i \in \mathbb{N}$, $1 < m_2 < \cdots < m_I$ and $x_r = e^{-\mathrm{i}\pi\cos(\theta_r)}$. The vector $\mathbf{a}_r$ is a generalized Vandermonde vector and for this reason we call (3) a generalized Vandermonde matrix factorization (GVMF), as will be elaborated on in Section III. This is in contrast to the ULA case where $\mathbf{a}_r = [1 \; x_r \; x_r^2 \; \dots \; x_r^{I-1}]^T$ is a standard Vandermonde vector. In DOA applications, we say that the angles $\theta_1, \dots, \theta_R$ are identifiable if they can be determined, observing only $\mathbf{X}$. Similarly, in source separation applications, we say that $\mathbf{S}$ is identifiable if it can be determined (up to intrinsic column scaling and permutation ambiguities), observing only $\mathbf{X}$. Unlike the ULA case, the identifiability properties of the angles and the signal matrix are not well-understood in the NLA case. However, if the signal matrix can be assumed to be columnwise orthogonal, then a GVMF problem can in certain cases be converted to a standard Vandermonde factorization problem. In short, if $\mathbf{S}$ is columnwise orthogonal, then

$$\mathbf{X}\mathbf{X}^H = \mathbf{A}\boldsymbol{\Lambda}\mathbf{A}^H \Leftrightarrow \mathrm{vec}(\mathbf{X}\mathbf{X}^H) = (\mathbf{A}^* \odot \mathbf{A})\mathrm{vecd}(\boldsymbol{\Lambda}), \tag{5}$$

where $\boldsymbol{\Lambda} = \mathbf{S}^T\mathbf{S}^*$ is a diagonal matrix, * denotes the conjugate, ^{H} denotes the conjugate-transpose, $\mathrm{vec}(\mathbf{X}\mathbf{X}^H)$ denotes the vectorization of matrix $\mathbf{X}\mathbf{X}^H$, $\mathrm{vecd}(\boldsymbol{\Lambda})$ denotes the vectorization of the diagonal part of the diagonal matrix $\boldsymbol{\Lambda}$, and $\odot$ denotes the Khatri–Rao (columnwise) Kronecker product. By carefully choosing the integers $m_1, \dots, m_I$ in (4), a submatrix of $\mathbf{A}^* \odot \mathbf{A}$ corresponds to a Vandermonde matrix (e.g., [7], [8], [15], [17]). This means that when $\mathbf{S}$ is columnwise orthogonal, then (5) can, under certain conditions, be interpreted as a classical single snapshot ($K = 1$) Vandermonde matrix factorization problem for which identifiability results and algorithms can be found in the literature. Note that in this paper we only assume that $\mathbf{S}$ has full column rank, implying that relation (5) is not satisfied and consequently the identifiability conditions derived in for instance [8], [15] will no longer be satisfied.

A natural extension of the NLA that will also be considered in this paper is the NRA, in which $\mathbf{a}_r$ in (3) is of the form

$$\mathbf{a}_r = \mathbf{S}_{\text{sel}}(\mathbf{c}_r \otimes \mathbf{b}_r) \in \mathbb{C}^L, \tag{6}$$

where $\otimes$ denotes the Kronecker product, $\mathbf{S}_{\text{sel}} \in \{0,1\}^{L \times IJ}$ is a row-selection matrix and

$$\begin{aligned}\mathbf{b}_r &= \left[1 \; e^{-\mathrm{i}\pi m_2 \cos(\theta_r)\sin(\phi_r)} \; \dots \; e^{-\mathrm{i}\pi m_I \cos(\theta_r)\sin(\phi_r)}\right]^T \\ &= [1 \; x_r^{m_2} \; \dots \; x_r^{m_I}]^T \in \mathbb{C}^I, \end{aligned} \tag{7}$$

$$\begin{aligned}\mathbf{c}_r &= \left[1 \; e^{-\mathrm{i}\pi n_2 \sin(\theta_r)\sin(\phi_r)} \; \dots \; e^{-\mathrm{i}\pi n_J \sin(\theta_r)\sin(\phi_r)}\right]^T \\ &= [1 \; y_r^{n_2} \; \dots \; y_r^{n_J}]^T \in \mathbb{C}^J, \end{aligned} \tag{8}$$

in which $x_r = e^{-\mathrm{i}\pi\cos(\theta_r)\sin(\phi_r)}$, $y_r = e^{-\mathrm{i}\pi\sin(\theta_r)\sin(\phi_r)}$, $m_i, n_j \in \mathbb{N}$, $1 < m_2 < \cdots < m_I$ and $1 < n_2 < \cdots < n_J$. Note that when $\mathbf{b}_r = [1 \; x_r \; x_r^2 \; \dots \; x_r^{I-1}]^T$, $\mathbf{c}_r = [1 \; x_r \; x_r^2 \; \dots \; x_r^{J-1}]^T$ and $\mathbf{S}_{\text{sel}} = \mathbf{I}_{IJ}$ is the identity matrix, then (3) corresponds to the standard Uniform Rectangular Array (URA) model. Identifiability conditions for DOA estimation and source separation based on the URA model can for instance be found in [5], [10], [11]. For special NRA configurations, such as L-shaped arrays, identifiability conditions can be found in [14]. However, the development of dedicated identifiability conditions for DOA estimation and source separation for the general NRA case has not received much attention in the array processing literature.

Based on the link between GVMF and the NLA and NRA models discussed in the next section we will propose dedicated identifiability conditions for DOA estimation and source separation when NLAs or NRAs are used and the signal matrix $\mathbf{S}$ has full column rank.

III. Generalized Vandermonde Matrix Factorization (GVMF)

A. Definition

The GVMF of matrix $\mathbf{X} \in \mathbb{F}^{I \times K}$ is defined as follows

$$\mathbf{X} = \mathbf{A}\mathbf{S}^T \in \mathbb{F}^{I \times K}, \tag{9}$$

where $\mathbb{F}$ denotes $\mathbb{R}$ or $\mathbb{C}$ depending on the application, $\mathbf{S} \in \mathbb{F}^{K \times R}$ and $\mathbf{A} \in \mathbb{F}^{I \times R}$ is a generalized Vandermonde matrix (GVM), defined next. Let $\{m_1, \dots, m_I\}$ be a set of distinct integers with property $0 \le m_1 < m_2 \cdots < m_I$ and let $\{x_1, \dots, x_R\}$ be a set of distinct real or complex numbers, depending on the application. We say that $\mathbf{A}$ is a GVM when

$$\mathbf{A} = \begin{bmatrix} x_1^{m_1} & \cdots & x_R^{m_1} \\ \vdots & \ddots & \vdots \\ x_1^{m_I} & \cdots & x_R^{m_I} \end{bmatrix} \in \mathbb{F}^{I \times R}. \tag{10}$$

We call the elements in $\{x_1, \dots, x_R\}$ the generators of $\mathbf{A}$. Observe that $\mathbf{A}$ can be interpreted as a punctured Vandermonde matrix. Comparing (3) with (9) it is clear that the NLA factorization with $\mathbf{a}_r$ of the form (4) corresponds to a GVMF in which the generators are of the form $x_r = e^{-\mathrm{i}\pi\cos(\theta_r)}$. We note in passing that the model (9) with real valued generators with property $x_r < 0, \forall r \in \{1, \dots, R\}$ can potentially also be of interest for chemometrics applications (e.g., [9]) involving exponential decay functions when nonuniform sampling is used.

B. Link to Canonical Polyadic Decomposition

The GVMF (9) of $\mathbf{X}$ can be interpreted as a basic Vandermonde Matrix Factorization (VMF) with missing rows:

$$\mathbf{Y} = \mathbf{D}\mathbf{B}\mathbf{S}^T \in \mathbb{F}^{J \times K}, \tag{11}$$

where $J = m_I$, $\mathbf{D} \in \{0,1\}^{J \times J}$ is a binary diagonal with property $(\mathbf{D})_{ii} = 1$ if $i \in \{m_1, \dots, m_I\}$ and $(\mathbf{D})_{ii} = 0$

otherwise, and $\mathbf{B}$ is a (scaled) Vandermonde matrix of the form

$$\mathbf{B} = \begin{bmatrix} x_1 & \cdots & x_R \\ x_1^2 & \cdots & x_R^2 \\ \vdots & \ddots & \vdots \\ x_1^J & \cdots & x_R^J \end{bmatrix} \in \mathbb{F}^{J\times R}. \tag{12}$$

Note that $\mathbf{X} = \mathbf{RY}$, where $\mathbf{R} \in \{0,1\}^{I\times J}$ is a row selection matrix that selects the "observed" rows of $\mathbf{Y}$. The shift-invariance property of the Vandermonde matrix implies that

$$\mathbf{Z} = \begin{bmatrix} \underline{\mathbf{Y}} \\ \overline{\mathbf{Y}} \end{bmatrix} = \mathbf{D}_{\mathbf{Y}}(\mathbf{G} \odot \underline{\mathbf{B}})\mathbf{S}^T \in \mathbb{F}^{2(J-1)\times K}, \tag{13}$$

where $\underline{\mathbf{Y}} = \mathbf{Y}(1:J-1,:) \in \mathbb{F}^{(J-1)\times K}$, $\overline{\mathbf{Y}} = \mathbf{Y}(2:J,:) \in \mathbb{F}^{(J-1)\times K}$, $\underline{\mathbf{B}} = \mathbf{B}(1:J-1,:) \in \mathbb{F}^{(J-1)\times K}$, $\mathbf{G} = \left[\begin{smallmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_R \end{smallmatrix}\right] \in \mathbb{F}^{2\times R}$ and $\mathbf{D}_{\mathbf{Y}} = \left[\begin{smallmatrix} \underline{\mathbf{D}} \\ \overline{\mathbf{D}} \end{smallmatrix}\right] \in \{0,1\}^{2(J-1)\times 2(J-1)}$, where and $\underline{\mathbf{D}} = \mathbf{D}(1:J-1,:) \in \{0,1\}^{(J-1)\times(J-1)}$ and $\overline{\mathbf{D}} = \mathbf{D}(2:J,:) \in \{0,1\}^{(J-1)\times(J-1)}$. From (13) we observe that the GVMF of $\mathbf{X}$ can be interpreted as a canonical polyadic decomposition (CPD) of a tensor that has missing fibers [12]. Identifiability conditions developed for the CPD of a tensor that has missing fibers was also presented in [12] and can in principle be used to obtain identifiability conditions for the GVMF. However, they do not exploit that the "nonzero" part of $\mathbf{DB}$, corresponding to $\mathbf{A}$ in (9) is a GVM. In Section IV we will develop identifiability conditions for GVMF that better exploit the GVM structure of $\mathbf{A}$, and are easy to check.

C. Extension to higher-order GVMF

Consider the following higher-order extension of (9):

$$\mathbf{X} = \mathbf{AS}^T = \mathbf{S}_{\text{sel}}(\mathbf{C} \odot \mathbf{B})\mathbf{S}^T \in \mathbb{F}^{L\times K}, \tag{14}$$

where $\mathbf{A} = \mathbf{S}_{\text{sel}}(\mathbf{C} \odot \mathbf{B})$ in which $\mathbf{S}_{\text{sel}} \in \{0,1\}^{L\times IJ}$ is a row-selection matrix, $\mathbf{B} \in \mathbb{F}^{I\times R}$ and $\mathbf{C} \in \mathbb{F}^{J\times R}$ are generalized Vandermonde matrices. (The extension to more than two GVMs is analogous and will not be discussed.) Comparing (3) with (14) it is clear that the NRA factorization with $\mathbf{a}_r$ of the form (6) corresponds to a GVMF in which the generators of $\mathbf{B}$ are of the form $x_r = e^{-\mathrm{i}\pi\cos(\theta_r)\sin(\phi_r)}$, $r \in \{1,\ldots,R\}$ and the generators of $\mathbf{C}$ are of the form $y_r = e^{-\mathrm{i}\pi\sin(\theta_r)\sin(\phi_r)}$, $r \in \{1,\ldots,R\}$.

IV. Identifiability conditions for GVMF

A. Generic identifiability conditions

Based on a property of the determinant of square generalized Vandermonde matrices, the rank property of Fourier matrices and the identifiability properties of structured bilinear matrix factorizations with parsimonious parametrizations, generic identifiability conditions for GVMF can be derived. In more detail, our starting point is Theorem IV.1 which guarantees the generic uniqueness of the following structured decomposition of an $I \times K$ matrix $\mathbf{Y}$:

$$\mathbf{Y} = \sum_{r=1}^{R} \mathbf{a}(\boldsymbol{\zeta}_r)\mathbf{s}_r^T, \quad \mathbf{s}_r \in \mathbb{F}^K, \quad \boldsymbol{\zeta}_r \in \mathbb{F}^l, \tag{15}$$

where the vector function $\mathbf{a} : \mathbb{F}^l \to \mathbb{F}^I$ is known and constructed as explained below. We say that decomposition (15) is generically unique if it is unique for a generic choice of $\boldsymbol{\zeta}_1,\ldots,\boldsymbol{\zeta}_R$, i.e., unique for all $\{\boldsymbol{\zeta}_r\}_{r=1}^R$ except for a set of Lebesgue measure zero in $\mathbb{F}^l$. The vector function $\mathbf{a}$ is structured as follows:

$$\begin{aligned} &\mathbf{a}(\cdot) = \mathbf{r}(\mathbf{f}(\cdot)), \\ &\mathbf{r}(\cdot) = (p_1(\cdot),\ldots,p_I(\cdot)), \\ &p_1,\ldots,p_I \ \text{ are polynomials in } l \text{ variables}, \\ &\mathbf{f}(\cdot) = (f_1(\cdot),\ldots,f_l(\cdot)), \\ &f_1,\ldots,f_l \text{ are functions analytic on } \mathbb{C}^l. \end{aligned} \tag{16}$$

Let $\mathbf{J}_{\mathbf{r}} \in \mathbb{F}^{I\times l}$ and $\mathbf{J}_{\mathbf{f}} \in \mathbb{F}^{l\times l}$ denote the Jacobian matrices of $\mathbf{r}$ and $\mathbf{f}$, respectively.

Theorem IV.1. *[2, Theorem 1] Assume that*

i) *the matrix* $[\mathbf{s}_1, \ \ldots \ ,\mathbf{s}_R]$ *has full column rank;*
ii) *there exists* $\boldsymbol{\zeta}^0 \in \mathbb{C}^l$ *such that* $\det \mathbf{J}_{\mathbf{f}}(\boldsymbol{\zeta}^0) \neq 0$*;*
iii) *the dimension of the subspace spanned by the vectors of the form* (16) *is at least* $\widehat{N}$*;*
iv) $\operatorname{rank} \mathbf{J}_{\mathbf{r}}(\mathbf{x}) \leq \widehat{l}$ *for a generic choice of* $\mathbf{x} \in \mathbb{C}^l$*;*
v) $R \leq \widehat{N} - \widehat{l} - 1$.

Then decomposition (15) *is generically unique.*

We will use Theorem IV.1 to obtain generic identifiability conditions for GVMF with generators of the form $x_r = e^{-\mathrm{i}\pi\cos(\theta_r)}$. Our result is not limited to this special case; we will also consider GVMFs with real valued generators, $x_r \in \mathbb{R}$. The following property of the determinant of square generalized Vandermonde matrices attributed to Mitchell [6] will be used to obtain $\widehat{N}$ in Theorem IV.1 when $\mathbf{A}$ is a generalized Vandermonde matrix with real generators.

Theorem IV.2. *[3, Theorem 5], [1, Theorem 3] Consider the square generalized Vandermonde matrix* $\mathbf{A} \in \mathbb{F}^{R\times R}$ *of the form* (10) *with* $I = R$*. Then the determinant of* $\mathbf{A}$ *is equal to*

$$det(\mathbf{A}) = \Big(\prod_{i>j}(x_i - x_j)\Big)S(x_1,\ldots,x_R), \tag{17}$$

where $S(x_1,\ldots,x_R) = \sum_k c_k x_1^{p_{k,1}} \cdots x_R^{p_{k,R}}$ *is a symmetric polynomial in* $x_1,\ldots,x_R$*, with nonnegative coefficients* c_k*. Moreover, the sum of the coefficients of* $S(x_1,\ldots,x_R)$ *is given by*

$$\sum_k c_k = \frac{\prod_{R\geq i>j>0}(m_i - m_j)}{\prod_{R>i>j\geq 0}(i-j)}. \tag{18}$$

Similarly, the following rank property of Fourier matrices, which according to [4] is attributed to Chebotarev, will be used to obtain $\widehat{N}$ in Theorem IV.1 when $\mathbf{A}$ is a generalized Vandermonde matrix with complex valued unit norm generators $|x_r| = 1$, i.e., $x_r = e^{-\mathrm{i}\pi\cos(\theta_r)}$.

Theorem IV.3. *[3, Theorem 6], [1, Theorem 4] Consider the Fourier matrix* $\mathbf{F}_p \in \mathbb{C}^{p\times p}$ *with entries* $(\mathbf{F}_p)_{ij} = \omega^{ij}$, $0 \leq i,j < p$*, where* $\omega = e^{\mathrm{i}2\pi/p}$*. If* p *is a prime number, then the Fourier matrix* $\mathbf{F}_p$ *does not contain any singular square submatrix.*

We are now ready to state a generic identifiability condition for GVMF.

Proposition IV.4. *Consider the GVMF of* $\mathbf{X} \in \mathbb{F}^{I\times K}$ *given by* (9)*, where* $\mathbf{A} \in \mathbb{F}^{I\times R}$ *is a generalized Vandermonde matrix of the form* (10) *with* $\mathbb{F} = \mathbb{R}$ *or* $\mathbb{F} = \mathbb{C}$ *and* $x_1, \dots, x_R \in \mathbb{R}$*, or* $x_1 = e^{-\mathrm{i}\pi\cos(\theta_1)}, \cdots, x_R = e^{-\mathrm{i}\pi\cos(\theta_R)}$ *with* $\theta_1, \dots, \theta_R \in \mathbb{R}$*, respectively. If* $\mathbf{S}$ *is full column rank, and*

$$R \leq I - 2, \tag{19}$$

then the GVMF of $\mathbf{X}$ *is generically unique.*

Proof. Consider the conditions of Theorem IV.1:

i) We assume that the unstructured matrix $\mathbf{S} = [\mathbf{s}_1, \ \dots \ , \mathbf{s}_R]$ has full column rank, which is generically true when $R \leq K$.

ii) Consider first the case where $x_1, \dots, x_R \in \mathbb{R}$. Define the functions $\mathbf{f}(\boldsymbol{\zeta}) = \zeta : \ \mathbb{R} \to \mathbb{R}$ and $\mathbf{r}(z) = (z^{m_1}, z^{m_2}, \dots, z^{m_I}) : \ \mathbb{R} \to \mathbb{R}^I$. The columns of $\mathbf{A}$ are sampled values of $\mathbf{a}(\boldsymbol{\zeta}) = \mathbf{r}(\mathbf{f}(\boldsymbol{\zeta}))$ at points $\boldsymbol{\zeta}_1 = x_1, \dots, \boldsymbol{\zeta}_R = x_R$. The Jacobian of $\mathbf{f}$ is equal to 1, i.e., it is not identically zero.

Consider now the case where $|x_1| = \cdots = |x_R| = 1$. As before, we define the functions $\mathbf{f}(\boldsymbol{\zeta}) = e^{\mathrm{i}\zeta} : \ \mathbb{R} \to \mathbb{C}$ and $\mathbf{r}(z) = (z^{m_1}, z^{m_2}, \dots, z^{m_I}) : \ \mathbb{C} \to \mathbb{C}^I$, so that the columns of $\mathbf{A}$ are sampled values of $\mathbf{a}(\boldsymbol{\zeta}) = \mathbf{r}(\mathbf{f}(\boldsymbol{\zeta}))$ at points $\boldsymbol{\zeta}_1 = \arg(x_1), \dots, \boldsymbol{\zeta}_R = \arg(x_R)$, where $\arg(x_r)$ denotes the argument of x_r. The Jacobian of $\mathbf{f}$ is equal to $\mathrm{i}\arg(x)e^{\mathrm{i}\arg(x)}$, i.e., it is not identically zero.

Finally, we consider the case where $x_1 = e^{-\mathrm{i}\pi\cos(\theta_1)}, \cdots, x_R = e^{-\mathrm{i}\pi\cos(\theta_R)}$. As before, we define the functions $\mathbf{f}(\boldsymbol{\zeta}) = e^{-\mathrm{i}\pi\cos(\zeta)} : \ \mathbb{R} \to \mathbb{C}$ and $\mathbf{r}(z) = (1, z^{m_2}, \dots, z^{m_I}) : \ \mathbb{C} \to \mathbb{C}^I$, so that the columns of $\mathbf{A}$ are sampled values of $\mathbf{a}(\boldsymbol{\zeta}) = \mathbf{r}(\mathbf{f}(\boldsymbol{\zeta}))$ at points $\boldsymbol{\zeta}_1 = \theta_1, \dots, \boldsymbol{\zeta}_R = \theta_R$. The Jacobian of $\mathbf{f}$ is equal to $\mathrm{i}\pi\sin(\theta)e^{-\mathrm{i}\pi\cos(\theta)}$, i.e., it is not identically zero.

iii) Consider first the case where $x_1, \dots, x_R \in \mathbb{R}$. Let $\widetilde{\mathbf{A}} \in \mathbb{R}^{I\times I}$ be a generalized Vandermonde matrix with real valued generators $x_1, \dots, x_I \in \mathbb{R}$ with property $0 < x_1 < x_2 < \cdots < x_I$. Relations (17) and (18) in Theorem IV.2 ensure that $\det(\widetilde{\mathbf{A}}) > 0$, i.e., we can set $\widehat{N} = I$ and the vectors $\mathbf{r}(z) \in \mathbb{C}^I$ span the whole space.

Consider now the case where $x_1 = e^{-\mathrm{i}\pi\cos(\theta_1)}, \cdots, x_R = e^{-\mathrm{i}\pi\cos(\theta_R)}$. Pick a prime number $p > m_I R$ such that

$$-1 \leq \frac{-2(R-1)}{p} < \frac{-2(R-2)}{p} < \cdots < \frac{-2}{p} \leq 0.$$

Choose θ_r as follows

$$\theta_r = \cos^{-1}(-2(r-1)/p) \Leftrightarrow \cos(\theta_r) = -2(r-1)/p).$$

Then

$$x_r^{m_i} = e^{-\mathrm{i}\pi\cos(\theta_r)m_i} = e^{\mathrm{i}2\pi m_i(r-1)/p} = \omega^{m_i(r-1)}$$

with $\omega = e^{\mathrm{i}2\pi/p}$. Note that $x_r^{m_i}$ corresponds to an entry of a $(p\times p)$ Fourier matrix $\mathbf{F}_p$ when $p > m_I R$. Theorem IV.3 now tells us that the vectors $\mathbf{r}(z) \in \mathbb{C}^I$ span the whole space, i.e., we can set $\widehat{N} = I$.

iv) We set $\widehat{l} = 1$.

v) Condition holds since $R \leq \widehat{N} - \widehat{l} - 1 = I - 2$. $\square$

Note that condition (19) is very close to a necessary bound for the case where $\mathbf{S}$ has full column rank, which is $R \leq I - 1$ [14, Theorem III.3]. In the context of array processing, Proposition IV.4 tells us that DOA estimation and source separation using a NLA is generically possible when $R \leq I - 2$ and $R \leq K$. Interestingly, this bound almost coincides with the necessary and sufficient generic bound for the ULA case when $\mathbf{S}$ has full column rank (implying $R \leq K$), which is $R \leq I - 1$ (e.g., [11]).

Proposition IV.5 below is an extension of Proposition IV.4 to the NRA case where the generators are of the form $x_r = e^{-\mathrm{i}\pi\cos(\theta_r)\sin(\phi_r)}$ and $y_r = e^{-\mathrm{i}\pi\sin(\theta_r)\sin(\phi_r)}$.

Proposition IV.5. *Consider the GVMF of* $\mathbf{X} \in \mathbb{C}^{L\times K}$ *given by* (14)*, where* $\mathbf{B} \in \mathbb{F}^{I\times R}$ *is a generalized Vandermonde matrix with generators* $x_r = e^{-\mathrm{i}\pi\cos(\theta_r)\sin(\phi_r)}$*,* $\mathbf{C} \in \mathbb{F}^{J\times R}$ *is a generalized Vandermonde matrix with generators* $y_r = e^{-\mathrm{i}\pi\sin(\theta_r)\sin(\phi_r)}$ *and* $\theta_r, \phi_r \in \mathbb{R}$ *for all* $r \in \{1, \dots, R\}$*. If* $\mathbf{S}$ *is full column rank, and*

$$R \leq L - 3, \tag{20}$$

then the GVMF of $\mathbf{X}$ *is generically unique.*

Proof. We again consider the conditions of Theorem IV.1:

i) We assume that the unstructured matrix $\mathbf{S} = [\mathbf{s}_1, \ \dots \ , \mathbf{s}_R]$ has full column rank, which is generically true when $R \leq K$.

ii) Define $\boldsymbol{\zeta} = [\theta, \ \phi]^T$ and the vector functions

$$\mathbf{f}(\boldsymbol{\zeta}) = (e^{-\mathrm{i}\pi\cos(\theta)\sin(\phi)}, {}^{-\mathrm{i}\pi\sin(\theta)\sin(\phi)}) : \ \mathbb{R}^2 \to \mathbb{C}^2,$$

$$\mathbf{r}(z_1, z_2) = \mathbf{S}_{\text{sel}} \left(\begin{bmatrix} 1 \\ z_2^{n_2} \\ \vdots \\ z_2^{n_J} \end{bmatrix} \otimes \begin{bmatrix} 1 \\ z_1^{m_2} \\ \vdots \\ z_1^{m_I} \end{bmatrix} \right) : \ \mathbb{C}^2 \to \mathbb{C}^L,$$

where $\mathbf{S}_{\text{sel}} \in \{0, 1\}^{L\times IJ}$ is the row-selection matrix in (14). The columns of $\mathbf{A}$ are sampled values of $\mathbf{a}(\boldsymbol{\zeta}) = \mathbf{r}(\mathbf{f}(\boldsymbol{\zeta}))$ at points $\boldsymbol{\zeta}_1 = [\theta_1, \ \phi_1]^T, \dots, \boldsymbol{\zeta}_R = [\theta_R, \ \phi_R]^T$. The determinant of the Jacobian of $\mathbf{f}$ is equal to $\pi^2 f_1 f_2 \sin(2\phi)$, i.e., it is not identically zero.

iii) We will show that the vectors $\mathbf{r}(z_1, z_2) \in \mathbb{C}^L$ span the whole space, i.e., we can set $\widehat{N} = L$. Observe that when $y = x^{m_I+1}$, then

$$\begin{aligned} \mathbf{a} &= \mathbf{S}_{\text{sel}}(\mathbf{c} \otimes \mathbf{b}) = \mathbf{S}_{\text{sel}}([1 \ y^{n_2} \dots \ y^{n_J}]^T \otimes [1 \ x^{m_2} \ \dots \ x^{m_I}]^T) \\ &= \mathbf{S}_{\text{sel}}([1 \ x^{(m_I+1)n_2} \ \dots \ x^{(m_I+1)n_J}]^T \otimes [1 \ x^{m_2} \ \dots \ x^{m_I}]^T) \\ &= \mathbf{S}_{\text{sel}}[1 \ x^{m_2} \ \dots \ x^{m_I(m_I+1)n_J}]^T \end{aligned} \tag{21}$$

is a generalized Vandermonde vector with generator x. The goal is now to choose pairs $\{\theta_r, \phi_r\}$ such that an $(R\times R)$ submatrix of $\mathbf{S}_{\text{sel}}(\mathbf{C} \odot \mathbf{B})$ corresponds to an $(R\times R)$ submatrix of a Fourier matrix $\mathbf{F}_p$ for some prime number $p > LR$.

We will select θ_r and ϕ_r such that for sufficiently large prime number $p > LR$ we have

$$\cos(\theta_r)\sin(\phi_r) = -\frac{2(r-1)}{p} \Leftrightarrow \tag{22}$$

$$\sin(\phi_r) = -\frac{2(r-1)}{p}\frac{1}{\cos(\theta_r)}, \tag{23}$$

$$\sin(\theta_r)\sin(\phi_r) = -\frac{(m_I+1)2(r-1)}{p} \Leftrightarrow \quad (24)$$

$$\sin(\phi_r) = -\frac{2(r-1)}{p}\frac{(m_I+1)}{\sin(\theta_r)}. \quad (25)$$

From (23) and (25) we conclude that

$$\frac{1}{\cos(\theta_r)} = \frac{m_I+1}{\sin(\theta_r)} \Leftrightarrow \tan(\theta_r) = \frac{\sin(\theta_r)}{\cos(\theta_r)} = m_I+1. \quad (26)$$

From (23) and (26) we conclude that by choosing the angles as follows

$$\theta_r = \tan^{-1}(m_I+1) \text{ and } \phi_r = \sin^{-1}\left(-\frac{2(r-1)}{p}\frac{1}{\cos(\theta_r)}\right)$$

then, due to (22) and (24), we obtain

$$x_r^{m_i} = e^{-\mathrm{i}\pi\cos(\theta_r)\sin(\phi_r)m_i} = e^{\mathrm{i}2\pi m_i\frac{(r-1)}{p}} = \omega^{m_i(r-1)}, \quad (27)$$

$$\begin{aligned} y_r^{n_i} &= e^{-\mathrm{i}\pi\sin(\theta_r)\sin(\phi_r)n_i} = e^{\mathrm{i}2\pi n_i(m_I+1)\frac{(r-1)}{p}} \\ &= \omega^{n_i(m_I+1)(r-1)} = x_r^{n_i(m_I+1)}, \end{aligned} \quad (28)$$

where $\omega = e^{\mathrm{i}2\pi/p}$. From (21), (27) and (28), we can conclude that $\mathbf{A} = \mathbf{S}_{\text{sel}}(\mathbf{C}\odot\mathbf{B})$ contains an $(R\times R)$ submatrix that corresponds to an $(R\times R)$ submatrix of an $(p\times p)$ Fourier matrix $\mathbf{F}_p$ when $p > LR$. Theorem IV.3 now tells us that the vectors $\mathbf{r}(z_1,z_2)\in\mathbb{C}^L$ span the whole space, i.e., we can set $\widehat{N} = L$.

iv) We set $\widehat{l} = 2$.

v) Condition holds since $R \le \widehat{N} - \widehat{l} - 1 = L-3$. $\square$

Let us compare Proposition IV.5 for NRA with the best known result for URAs when $\mathbf{S}$ has full column rank. In [11] it was shown that DOA estimation and source separation problems can generically be solved using a URA when $\min(IJ-3,K)\ge R$. Note that this corresponds to the bound (20) for NRA when $L = IJ$. Hence, in terms of identifiability we do not expect to lose anything when using an NRA instead of a URA for DOA estimation or source separation.

B. Remark on deterministic identifiability conditions

We briefly mention that using the results in [13], deterministic identifiability conditions for GVMF can be derived that exploit polynomial structure of the vectors $\mathbf{a}$ of the form (4) or (6). Consider monomial equality constraints of the form

$$a_{\alpha_1}\cdots a_{\alpha_{L_n}} - a_{\beta_1}\cdots a_{\beta_{L_n}} = 0, \quad (29)$$

where a_{α_l} denotes the α_l-th entry of $\mathbf{a}$, a_{β_l} denotes the β_l-th entry of $\mathbf{a}$ and L_n denotes the degree of the monomials in (29), where $L_1 > \cdots > L_N$. Assume that vector $\mathbf{a}$ satisfies M_n monomial equality constraints of degree L_n of the form (29). Since $\mathbf{S}$ is assumed to have full column rank, we can without loss of generality assume that $\mathbf{S}$ is nonsingular ($K = R$). There exists a vector $\mathbf{w}\in\mathbb{C}^R$ such that $a_{ir} = \mathbf{e}_i^{(I)T}\mathbf{X}\mathbf{w} = \mathbf{e}_i^{(I)T}\mathbf{A}\mathbf{S}^T\mathbf{w}$, where $\mathbf{e}_i^{(I)}\in\{0,1\}^I$ denotes the unit vector with unit entry in position i. Relation (29) implies that $\prod_{l=1}^{L_n}(\mathbf{e}_{p_{l,m}}^{(I)T}\mathbf{X}\mathbf{w}) - \prod_{l=1}^{L_n}(\mathbf{e}_{s_{l,m}}^{(I)T}\mathbf{X}\mathbf{w}) = \mathbf{p}_{L_n}^{(n)T}\cdot\left(\mathbf{w}\otimes\cdots\otimes\mathbf{w}\right) = 0$, where $\mathbf{p}_{L_n}^{(m)} = \bigotimes_{l=1}^{L_n}(\mathbf{X}^T\mathbf{e}_{p_{l,m}}^{(I)}) - \bigotimes_{l=1}^{L_n}(\mathbf{X}^T\mathbf{e}_{s_{l,m}}^{(I)})\in\mathbb{C}^{R^L}$. Stacking yields $\mathbf{P}^{(M_n,L_n)}\cdot(\mathbf{w}\otimes\cdots\otimes\mathbf{w}) = \mathbf{0}$, where $\mathbf{P}^{(M_n,L_n)} = [\mathbf{p}_{L_n}^{(1)},\ldots,\mathbf{p}_{L_n}^{(M_n)}]^T\in\mathbb{C}^{M_n\times R^{L_n}}$. It can now be verified if the dimension of

$$\ker\begin{bmatrix}\mathbf{I}_{R^{L_1-L_1}}\otimes\mathbf{P}^{(M_1,L_1)}\\ \vdots \\ \mathbf{I}_{R^{L_1-L_N}}\otimes\mathbf{P}^{(M_N,L_N)}\end{bmatrix}\cap\pi_S^{(L_1,R)}$$

is minimal (i.e., R), where 'ker' denotes the kernel of the matrix and $\pi_S^{(L_1,R)}$ denotes the set of vectorized ("flattened") versions of the symmetric tensors in the vector space of all symmetric L_1-th order tensors defined on $\mathbb{C}^R$. Then the GVMF of $\mathbf{X}$ is unique; see [13] for details.

V. Conclusion

In this paper we studied the identifiability properties of the GVMF. We presented generic identifiability conditions and briefly explained how deterministic identifiability conditions can be derived. The results demonstrate that when the signal matrix has full column rank, then DOA and source identification based on a NLA/NRA is possible under roughly the same conditions as when an ULA/URA is used.

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