SINGLE-SNAPSHOT DOA ESTIMATION VIA WEIGHTED HANKEL-STRUCTURED MATRIX COMPLETION

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ABSTRACT

In this paper, we consider the problem of estimating direction of arrivals (DOA) using a single snapshot of sparse linear array (SLA); the employed SLA is a sampled version of a uniform linear array (ULA). For the estimation task, we propose a two-step algorithm: (i) we first interpolate for the missing samples of the SLA to form a complete ULA by converting the samples into Hankel matrix and solving a weighted low-rank minimization. (ii) Next, we estimate the DOAs using a subspace method, like Prony. In step (i), the matrix completion problem is approached by adding left and right weight matrices to the Hankel matrix obtained by lifting the antenna observations. Simulation results show that the proposed method has superior accuracy in DOA estimation compared to the other methods proposed in the literature, such as atomic-norm minimization and off-the-grid approaches.

Index Terms— Direction of arrival; Matrix completion; Non-uniform sampling; Off-the-grid DOA estimation; Super-resolution.

1. INTRODUCTION

Estimating the direction of arrivals (DOA) using measurements on sensor arrays is required in a variety of applications, spanning from radar and sonar applications to biomedical engineering. Given its relevance, the DOA estimation problem has been studied for decades. Estimation of the auto-correlation function is among the primary methods for DOA estimation when multiple snapshot measurements are available [1]. The advent of the compressed sensing (CS) allowed for taking advantage of the natural sparsity of the sources in certain domains (e.g., spatial domain) in order to reduce the number of required snapshots to one [2], which is commonly known as single snapshot DOA estimation. As the standard compressed sensing methods are developed for sparse vectors, a pre-defined grid for the targets had to be assumed in [2]. In practice, however, the targets might not lie on the grid which leads to grid mismatch error [3]. To alleviate the mismatch error, various grid selection approaches are proposed [4]; however, in this fashion, we can never fully remove this type of error [5]. Recently, grid-less sparse DOA estimation methods have been proposed in the literature that estimate a sparse mixture of continuous single-frequency signals using both the uniform linear arrays (ULA) [6] and the sparse linear arrays (SLA) [7]. Although powerful, these methods require a minimum angular separation between the direction of sources. Another grid-less approach, EMaC, is introduced in [8] that estimates DOAs from a SLA. Similarly, EMaC requires a minimum angular source separation, but a smoother one; hence, it is able to retrieve more sources under similar conditions, especially in challenging source setups (almost co-located sources). In [8], a matrix completion method is devised for the DOA estimation problem; in particular, the measurements of a SLA are transformed into a Hankel-structured matrix with missing samples. The missing elements correspond to the ULA samples not included in the SLA. After interpolating the missing elements (matrix completion), [8] estimated the DOAs from the completed ULA. To improve the performance of matrix completion underlying a SLA, an adaptive non-uniform sampling method is proposed in [9] that relies on a concept called leverage scores. A more practical and improved two-snapshot version of this method is presented in [10]: the first snapshot is employed to estimate the leverage scores and decide for the most informative array elements (not necessarily available among the samples). In the second snapshot, the measurements at the determined array elements are processed to estimate the DOAs. Note, however, that in most practical DOA estimation problems, the array (SLA in this case) is fixed and could not be adaptively changed. Leverage scores are also used in the matrix completion technique of [11]; the method transforms the scores into weight matrices in a suitable way that are later used in a weighted minimization task. It should be highlighted that the methods uses the leverage scores as prior information.

Contributions: In this paper, we propose a single-snapshot DOA estimation algorithms in which the leverage scores and the DOA estimation are achieved over the same array positions. We should highlight that the previous approaches relying on leverage scores for DOA estimation relied on a mechanism in which the antenna array elements could be adaptively tuned, which is unfeasible in many practical scenarios. Unlike the existing work on leverage scores, our method in this work does not rely on leverage scores as prior knowledge.

Instead, we design weight matrices tailored to the EMaC algorithm that best match the existing array position based on the leverage scores. Then, we solve a weighted nuclear-norm minimization for matrix completion (transforming the SLA into a ULA), and estimate the DOAs accordingly. For this reason, we refer to the proposed approach as *Weighted EMaC (WEMaC)*. To solve the nuclear-norm minimization involved in the matrix completion step, we introduce an alternating direction method of multipliers (ADMM) scheme. Finally, to find the DOAs, we apply prony's method [12] on the ULA. Our simulation results confirm that the proposed approach outperforms some of the existing methods such as a grid-based CS method (BP), EMaC, and atomic-norm minimization (ANM).

2. SYSTEM MODEL AND NOTATIONS

Assume an array with m elements among n equi-distant locations (distance d_s between each two neighbors) across a line. We call this a SLA if $m < n$, and ULA if $m = n$. The received signal with wavelength λ at the k-th element of a SLA (or ULA) from a far-field source at angle ϕ with respect to the array could be modeled as $b\, a_k(\phi)$ where $a_k(\phi) = \exp\big(-\mathrm{j}\frac{2\pi}{\lambda}kd_s\sin(\phi)\big)$ is the phase shift of signals relative to the reference element (e.g., the first) of the array, and $b \in \mathbb{C}$ stands for the amplitude and initial phase of the signal. Consequently, for r sources at angles $\phi_1, \ldots, \phi_r$, the received signal is given by

$$y_k = \sum_{\ell \in [r]} b_\ell a_k(\phi_\ell). \tag{1}$$

It is common to define the continuous value $\tau_\ell := \frac{d_s}{\lambda}\sin(\phi_\ell)$. Now, the DOA estimation problem can be formulated as the task of estimating the pairs $\{\tau_\ell, b_\ell\}_{\ell\in[r]}$ from the set of observations $\boldsymbol{y}_\Omega = [y_{\Omega_r}, \ldots, y_{\Omega_m}]$, where $\Omega \subset [n]$ consists of the m available locations of SLA array elements.

Notation: Column vectors and matrices are denoted by lowercase and uppercase boldface letters, respectively (e.g. $\mathbf{x}$ and $\mathbf{M}$). $\mathcal{H}^d : \mathbb{C}^n \to \mathbb{C}^{d\times(n-d+1)}$ is the Hankel lifting operator that maps the input vector $\mathbf{x}$ into a Hankel matrix $\mathbf{M} = \mathcal{H}^d(\mathbf{x})$, such that $\mathbf{M}_{(i,j)} = \mathbf{x}_{|i+j-1|}$ In other words,

$$\mathbf{M} = \mathcal{H}^d(\mathbf{x}),\ \ \mathbf{M}_{(i,j)} = \mathbf{x}_{|i+j-1|}, \tag{2}$$

${\mathcal{H}^d}^\dagger$ is the inverse Hankel operator. We denote the rank operator, the Hermitian operator (i.e. conjugate-transpose), the Frobenius norm and the nuclear-norm of a matrix $\boldsymbol{X}$ by $\mathrm{rank}(\boldsymbol{X})$, $\boldsymbol{X}^{\mathrm{H}}$, $\|\boldsymbol{X}\|_{\mathrm{F}}$ and $\|\boldsymbol{X}\|_*$, respectively. Also, $\mathbb{1}_{\boldsymbol{x},\Omega}$ refers to the indicator function where it equals to $\boldsymbol{x}_i$ for $i \in \Omega$ and infinity otherwise. $\mathcal{P}_\Omega : \mathbb{C}^n \to \mathbb{C}^m$ with $m \le n$ is the projection operator that discards the elements of the input, the indices of which are not inside Ω. n simply shows the set $\{1, \ldots, n\}$. We define $\boldsymbol{e}_i^n$ as the ith canonical basis of $\mathbb{R}^n$; similarly, $\boldsymbol{A}_k = \frac{\mathcal{H}^d(\boldsymbol{e}_i^n)}{\|\mathcal{H}^d(\boldsymbol{e}_i^n)\|_{\mathrm{F}}}$ are basis matrices for the space of Hankel matrices with size $d \times (n-d+1)$.

3. DOA ESTIMATION BY WEIGHTED MATRIX COMPLETION

In WEMaC, we estimate the set $\{\tau_\ell, b_\ell\}_{\ell\in[r]}$ from the observed data $\boldsymbol{y}_\Omega$, in three steps: (i) we first lift the observation through the Hankel operator, then (ii) we reconstruct noiseless measurements over the ULA through a weighted matrix completion, and (iii) we finally estimate the DOAs by employing Prony's method which performs well in absence of noise. Under noisy settings, this last step could be replaced with more robust approaches. A conceptual representation of the proposed approach is provided in Fig. 1.

In step (ii), we make use of leverage scores which were originally introduced in [9] for an adaptive sampling strategy. The leverage scores roughly represent the importance of each sample in the overall matrix completion problem. It is shown that random sampling of matrix elements with probabilities proportional to the leverage scores reduce the number of required samples for prefect recovery in standard matrix completion [9] and also Hankel matrix structures [10]. However, leverage scores themselves depend on the full matrix (ULA samples in our case) and their availability as prior information is not feasible. In adaptive sampling techniques, one observes few matrix entries, approximates the leverage scores and captures new samples based on the leverage scores.

As an alternative solution to prior knowledge, an adaptive sampling technique is proposed in [10] to estimate the scores through the snapshots. One restriction of the method is the need for multi-snapshots (at least two) which makes it inapplicable in some practical settings. Here, we instead aim to modify the reconstruction strategy based on the leverage scores using the available samples. To do that, we introduce two left and right weight matrices $\boldsymbol{L}$ and $\boldsymbol{R}$ and minimize the rank (nuclear-norm) of $\boldsymbol{L}\mathcal{H}^d(\boldsymbol{x})\boldsymbol{R}$; the weight matrices are tuned with respect to the log-likelihood of the sampling probability, so that the leverage scores are proportional to the sampling distribution. In other words, we incorporate the weight matrices to the Hankel structure matrix completion and propose the WEMaC for completion of the array as follows

$$\widehat{\boldsymbol{y}} = \underset{\boldsymbol{g}\in\mathbb{C}^n}{\mathrm{argmin}}\ \|\boldsymbol{L}\mathcal{H}^d(\boldsymbol{g})\boldsymbol{R}^{\mathrm{H}}\|_*\ \ \text{s.t.}\ \mathcal{P}_\Omega(\boldsymbol{g}) = \boldsymbol{y}_\Omega, \tag{3}$$

where $\boldsymbol{L} \in \mathbb{C}^{d\times d}$ and $\boldsymbol{R} \in \mathbb{C}^{(n-d+1)\times(n-d+1)}$. To solve the optimization problem in (3), we need to find good candidates for weight matrices $\boldsymbol{L}$ and $\boldsymbol{R}$ in the sense of leverage scores for the weighted structure.

Definition 1. Assume $\boldsymbol{L}$ and $\boldsymbol{R}$ are complex square matrices of size $d \times d$ and $(n-d+1)\times(n-d+1)$, respectively. Let $\boldsymbol{U}_{d\times r}\boldsymbol{\Sigma}_{r\times r}(\boldsymbol{V}_{(n-d+1)\times r})^{\mathrm{H}}$ be the SVD of $\boldsymbol{L}\mathcal{H}^d(\boldsymbol{x})\boldsymbol{R}^{\mathrm{H}}$

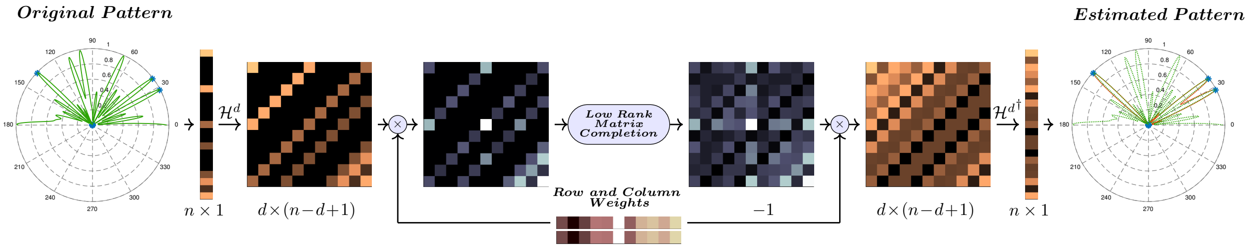


Fig. 1: The overview of the proposed DOA estimation method: *WEMaC*.

where its rank is r for $\boldsymbol{x}\in\mathbb{C}^n$. The weighted leverage scores μ_k for each $k\in[n]$ is defined as

$$\mu_k := \frac{n}{r}\max\{\|\mathcal{P}_U(\boldsymbol{A}_k)\|_{\mathrm{F}}^2, \|\mathcal{P}_V(\boldsymbol{A}_k)\|_{\mathrm{F}}^2\}, \quad k\in[n], \quad (4)$$

where $\mathcal{P}_U$ and $\mathcal{P}_V$ are projection operators that map a given $\boldsymbol{Y}\in\mathbb{C}^{d\times(n-d+1)}$ into

$$\mathcal{P}_U(\boldsymbol{Y}) = \boldsymbol{L}^{\mathrm{H}}\boldsymbol{U}\left(\boldsymbol{U}^{\mathrm{H}}\boldsymbol{L}\boldsymbol{L}^{\mathrm{H}}\boldsymbol{U}\right)^{-1}\boldsymbol{U}^{\mathrm{H}}\boldsymbol{L}\boldsymbol{Y}, \quad (5a)$$

$$\mathcal{P}_V(\boldsymbol{Y}) = \boldsymbol{Y}\boldsymbol{R}^{\mathrm{H}}\boldsymbol{V}\left(\boldsymbol{V}^{\mathrm{H}}\boldsymbol{R}\boldsymbol{R}^{\mathrm{H}}\boldsymbol{V}\right)^{-1}\boldsymbol{V}^{\mathrm{H}}\boldsymbol{R}. \quad (5b)$$

In [10], sample complexity of $\mathcal{O}\big(\sum_k^n \mu_k r^2\log^3(n)\big)$ is shown to guarantee the perfect recovery using optimization (3) by sampling each element of the ULA proportional to its corresponding score. Let p_k represent the probability of observing the k-th element of the array.To align the sampling probabilities with the observed indices in Ω, we maximize the likelihood of obtaining the current samples over the probabilities $\{p_k\}_{k\in[n]}$ i.e. $\Big(\prod_{k\in\Omega}p_k\Big)\Big(\prod_{k\notin\Omega}(1-p_k)\Big)$. It is easy to see that $p_k=1,\ k\in\Omega$ and $p_k=0,\ k\notin\Omega$ maximizes the likelihood; however, it is necessarily consistent with the leverage scores e.g. [10, Theorem 1]. Instead, if we consider the lower-bounds in [10, Theorem 1], the highest likelihood happens if

$$p_k = \begin{cases} 1 & k\in\Omega \\ \min\{1\ ,\ c(r,n)\mu_k\} & k\notin\Omega. \end{cases} \quad (6)$$

It should be noted that we expect $c(r,n)\mu_k$ to be small for $k\notin\Omega$; otherwise, the overall likelihood shall not be enough to consider Ω a typical outcome of the random sampling. Consequently, we use the condition $c(r,n)\mu_k<1$ for tuning the weights; nevertheless, its validity should be rechecked after tuning the weight matrices. We further approximate $\sum_{k\notin\Omega}\log(1-p_k)\approx-\sum_{k\notin\Omega}p_k$ for the case p_k is small enough which is valid for $k\notin\Omega$. Then, we determine $\boldsymbol{L},\boldsymbol{R}$ by minimizing the log-likelihood as

$$\boldsymbol{L}^*,\boldsymbol{R}^* = \underset{\boldsymbol{L},\boldsymbol{R}}{\mathrm{argmax}} -\sum_{k\notin\Omega}p_k \equiv \underset{\boldsymbol{L},\boldsymbol{R}}{\mathrm{argmin}}\sum_{k\notin\Omega}\mu_k. \quad (7)$$

Here, to compute μ_ks in (4), we need to have access to the matrices $\boldsymbol{U}$ and $\boldsymbol{V}$ of the optimal solution $\boldsymbol{x}$ which is unavailable. Both for the sake of simplicity and to solve this issue, we restrict the weight matrices to be diagonal; this enables us to invoke the following Lemma.

Lemma 3.1. *Let left and right weight matrices be restricted to non-negative diagonal matrices* $\boldsymbol{L}=\mathrm{diag}\big(\sqrt{L_1},\dots,\sqrt{L_d}\big)$, *and* $\boldsymbol{R}=\mathrm{diag}\big(\sqrt{R_1},\dots,\sqrt{R_{n-d+1}}\big)$, *where* L_i *for* $i\in[d]$ *and* R_j *for* $j\in[n-d+1]$ *are sorted in ascending order. Then, the leverage scores in Definition 1 are bounded as*

$$\frac{\mu_k r}{n}\leq\max\left\{\frac{\|\boldsymbol{L}\boldsymbol{A}_k\|_{\mathrm{F}}^2}{\sum_{i\in[N]}L_i}, \frac{\left\|\boldsymbol{A}_k\boldsymbol{R}^{\mathrm{T}}\right\|_{\mathrm{F}}^2}{\sum_{j\in[N]}R_j}\right\}, \quad (8)$$

where $N=\min\left\{\lfloor\frac{1}{\|\boldsymbol{U}^{\mathrm{H}}\|^2}\rfloor,\lfloor\frac{1}{\|\boldsymbol{V}^{\mathrm{H}}\|^2}\rfloor\right\}$.

Proof. Due to the page limitations, the proof is provided online. [1] □

Based on Lemma 3.1, we can replace μ_ks by their upper-bound in (8). As this upper-bound remains unchanged by constant scaling of the weight matrices, we impose $\|\boldsymbol{L}\|_{\mathrm{F}}^2=n-d+1$ and $\|\boldsymbol{R}\|_{\mathrm{F}}^2=d$ to provide uniqueness of the solution. Therefore, (7) can be rewritten as

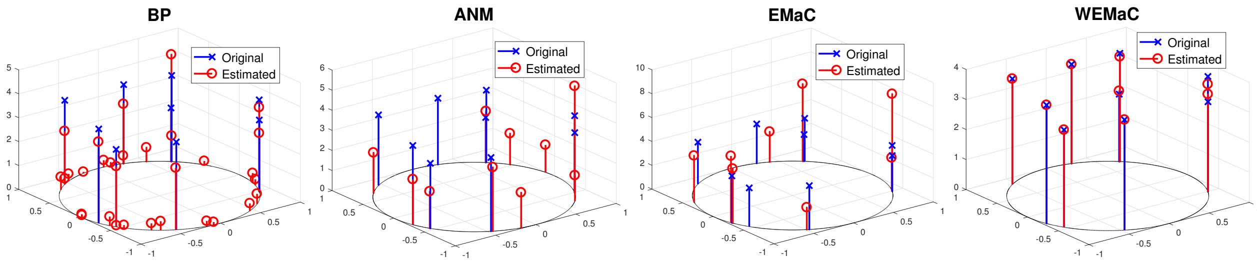
$$\begin{aligned}\{L_i\}_{i\in[d]},\{R_j\}_{j\in[n-d+1]} &= \underset{L_i,R_j\in\mathbb{R}_+}{\mathrm{argmin}} \\ &\sum_{k\notin\Omega}\frac{\max\left\{\sum_{i\in[d]}\sum_{i\in[n-d+1]}R_j\,\mathbb{1}_{i\leq k\leq d+i-1}\right\}}{\min\{d,n-d+1,k,n-k+1\}} \\ &\text{s.t. } \textstyle\sum_{i\in[d]}L_i=n-d+1,\sum_{i\in[n-d+1]}R_j=d.\end{aligned} \quad (9)$$

We should emphasize that (9) is a convex optimization problem which can be solved tractably.

As shown in [13], the nuclear norm of matrix $\boldsymbol{A}$, i.e. $\|\boldsymbol{A}\|_*$ can be replaced with the $\min_{\boldsymbol{U},\boldsymbol{V}}\|\boldsymbol{U}\|_{\mathrm{F}}^2+\|\boldsymbol{V}\|_{\mathrm{F}}^2$ subject to $\boldsymbol{A}=\boldsymbol{U}\boldsymbol{V}^{\mathrm{H}}$. Accordingly, (3) transforms to

$$\begin{aligned}&\min_{\boldsymbol{U},\boldsymbol{V},\boldsymbol{g}\in\mathbb{C}^n}\ \|\boldsymbol{U}\|_{\mathrm{F}}^2+\|\boldsymbol{V}\|_{\mathrm{F}}^2 \\ &\quad\text{s.t. } \mathcal{P}_\Omega(\boldsymbol{g})=\boldsymbol{y}_\Omega,\ \ \boldsymbol{L}\mathcal{H}^d(\boldsymbol{g})\boldsymbol{R}^{\mathrm{H}}=\boldsymbol{U}\boldsymbol{V}^{\mathrm{H}}.\end{aligned} \quad (10)$$

[1] See http://sharif.ir/∼aamini/Papers/EUSIPCO2022_Hankel.pdf.



(a) (b) (c) (d)

Fig. 2: The performance of different methods in DOA estimation of a challenging setup: among the 9 considered sources, two pairs are almost co-located. The 360° angular space and received power from the targets are shown by the base circle and the height of the bars, respectively.

Algorithm 1 Weighted Interpolation using ADMM

Input: Sampling indices $\Omega \subset [n]$, corresponding samples $\boldsymbol{y}_\Omega \in \mathbb{C}^m$, and parameter ρ for the augmented Lagrangian form.
Output: Completed Vector $\widehat{\boldsymbol{y}} \in \mathbb{C}^n$.
procedure WEIGHTED INTERPOLATION($\boldsymbol{y}_\Omega, \rho$)
    Solve (9) to find $\boldsymbol{L}$ and $\boldsymbol{R}$
    Solve ADMM problem in (11) (using ρ, $\boldsymbol{L}$, and $\boldsymbol{R}$ as its inputs) to find the interpolated ULA data $\widehat{\boldsymbol{y}}$
    return $\widehat{\boldsymbol{y}}$
end procedure

To apply the ADMM technique, we build the augmented Lagrangian of the cost function as

$$\begin{aligned}\mathcal{L}_\rho(\boldsymbol{U},\boldsymbol{V},\boldsymbol{g},\boldsymbol{\Lambda}) = \|\boldsymbol{U}\|_{\mathrm{F}}^2 + \|\boldsymbol{V}\|_{\mathrm{F}}^2 + \mathbb{1}_{\boldsymbol{g},\Omega} \\ + \rho\|\boldsymbol{L}\mathcal{H}^d(\boldsymbol{g})\boldsymbol{R}^{\mathrm{T}} - \boldsymbol{U}\boldsymbol{V}^{\mathrm{H}} + \boldsymbol{\Lambda}\|_{\mathrm{F}}^2,\end{aligned} \quad (11)$$

where the Lagrange multiplier $\boldsymbol{\Lambda}$ has the same size as $\mathcal{H}^d(\boldsymbol{g})$, ρ is an arbitrary positive scalar, and $\mathbb{1}_{\boldsymbol{g},\Omega}$ is the indicator function. For the ADMM, we should iteratively update $\boldsymbol{U}$, $\boldsymbol{V}$, $\boldsymbol{g}$, and $\boldsymbol{\Lambda}$. Due to lack of space, the equations are also provided in the manuscript containing the proof of Lemma 3.1. Note that (10) is bi-linear in terms of $\boldsymbol{U}$ and $\boldsymbol{V}$, but not necessarily a convex problem overall. However, the ADMM is guaranteed to converge to correct result if the penalty parameter ρ is sufficiently large [14, 15].

4. SIMULATIONS

To evaluate the performance of proposed method WEMaC, we compare it with ANM [7], EMaC [8], and BP methods which are single-snapshot DOA estimation approaches. ANM and EMaC, similar to the WEMaC, consist of two steps: (i) an interpolation step to reconstruct the ULA output and (ii) an estimation step of the DOAs by applying a super-resolution method (such as [6] or [12]). Therefore, we analyze the performance of the algorithms in terms of

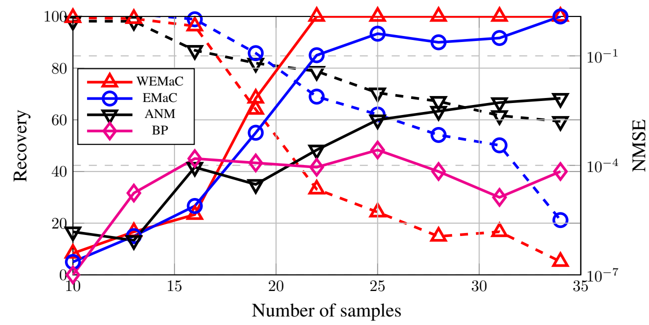


Fig. 3: The DOA recovery (solid curves) and the SLA interpolation performance (dashed curves) in terms of normalized mean-square error are depicted with respect to the array size. A source setup with 6 sources with two pairs of almost co-located sources are considered.

both (i) the mean-square error of array interpolation and (ii) the DOA estimation accuracy. We further study the effect of actual array size (or the number of elements m) on the performance of the algorithms. In our simulations, we set the spacing between adjacent array elements in the ULA as $\lambda/2$ where λ is the wavelength. We choose n to be an odd integer and set the pencil parameter d such that the resulting Hankel matrix is a square (this is possible for odd n). The SLA is constructed by uniformly sampling the ULA in all simulations. For the BP grid-based method, we uniformly divide the interval $[-1, 1]$ into 2^{12} bins. This interval speaks for the sin-transformed angles.To measure the estimation accuracies, we call the estimated angle $\widehat{\theta}$ of a source at θ successful if $|\sin(\theta) - \sin(\widehat{\theta})| \leq 0.005$. Furthermore, in the ADMM method, parameter $\rho = 10^3$ is chosen as a large number.

DOA Estimation: The SLA array is constructed by choosing $m = 25$ locations uniformly at random from 101 possible locations of the ULA array. As a challenging scenario, we consider 9 sources, two pairs of which are almost co-located. The ground-truth DOAs and the estimated ones are plotted in Figure 2 where the based circle and the height of bars indicate the angles and powers, respectively. We observe that the existing methods not only are unable to detect or discriminate the almost co-located sources, but also they do estimate considerable ghost sources. At the same time, WEMaC managed to spot all sources and differentiate between the almost co-located sources.

Effect of number of antennas: In this section, we investigate the performance of WEMaC, EMaC, ANM, and BP algorithms in terms of array interpolation and DOAs recovery versus the number of array elements. We consider an array with aperture size $100\lambda/2$ and form the SLA with 10 to 34 elements which are randomly chosen from the ULA with uniform probability. Further, we consider a demanding source setup with 6 sources located at $[-60.48°$, $-45.69°$, $-16.70°$, $-15.51°$, $19.44°$, $20.66°]$ (two pair of them are seemingly co-located) with corresponding amplitudes $[3.35, 2.30, 3.16, 2.07, 3.68, 2.37]$. Each curve represents the results averaged over 100 random realizations of the SLA element selection.

The percentage of DOA recoveries in terms of the SLA size (number of antennas) are depicted in Figure 3-(solid lines). While EMaC based approaches outperform ANM and BP, the performance of the WEMaC is a cut above the EMaC, so that WEMaC detects all DOAs after a certain SLA size. We also plot the normalized mean-square errors of the interpolation method (estimating the missing samples) in Figure 3 using dashed curves. We observe that the interpolation error of WEMaC is better than EMaC and ANM almost for any array size.

5. CONCLUSION

We proposed a single snapshot DOA estimation based on the measurements of a SLA. The method works by first estimating the measurements at the missing elements of the array by means of a weighted nuclear-norm minimization over a Hankel structure. Then, the completed measurements mimicking a ULA are processed by the simple Prony's method to find the DOAs. The weight matrices in the interpolation stage are tuned based on the leverage scores.

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