Density Aware Blue-Noise Sampling on Graphs

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***Abstract*—Efficient sampling of graph signals is essential to graph signal processing. Recently, blue-noise was introduced as a sampling method that maximizes the separation between sampling nodes leading to high-frequency dominance patterns, and thus, to high-quality patterns. Despite the simple interpretation of the method, blue-noise sampling is restricted to approximately regular graphs. This study presents an extension of blue-noise sampling that allows the application of the method to irregular graphs. Before sampling with a blue-noise algorithm, the approach regularizes the weights of the edges such that the graph represents a regular structure. Then, the resulting pattern adapts the node's distribution to the local density of the nodes. This work also uses an approach that minimizes the strength of the high-frequency components to recover approximately bandlimited signals. The experimental results show that the proposed methods have superior performance compared to the state-of-the-art techniques.**

***Index Terms*—Graph signal reconstruction, blue-noise, graph signal sampling, graph signal processing.**

I. Introduction

The sampling and reconstruction of graph signals have attracted the attention of the scientific community [1]; the earlier work on these tasks focused on strictly bandlimited signals and used the least-squares errors minimization reconstruction [1]. To generate high-quality sampling patterns, the sampling methods often rely upon expensive eigendecomposition. While methods that avoid eigendecomposition have been proposed, they still present different drawbacks such as high complexity, the need for parameter tuning and the limitations of strictly bandlimited signals models. Recently, the work in [2] used the graph Laplacian regularization to recover approximately bandlimited signals, which are a better representation of real world signals where a high-frequency component may arise. The authors in [2] minimize the worst-case error and avoid eigendecomposition using a clever approach based on the Gershgorin Disc theorem.

Blue-noise (BN) sampling of graph signals was introduced in [3] as an intuitive method applied entirely in the vertex domain. The algorithm is based on the concepts of spatial dithering, and it generates patterns by promoting separation between sampling nodes; this characteristic leads to patterns with high-frequency dominance, thus the name blue-noise, since the color blue is associated with the higher frequencies. Since, in irregular graphs, the uniform BN patterns fail to represent the graph distribution, the benefits of BN are restricted to highly regular graphs. This work extends BN sampling to signals on irregular graphs using regularization of the weight of the edges such that the graph defined by these new edges presents an approximately regular structure. In the mapped graph, a blue-noise sampling algorithm generates a uniform pattern such that the distribution of sampling nodes represents that of the original graphs.

Under the assumption that smooth signals present low-frequency dominance, given the sampled signal, this work uses a reconstruction method that aims to minimize the presence of high-frequency components on the recovered signals, relying on the optimization problem solved in [4]. The method allows the recovery of approximately bandlimited signals without using eigendecomposition and avoids the estimation of the signal's bandwidths.

This paper is organized as follows. In Section II the notation and blue-noise sampling are introduced. Section III presents the regularization of the weights for the sampling of irregular graphs and the greedy algorithm to select the regularization parameter. Section IV defines the reconstruction approach. Section V presents a set of experiments comparing the proposed methods against existing techniques, and Section VI presents a set of conclusions

II. Graph Signal Sampling and Reconstruction

A graph $G(V(G), E(G))$ consists of a set of nodes $V(G)$ and a set of edges $E(G)$. The strength of the connection between nodes is contained in the weight matrix, $\mathbf{W}$, where $\mathbf{W}(u,v) > 0$ if the node u is adjacent to v ($u \sim v$). The nodes adjacent to a node v is denoted by $\mathcal{N}(v) = \{u \in V(G) |\ \mathbf{W}(u,v) > 0\}$ also known as the neighborhood of v. A graph signal is a function that maps the nodes' information to the real numbers, $\boldsymbol{x} : V(G) \to \mathbb{R}^N$, and it is represented by a vector $\boldsymbol{x} \in \mathbb{R}^N$, whose entry $\boldsymbol{x}(v)$ represents the value of the signal on the node v.

The combinatorial Laplacian, $\mathbf{L}$, of a graph is given by $\mathbf{L} = \mathbf{D} - \mathbf{W}$, where $\mathbf{D}$ is the diagonal degree matrix whose entries are $\mathbf{D}(v,v) = \sum_{u \in V(G)} \mathbf{W}(u,v)$; and the normalized Laplacian is given by $\mathcal{L} = \mathbf{D}^{-1/2}\mathbf{L}\mathbf{D}^{-1/2}$. The graph Lapacian is often used as a variation operator as it allows to capture the smoothness of a signal by measuring how it varies according to the underlying structure of the graph [5]. Then, the variation of a signal $\boldsymbol{x}$ at a node v is computed in terms of the combinatorial Laplacian by $\| \ (\mathbf{L}\boldsymbol{x})_v \ \|_2 =$

$\left[\sum_{u\in\mathcal{N}(v)} \mathbf{W}(v,u)|\boldsymbol{x}(v)-\boldsymbol{x}(u)|^2\right]^{\frac{1}{2}}$ [5]. Smooth signals are expected to have low values of $\| (\mathbf{L}\boldsymbol{x})_v \|_2$, on every node [5].

The eigendecomposition of the Laplacian, $\mathbf{L}=\mathbf{U}\mathbf{\Lambda}\mathbf{U}^\top$, is often used as the Graph Fourier Transforms (GFT) [5], such that the eigenvectors $\mathbf{U}=[\mathbf{u}_1,\mathbf{u}_2,...,\mathbf{u}_N]$ represent a Fourier-like basis and the eigenvalues $0=\mu_1\leq\mu_2\leq...\leq\mu_N$ correspond to the frequencies, where $N=|V(G)|$ is the number of nodes in G. Then, the GFT of the graph signal $\boldsymbol{x}$ is given by $\widehat{\boldsymbol{x}}=\mathbf{U}^\top\boldsymbol{x}$, and the bandwidth, ω, of the signal $\boldsymbol{x}$ is defined by the non-zero components of $\widehat{\boldsymbol{x}}$ [5]. A signal is said to be k-bandlimited if it satisfies $\boldsymbol{x}=\mathbf{U}_k\widehat{\boldsymbol{x}}_k$, where $\widehat{\boldsymbol{x}}_k$ are the first k elements of $\widehat{\boldsymbol{x}}$ and $\mathbf{U}_k$ the first k vectors in $\mathbf{U}$.

A. Graph Signal Sampling

The problem of obtaining a unique and stable reconstruction given the signal on a subset of observed nodes, $S=\{s_1,s_2,...s_m\}$ with $m<N$ is known as reconstruction problem. The sampled signal is defined as $\boldsymbol{x}(S)=\mathbf{M}\boldsymbol{x}$, where $\mathbf{M}=[\delta_{s_1},\delta_{s_2},...,\delta_{s_m}]^\top$ is a sampling operator and δ_{s_i} is a Kronecker column vector centered at s_i. There are different approaches to recover signals from its values on S [2], [6], [7]. Strictly bandlimited signals ca be recovered by solving the least squares errors minimization:

$$\boldsymbol{x}=\underset{z\in\text{span}(\mathbf{U}_k)}{\text{argmin}} \| \mathbf{M}\boldsymbol{z}-\boldsymbol{x}(S) \|_2 = \mathbf{U}_k(\mathbf{M}\mathbf{U}_k)^\dagger\boldsymbol{x}(S), \quad (1)$$

where $\mathbf{U}_k$ contains the first k eigenvectors of the GFT such that $\mu_k\leq\omega$ and $(\mathbf{M}\mathbf{U}_k)^\dagger\in\mathbb{R}^{k\times m}$ is the Moore-Penrose pseudo-inverse of $\mathbf{M}\mathbf{U}_k$ [6], [8]. Note that for $m\leq k$, the system is underdetermined and there are no guarantees of a unique reconstruction [8]. There exist two sampling schemes that minimize recovery errors: random and deterministic. Random sampling methods select the sampling nodes according to a probability distribution [8]. On the other hand, deterministic methods minimize an objective function by adding one node at a time optimizing some objective function on each iteration [1]. Naturally, for k-bandlimited sampling consist on the optimal row selection of $\mathbf{U}_k$.

Smooth signals that are approximately bandlimited can be recovered by solving the Laplacian regularization:

$$\begin{aligned}\boldsymbol{x}_{\text{rec}} &= \underset{\boldsymbol{z}}{\text{argmin}} \quad \| \mathbf{M}\boldsymbol{z}-\boldsymbol{x}(S) \|_2 + \phi(\boldsymbol{z}^\top\mathbf{L}^q\boldsymbol{z}) \\ &= (\mathbf{M}^\top\mathbf{M}+\phi\mathbf{L})^{-1}\mathbf{M}^\top\boldsymbol{x}(S), \end{aligned} \quad (2)$$

where ϕ is a parameter that balances the data's fidelity and the signal's smoothness. The work on [2] proposes a sampling approach that minimizes the worst-case error of the reconstruction that avoids eigendecomposition by using the Gershgorin Disc theorem.

This paper recovers a signal by minimizing the variation of them within a certain neighborhood od the nodes, and uses a sampling method based on the spatial characteristics of the graph to minimize recovery errors.

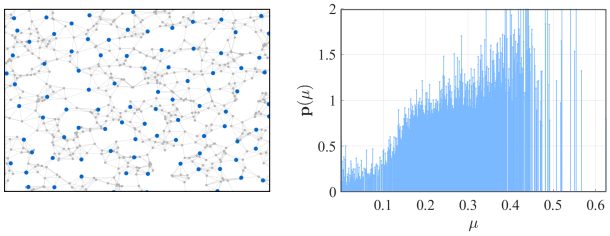

Fig. 1. Spatial and spectral characteristics of blue-noise sampling patterns with density $d=0.15$ on a sensor graph with 1000 nodes: (left) a blue-noise sampling pattern on the graph, (right) the average power spectrum of 100 blue-noise sampling patterns.

B. Blue-Noise Sampling

In order to introduce blue-noise, first the notion of distance on a graph needs to be defined. The distance between two nodes $\mathbf{\Gamma}(u_a,u_b)$ is defined as the path with minimum length between them, where a path is a sequence of edges that joins a sequence of nodes $(u_a,u_1,...,u_n,u_b)$.

Blue-noise was originally introduced as a dithering technique [9]–[11], in graphs, blue-noise sampling distributes the sampling nodes as homogeneously as possible, such that the selected nodes are separated by a minimum distance of λ_b, known as the principal path-length of the pattern. Naturally, the value of λ_b is inversely proportional to the number of samples. Blue-noise patterns are also characterized by a high-frequency dominance on their power spectrum. Figure 1 (right) shows the graph power spectrum of 100 ideal blue-noise sampling patterns on a Sensor Network graph with $N=1000$. In this plot, the high-frequency components are substantially larger than low-frequency components; this phenomenon is due to the maximization of the minimum separation between samples which results in patterns that look like the one depicted in Fig. 1 (left).

In color theory, the red color is associated with lower frequencies, hence, the redness of a pattern measures the strength of the lower frequency components of the pattern:

$$R_{\boldsymbol{s}}=\frac{1}{\| \widehat{\boldsymbol{s}} \|_2^2}\sum_{\ell=2}^{N}\frac{\widehat{\boldsymbol{s}}^2(\ell)}{\mu_\ell}=\frac{1}{m}\sum_{\ell=2}^{N}\frac{\widehat{\boldsymbol{s}}^2(\ell)}{\mu_\ell}, \quad (3)$$

where $\boldsymbol{s}$ denotes the binary signal that represents the sampling set, with $\boldsymbol{s}=1$ if $v\in S$ and $\boldsymbol{s}=0$ otherwise, and $\widehat{\boldsymbol{s}}$ is the Graph Fourier Transform of $\boldsymbol{s}$. Note that good blue-noise patterns should minimize $R_{\boldsymbol{s}}$.

The redness of a pattern is connected to the spatial characteristics of the graph in [3], using the concepts of graphs partitions and the isoperimetric dimension. For a graph $G=(V(G),E(G))$, a partition is a division of the nodes in the graph $V(G)$ into $|\mathcal{P}|$ subsets, $\mathcal{P}=\{V(\Omega_1),V(\Omega_2),...,V(\Omega_{|\mathcal{P}|})\}$ and Ω_j is the subgraph induced by the nodes on $V(\Omega_j)$. The isoperimetric dimension, δ, measures how similar is the graph G to a grid-like graph [3].

The work in [3] considers that a pattern is generated from a partition $\mathcal{P}$ by locating only one sampling node v per partition element, such that all the neighbors of the sample exist on

the same partition. It was also shown that the redness of the sampling pattern is connected to the characteristics of the partition from which it is generated by:

$$R_{\boldsymbol{s}} \leq \frac{(\mu_2+\mu_N)^2(1-|\mathcal{P}|/N)^2}{4\mu_2\mu_N \min_j \left\{\frac{C_{\delta_j}}{vol(\Omega_j)^{2/\delta_j}}\right\}}, \tag{4}$$

where $vol(\Omega_j) = \sum_{u\in\Omega_j} \mathbf{D}(u,u)$, δ_j is the isoperimetric dimension of the induced subgraph Ω_j, and C_{δ_j} is a constant that depends only on δ_j. The work in [3] proved that patterns that minimize the redness maximize the quality of the patterns.

It was shown in [3] that on graphs satisfying $\delta_1 = \delta_2 = ... = \delta_{|\mathcal{P}|} = \delta_G$, partitions with equal volume, $vol(\Omega_i) = vol(\Omega_j) \quad \forall i,j$, minimize the redness of the pattern and thus have high quality. For graphs with non-regular degree distributions, the isoperimetric dimension can vary from one partition to another. In this case, the patterns generated by an equal partition of an irregular graph do not minimize the right-hand side of (4).

III. Density Aware Blue-Noise Sampling

To define irregular graphs, it is necessary to introduce the notion of density. The density of a single node is computed as:

$$\boldsymbol{\rho}(v) = \frac{\mathbf{D}(v,v)}{|V(G)|}. \tag{5}$$

Formally, Chung [12] defines approximately regular graphs as a graph where all but n vertices have the same degree. While there are different characterizations of irregular graphs, here a graph is considered to be irregular if the standard deviation, σ, of the nodes' density, $\boldsymbol{\rho}(v)$, satisfies $\sigma \geq 0.1\rho$, where ρ is the density of G.

This work seeks to generate a good quality sampling pattern on irregular graphs using blue-noise sampling. To do so, the sampling approach proposed on [3] is applied to a modified version of the graph that is approximately regular, where the isoperimetric dimension varies less between partitions. The modified graph is denoted G_{ar}, and it consists of the same set of nodes $V(G_{ar}) = V(G)$ and edges $E(G_{ar}) = E(G)$ but the weights of the edges are modified according to:

$$\mathbf{W}_{ar}(u,v) = \mathbf{W}(u,v)\max\left\{\boldsymbol{\rho}(u),\boldsymbol{\rho}(v)\right\}^{\alpha}, \tag{6}$$

where $\boldsymbol{\rho}(i)$ is the density of the node i and α is a variable that controls the uniformity of G_{ar}. The weight of the edges is amplified or reduced depending on the nodes' density and the value of α. In general, (6) increases the distance between the nodes in denser parts of the graph by modifying the weight of the edges. The edge's weight is multiplied by the maximum of the density of the nodes it connects to ensure that denser nodes are spread in the resulting graph G_{ar}. The regularization parameter, α, controls how sparse the regions become.

Ideally, for blue-noise sampling, the value of α should be selected such that it maps G into an approximately regular graph G_{ar}. Therefore, α should minimize the standard deviation of the density vector of the graph, such that it tends to be regular; this is equivalent to minimizing the standard deviation of $\mathbf{D}_{ar}(u,u)$. Then, the optimal value of α solves the next optimization problem:

$$\begin{aligned} \underset{\alpha}{\text{minimize}} \quad & \sigma(\mathbf{D}_{ar}\mathbf{1}) \\ \text{s.t.} \quad & \mathbf{D}_{ar}(v,v) = \sum_{u\sim v} \mathbf{W}(u,v)\max\{\boldsymbol{\rho}(u),\boldsymbol{\rho}(v)\}^{\alpha}, \end{aligned} \tag{7}$$

where $\sigma(\cdot)$ denotes the standard deviation, $\mathbf{1}$ a column vector of ones with N rows, $\mathbf{D}_{ar}(v,v)$ denotes the degree of node v in the mapped graph, $\mathbf{W}(u,v)$ is the weight of the edge $u\sim v$, α is the density regularization parameter, and $\rho(v)$ is the nodes' density given by (5).

A. Regularized Void and Cluster

Since the objective function in (7) is a non-convex function of α, which is defined by the sum of weighted exponentials, the problem is not convex, this work uses a greedy approach based on the method proposed in [13] to find the value of α. For simplicity, this work denotes $f_\alpha = \sigma(\mathbf{D}_{ar})$. To find a suitable value of α, the proposed approach starts with $\alpha = 0$ and modifies its value by adding to it a small constant δ_g (step size) until the next condition holds, $f_{\alpha-\delta_g} - f_\alpha \geq 2(f_\alpha - f_{\alpha+\delta_g})$. The first value of α for which this condition is satisfied represents the value of α for which the function f_α stops dropping sharply or starts to increase. The algorithm proceeds as follows: first, it computes the standard deviation of the nodes' degree on the original graph (f_0). Then, sequentially the algorithm computes $\mathbf{W}_{ar}$ using (6) for $\alpha_r = \alpha_{r-1} + \delta_g$. For each value of α_r, the algorithm verifies the condition. If this condition holds, $\mathbf{W}_{ar} = \mathbf{W}_{\alpha_r}$ and $\alpha_g = \alpha_r$, otherwise the algorithm continues its greedy search. We define the generalized Void and Cluster (GVAC) as the algorithm where the Void and Cluster (VAC) sampling algorithm [3] is applied to the graph G_{ar} with weight matrix $\mathbf{W}_{ar}$. Void and Cluster is an iterative algorithm that generates an initial random pattern and repeatedly removes nodes from the tighter cluster and places them into the loosest void [3].

IV. Recovery of Approximately Smooth Signals

The spectrum of smooth signals on graphs is characterized by a low-frequency dominance. Therefore, the reconstruction model should aim to maximize these components and attenuate higher frequency components. To this end, we introduce the concept of blueness of a graph signal which captures the strength of the high-frequency components:

$$B_{\boldsymbol{x}} = \sum_{i=1}^{N} \mu_i^q \widehat{\boldsymbol{x}}_i^2, \tag{8}$$

where μ_i is the i^{th} eigenvalue of the shift operator, $\widehat{\boldsymbol{x}}$ is the Fourier Transform of the signal, and q is a factor that penalizes high-frequency components. Intuitively, for higher values of q, the high-frequencies will have a larger impact on the value of $B_{\boldsymbol{x}}$ than for smaller values of q. Then if the reconstruction approach seeks to minimize $B_{\boldsymbol{x}}$, the reconstructed signal $\boldsymbol{x}$ will contain fewer frequency components as the value of

q increases. When $q = 1$, $B_{\boldsymbol{x}} = \sum_{i=1}^{N} \mu_i \widehat{\boldsymbol{x}}_i^2 = \widehat{\boldsymbol{x}}^\top \Lambda \widehat{\boldsymbol{x}} = (\mathbf{U}^\top \boldsymbol{x})^\top \Lambda (\mathbf{U}^\top \boldsymbol{x}) = \boldsymbol{x}^\top \mathbf{L} \boldsymbol{x}$. The general form of blueness for $q \geq 1$, is $B_{\boldsymbol{x}} = \boldsymbol{x}^\top \mathbf{L}^q \boldsymbol{x}$. Then, the recovery of a graphs signal using Blueness Minimization can be formulated as the next optimization problem:

$$\begin{aligned} \boldsymbol{x}_{\text{rec}} = \underset{\boldsymbol{z}}{\text{argmin}} \quad & \boldsymbol{z}^\top \mathbf{L}^q \boldsymbol{z} \\ \text{s.t.} \quad & \mathbf{M}\boldsymbol{z} = \boldsymbol{x}(S). \end{aligned} \tag{9}$$

Note that while the objective in (9) is a convex problem, its closed-form requires the inversion of $\mathbf{L}^q$, which does not exist since $\mathbf{L}^q$ is a singular matrix. A perturbation $\mathbf{\Psi} = \epsilon \mathbf{I}$ with $\epsilon \in \mathbb{R}^+$ can be added to $\mathbf{L}^q$ such that the resulting matrix is invertible and meaningful. Using $\mathbf{L}^q + \mathbf{\Psi}$ instead of $\mathbf{L}^q$ in the problem in (9), the problem becomes the Sobolev norm minimization with close form [4]:

$$\boldsymbol{x}_{\text{rec}} = (\mathbf{L}^q + \mathbf{\Psi})^{-1} \mathbf{M}^\top (\mathbf{M} (\mathbf{L}^q + \mathbf{\Psi})^{-1} \mathbf{M}^\top)^{-1} \boldsymbol{x}(S). \tag{10}$$

Equation 10 is denoted as Blueness Minimization Reconstruction (BMR) and it offers two principal benefits. First, a cutoff frequency or bandwidth of the signal is not required. Secondly, since the method assumes signal smoothness it can recover approximately bandlimited signals.

V. Experiments

In the experiments, the performance of GVAC and the BMR is evaluated and compared to the state-of-the-art techniques in artificial signals using two artificial graphs as well as a real traffic network:

- $G1$: Small World Network graph with $N = 1000$, with rewiring probability $p = 0.8$ and average degree $\overline{d} = 80$.
- $G2$: The Minnesota traffic graph with $N = 2,642$ [14].

The artificial signals were generated according to the next models:

- SM1: the signals are noisy bandlimited. The signals were generated at random with a bandwidth of $k = 0.05N$, where the GFT coefficients are drawn from the Gaussian distribution $\mathcal{N}(1, 0.5^2)$. The sampled signal is contaminated with additive Gaussian noise such that the signal-to-noise ratio is $SNR = 20$dB.
- SM2: the true signals are approximately bandlimited. The GFT coefficients are drawn from the Gaussian distribution $\mathcal{N}(1, 0.5^2)$. Then the signal is passed through a Butterworth filter H of order $n = 8$ for $G1$ and $n = 1$ for $G2$, and $\omega_c = \mu_k$ with $k = 0.05N$ such that $H(\mu_i) = \left(1 + \left(\frac{\mu_i}{\mu_k}\right)^{2n}\right)^{-1}$.

The sampling methods proposed by Chen *et al.* [15] (E-optimal), Anis *et al.* [6] (SP), Sakiyama *et al.* [7] (ED-Free), Bai *et al.* [2] (BS-GDA), Parada *et al.* (VAC) [3] and Random sampling were compared against the proposed method, Generalized Void and Cluster (GVAC). For the approximately bandlimited signals (SM2), the filter's cut frequency $\omega = \omega_c$ was defined as the bandwidth of the signal for methods that required this parameter.

For each graph, 100 signals were generated using both signal models. Then, the signals were sampled using the different sampling approaches. Next, each signal was reconstructed using the following reconstruction methods: least-squares (LS) [6], GLR [2], the Splines method on [16] and BMR method in (10). For the BMR method, ϵ varied from 1×10^{-4} to 1×10^{3}, and the best result is reported for each sampling method. For Splines, the regularization parameter varied between 1×10^{-2} and 500, the best result is reported [16]. For SP [6], ED-Free [7] and BS-GDA [2] the parameter selections was done as suggested by the authors.

Figure 2 presents the average MSE of the 100 signals in terms of the size of the sampling set using BMR for the signals' recovery, it is clear that the performance of the proposed sampling method (GVAC) is comparable or better to deterministic approaches [6], [7], [15]. Note that the uniform patterns generated by VAC [3] perform similarly to random, proving the need for density regularization.

The comparison between different reconstruction methods is shown in Fig. 3. For the LS [6] the results using SP [6] sampling are reported. For the GLR [2] the results using BS-GDA [2] are reported. Finally, for Splines, and BMR the results using GVAC and the graphs' respective α value are reported. For the signal model SM1, since the signals are strictly bandlimited, the LS reconstruction [6] performs consistently better than other methods. For BMR, as q increases, the performance becomes closer to LS. For approximately bandlimited signals SM2, the proposed reconstruction method with $q = 8$ show the best overall performance as well.

Figure 4 shows the spatial representation of one repetition of the first experiment using BMR on a signal generated using SM2 on $G4$. The top row of Fig. 4 illustrates the absolute error between the reconstructed signal and the original signal. Finally, the bottom row of Fig. 4 presents the sampling patterns with density $d = 0.08$ used to obtain the signals' reconstruction. Note that the uniform pattern generated using VAC [3] fails to represent the tight cluster on the graph, resulting in higher reconstruction errors.

VI. Conclusions

This paper generalizes blue-noise sampling to irregular graphs. The proposed method first maps the graph into an approximately regular graph by regularizing the edges' weight. Then, a blue-noise algorithm (VAC) is used to sample the mapped graph. The experimental results showed that the proposed sampling approach achieves lower reconstruction errors on irregular graphs, than the original blue-noise sampling proposed in [3] and its performance is comparable to those that focus on the spectral characteristics of the graph, confirming the need to include the distribution of the nodes in the sampling process.

Based on the assumption that the signals of interest are smooth, this work introduced a reconstruction approach that minimizes the high-frequency components of the reconstructed signal's spectrum, named Blueness Minimization. The experi-

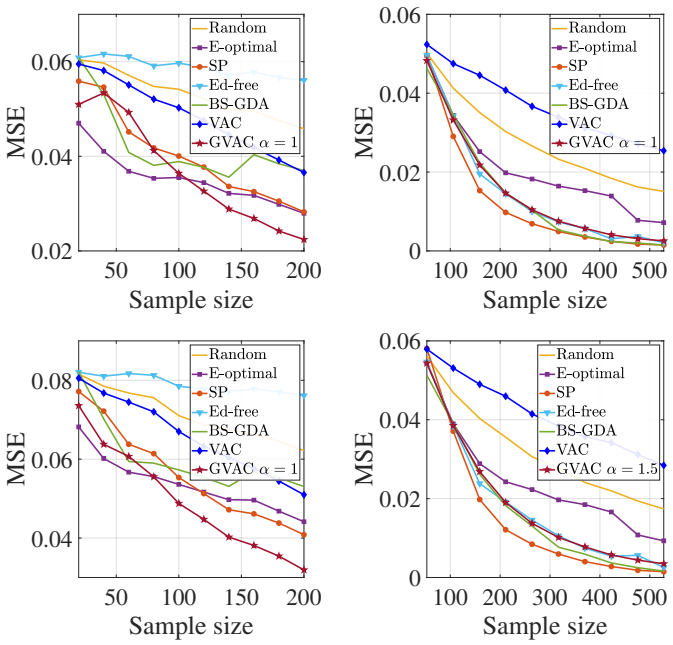

Fig. 2. Average MSE vs. sample size of 100 different artificial signals using the BMR method in (10) with $q = 1$ and different sampling schemes. The top row shows the results for SM1 in (a) Graph $G1$ and (b) Graph $G2$. The bottom row shows the results for SM2 in (e) Graph $G1$ and (f) Graph $G2$.

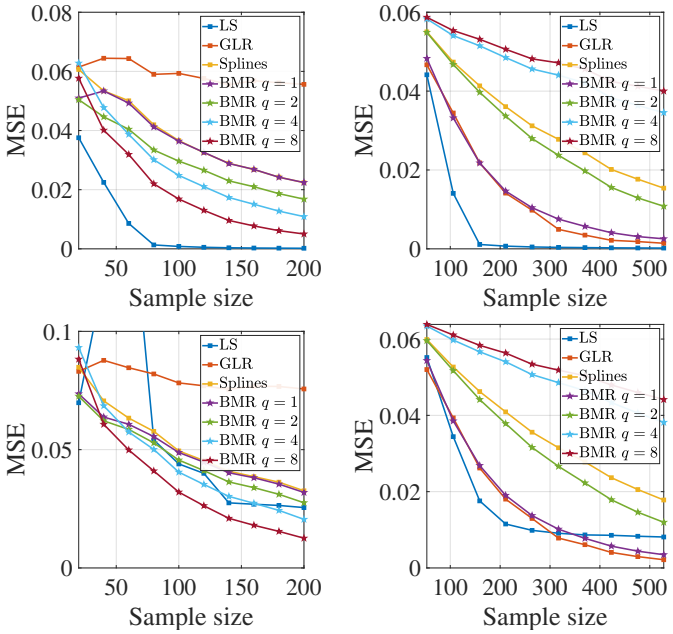

Fig. 3. Average MSE vs. sample size of 100 different artificial signals using different reconstruction schemes. The top row shows the results for SM1 in (a) Graph $G1$ and (b) Graph $G2$. The bottom row shows the results for SM2 in (e) Graph $G1$, (f) Graph $G2$.

mental results show that this approach recovers approximately bandlimited signals more efficiently.

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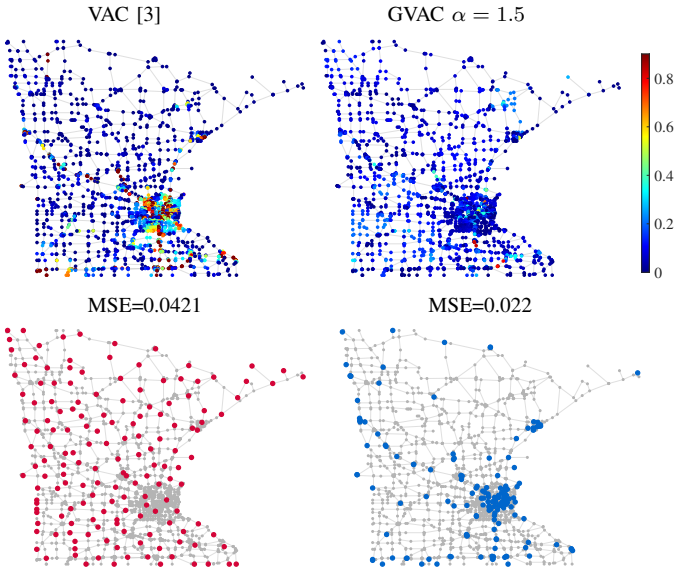

Fig. 4. (Top) absolute error between the signal and reconstructed signal using different sampling methods and (Bottom) the respective sampling patterns with density $d = 0.08$.

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