Global MDL Minimization-based Method for Detection of the Number of Sources in Presence of Unknown Nonuniform Noise

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***Abstract*—The classical Minimum Description Length (MDL) approach for detection of the number of sources fails in the presence of unknown nonuniform noise. In order to solve this problem, we propose to detect the number of sources by the global minimization of a newly built MDL criteria, named as the GM-MDL method. The proposed GM-MDL method first builds a new MDL objective function, which is a function of the number of sources and a whitening vector. Afterwards, the genetic algorithm (GA) is employed to find the global minimum solution of the newly built MDL objective function, which gives the estimates of the number of sources and the whitening vector. Simulation results demonstrate that the proposed GM-MDL method can estimate the number of sources correctly in the scenarios of unknown nonuniform and uniform noise. In addition, compared with the existing methods, the proposed GM-MDL method has significant improvement when the Worst Noise Power Ratio (WNPR) is large and/or the signal-to-noise ratio (SNR) is low. Furthermore, it also demonstrates a good performance in few snapshots.**

I. Introduction

Detection of the number of sources and direction-of-arrival (DOA) estimation are two important topics in array signal processing [1]. Generally, detection of the number of sources is a mandatory process prior to DOA estimation. It was first developed in the case of uniform noise. Most of detection methods can be categorized as ⓐHypothesis test-based methods [2]–[4]. ⓑ Information theoretic criteria(ITC)-based methods [5]–[13].

The ITC methods do not need subjective setting of the significance level for hypothesis test and they are simpler than the Hypothesis tests-based methods. Most of them are based on the eigenvalues of the covariance matrix of the received array signal, including Akaike information criterion (AIC) [5], minimum description length (MDL) criterion [6], predicted eigen-threshold (ET) [7], the Bayesian information criterion (BIC) [8], and their variants [9]–[13].

In practice, the noise powers at different sensors might be different due to the imperfect channel response and mutual coupling [14]. In this case, the noise is termed as nonuniform noise. Detection of the number of sources in the case of nonuniform noise was solved in [15]–[19]. In [15], a modified Gerschgorin disk estimator (GDE) was developed by using the unitary transformation of the covariance matrix, which increases the robustness to the noise model error. Its performance in a low SNR is limited because it replies on the separation of signal Gerschgorin disks from the noise Gerschgorin disks. The second order statistic of eigenvalues (SORTE) in [16] measures the gap among the eigenvalues. It is applicable to the case of the nonuniform noise, with the limitation that the sources are uncorrelated with each other. A nonuniform noise MDL (named as NU-MDL) was developed based on successive array element suppression in [17], which alleviates the effect of nonuniform noise to certain extent and correctly detects the number of source in small Worse Noise Power Ratio (WNPR). However, it fails in the case of large WNPR because nonunifrom noise remains in the rest of array elements after array element suppression. Detection of the number of sources based on signal subspace matching (SSM) was proposed in [18]. It is applicable to both white and colored noise, in the condition of moderate and high SNRs. In addition, an invariant SSM method [19] was specially developed for uniform arrays. It is noted that the above-mentioned methods detect the number of sources under certain limitations.

In this paper, in order to detect the number of sources with robustness to different scenarios, we propose a global MDL minimization-based method (named as GM-MDL), which is tailored to nonuniform noise. We first analyse that in the general case, the nonuniform noise causes virtual sources in addition to the true sources. As a result, the signal subspace of the array covariance matrix is expanded, which implies the conventional MDL method will overestimate the number of sources. Inspired by this fact, we build a new MDL objective function, of which the arguments are the unknown whitening vector and the number of sources. The global minimization of the new MDL objective function gives the correct estimation of source number and the correct whitening vector. Therefore, we employ genetic algorithm (GA) to implement the global

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searching in order to detect the number of sources. It is noted that, similar to the proposed GM-MDL method, the robust MDL method (named as rMDL) in [20] built an MDL objective function of which the arguments include the vector containing the parameters representing the deviations from the spatially white noise assumption. However, it involves the steering vectors of assuming sources, the white noise level, the source signal correlation matrix as arguments as well. Thus, the number of arguments involved in rMDL is much more than those in the proposed GM-MDL method.

In the simulation, we compare the proposed GM-MDL method with the MDL, NU-MDL, SSM methods. Simulation results demonstrate that the proposed GM-MDL method is more robust to low SNRs, few snapshots, and large WNPRs.

In Section II, the array model and the conventional MDL method are briefly introduced. Section III proves that the nonuniform noise causes virtual sources and thus expands the signal subspace of the array covariance matrix. Section IV proposes the GM-MDL method. Section V illustrates the performance of the GM-MDL method and compares it with those of other methods. The Conclusion is drawn in Section VI.

II. Background

A. Data Model

We consider an array is composed of M elements and K far-field and narrowband sources impinging on the array with DOAs of $(\theta_1, \theta_2, \cdots, \theta_K)$. The received signal vector $\mathbf{r}(t)$ is written as follows

$$\mathbf{r}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{n}(t), \tag{1}$$

where $\mathbf{s}(t)$ is the vector composed of K sources; $\mathbf{n}(t)$ is the vector of noises; They are expressed as $\mathbf{s}(t) = [s_1(t), \ \ldots \ , s_K(t)]^T$, and $\mathbf{n}(t) = [\mathrm{n}_1(t), \ \ldots \ , \mathrm{n}_M(t)]^T$, where $s_i(t)$ is the i-th source and $\mathrm{n}_m(t)$ is the noise at the m-th element which is Gaussian-distributed random process; $\mathbf{A} = [\mathbf{a}(\theta_1), \mathbf{a}(\theta_2) \quad \cdots \quad , \mathbf{a}(\theta_K)]$. Note that $\mathbf{a}(\theta_i)$ is the steering vector of the i-th the source with a DOA θ_i and it is written as

$$\mathbf{a}(\theta_i) = [1, \ e^{-j\frac{2\pi}{\beta}d_2 sin\theta_i}, \ldots, \ e^{-j\frac{2\pi}{\beta}d_{M-1} sin\theta_i}]^T, \tag{2}$$

where d_m is the spacing between the m-th element and the first one, β is the wavelength of the sources, and j is the imaginary unit.

In the condition that the signal and noise are uncorrelated, the expected covariance matrix of the received signal is then expressed as

$$\mathbf{R} = E[\mathbf{r}(t)\mathbf{r}^H(t)] = \mathbf{A}\mathbf{R}_s\mathbf{A}^H + \mathbf{R}_n, \tag{3}$$

where $E[\cdot]$ is the expectation operation, $\mathbf{R}_s = E[\mathbf{s}(t)\mathbf{s}^H(t)]$ is the source signal covariance matrix, and $\mathbf{R}_n = E[\mathbf{n}(t)\mathbf{n}^H(t)]$ is the noise covariance matrix.

In the case of nonuniform noises, the noise covariance matrix $\mathbf{R}_n$ is a diagonal matrix and it can be expressed as

$$\begin{gathered}\mathbf{R}_n = diag\left\{\mathbf{p}_n\right\} \\ \mathbf{p}_n = [\sigma_1^2, \sigma_2^2, \cdots, \sigma_M^2],\end{gathered} \tag{4}$$

where σ_m^2 is the noise power at the m-th element; $\sigma_1^2, \sigma_2^2, \cdots, \sigma_M^2$ are not all equal, and $diag\left\{\mathbf{x}\right\}$ represents a diagonal matrix with diagonal elements composed of $\mathbf{x}$.

B. MDL Method

The AIC and MDL methods both use the ITC for detection of the number of sources. In this paper, we consider the MDL method because it is a consistent estimator.

Define the log-likelihood function and penalty function as

$$L(k) = N(M-k)ln\left[\frac{\frac{1}{M-k}\sum_{i=k+1}^{M}\lambda_i}{(\prod_{i=k+1}^{M}\lambda_i)^{\frac{1}{M-k}}}\right] \tag{5}$$

$$P(k) = \frac{k}{2}(2M-k)lnN, \tag{6}$$

where k is a supposed number of sources, N is the number of snapshots, λ_i is the i-th eigenvalues of $\mathbf{R}$, and λ_i for $i = 1, \cdots, M$ is listed in a descend order.

The MDL objective function is given below

$$MDL(k) = L(k) + P(k). \tag{7}$$

The MDL method then estimates the number of sources as follows

$$\widehat{K} = \min_{k} \ MDL(k), \ k \in \{0, 1, \cdots, M-1\}. \tag{8}$$

III. Analysis of Virtual Sources Caused by Nonuniform Noise

We prove that the nonuniform noise leads to virtual sources, which causes that the conventional MDL method overestimates the number of sources.

Without loss of generality, we assume that σ_M^2 is the smallest among the noise powers at different antennas and $\sigma_1^2 \geq \sigma_2^2 \cdots \geq \sigma_l^2 > \sigma_{l+1}^2 = \sigma_{l+2}^2 = \cdots = \sigma_M^2$. Then, the noise covariance matrix $\mathbf{R}_n$ can be rewritten as

$$\mathbf{R}_n = \sigma_M^2\mathbf{I}_M + \sum_{m=1}^{l}\mathbf{Q}_m, \tag{9}$$

where $\mathbf{Q}_m = diag\left\{\mathbf{0}_{m-1}, \sigma_m^2 - \sigma_M^2, \mathbf{0}_{M-m}\right\}$, $m = 1, \cdots, l$, and $\mathbf{0}_m$ is a $1 \times m$ vector composed of zero.

Define $\mathbf{q}_m$ is an $M \times 1$ vector and $\mathbf{q}_m = [\mathbf{0}_{m-1}, 1, \mathbf{0}_{M-m}]^T$, $\mathbf{Q}_m$ can be then reformed as

$$\mathbf{Q}_m = (\sigma_m^2 - \sigma_M^2)\mathbf{q}_m\mathbf{q}_m^H. \tag{10}$$

Thus, according to Eqs.(3), (9), and (10), the covariance matrix $\mathbf{R}$ can be rewritten as

$$\mathbf{R} = \mathbf{A}\mathbf{R}_s\mathbf{A}^H + \sum_{m=1}^{l}(\sigma_m^2 - \sigma_M^2)\mathbf{q}_m\mathbf{q}_m^H + \sigma_M^2\mathbf{I}_M. \tag{11}$$

where $\mathbf{I}_M$ is an $M \times M$ identity matrix.

Therefore, we have

$$\mathbf{R} = \widetilde{\mathbf{A}}\widetilde{\mathbf{R}}_s\widetilde{\mathbf{A}}^H + \sigma_M^2\mathbf{I}_M, \tag{12}$$

where

$$\widetilde{\mathbf{A}} = [\mathbf{A}, \mathbf{q}_1, \cdots, \mathbf{q}_l], \tag{13}$$

$$\widetilde{\mathbf{R}}_s = \begin{bmatrix} \mathbf{R}_s & \mathbf{0}_{(M-l)l} \\ \mathbf{0}_{l(M-l)} & \mathbf{R}_{1s} \end{bmatrix}, \quad (14)$$

where $\mathbf{0}_{pq}$ is a $p \times q$ matrix with all elements equal to 0, and $\mathbf{R}_{1s} = diag\left\{(\sigma_1^2 - \sigma_M^2), \cdots, (\sigma_m^2 - \sigma_M^2), \cdots, (\sigma_l^2 - \sigma_M^2)\right\}$.

According to Eqs.(11)-(14), we can see that the second term on the right side of Eq.(11) can be treated as the effect of virtual sources. In addition, $\mathbf{q}_m$ and $\sigma_m^2 - \sigma_M^2$ for $m = 1, \cdots, l$ can be treated as the steering vector and power of the m-th virtual source, respectively. This, we obtain *Lemma 1* below.

Lemma 1: Without loss of generality, in the case that $\sigma_1^2 \geq \sigma_2^2 \geq \cdots \geq \sigma_l^2 > \sigma_{l+1}^2 = \sigma_{l+2}^2 = \cdots = \sigma_M^2$, l noise powers, which are larger than the smallest one, cause l virtual sources.

Lemma 1 indicates that nonuniform noise expands the signal subspace of the covariance matrix $\mathbf{R}$, resulting in the overestimate of the number of the true sources by the MDL method.

Assume an uniform linear array is composed of 5 elements and has half-wavelength inter-element spacing. There are two far-field sources impinging on the array with DOAs of 30° and 50°, respectively. The two sources have equal power. The SNR is defined as $\sigma_s^2/\overline{\sigma}_n^2$, where σ_s^2 is the signal power and $\overline{\sigma}_n^2$ is the average noise power defined as $\overline{\sigma}_n^2 = \frac{1}{M}\sum_{m=1}^{M}\sigma_m^2$. Fig.1 shows the estimate of the number of sources by the MDL method versus SNRs in the following cases:

1) Uniform noises.
2) Nonuniform Noise1: $\sigma_1^2 = 10$, and $\sigma_2^2 = \sigma_3^2 = \sigma_4^2 = \sigma_5^2 = 1$.
3) Nonuniform Noise2: $\sigma_1^2 = 10$, $\sigma_2^2 = 5$, and $\sigma_3^2 = \sigma_4^2 = \sigma_5^2 = 1$.

From Fig.1, we can see that in the case of uniform noises, the MDL method correctly estimates the number of sources when the SNR is not smaller than -5dB. However, in the case of Nonuniform Noise1, the MDL method overestimates the number of sources as 3 instead of the true one 2, even in high SNRs. This is because σ_1^2 is not equal to the rests of the noise powers, which causes one virtual source, as predicted by *Lemma 1*. In the case of Nonuniform Noise2, both σ_1^2 and σ_2^2 are unequal to the rests of noise powers. As predicted by *Lemma 1*, it causes 2 virtual sources. Therefore, in this case, the estimate of the number of sources in Fig.1 is 4 rather than the true one 2.

In [20], the phenomenon of virtual sources is noticed in the numerical experiments as well. However, Ref. [20] does not provide a proof of virtual sources.

IV. Global MDL Minimization-based Method

Inspired by *Lemma 1*, we propose a global MDL minimization-based method (named as GM-MDL), The GM-MDL method aims to find a whitening vector to whiten nonuniform noise as uniform one and thus eliminate the virtual sources, leading to correct estimate of true sources.

We first define a whitening vector $\mathbf{w}$, which is an M-dimensional vector of which all the elements are positive and real-valued. In addition, we denote a diagonal matrix

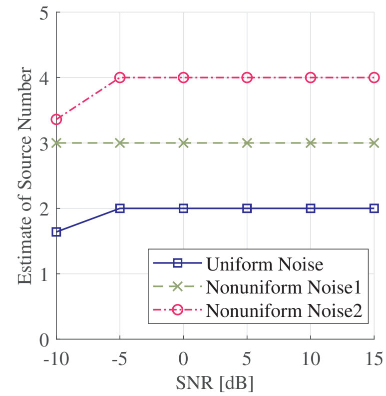

Fig. 1. Estimate of number of sources by MDL method in the cases of uniform and nonuniform noises

$\mathbf{W} = diag\{\mathbf{w}\}$, and construct a new covariance matrix as follows

$$\widetilde{\mathbf{R}} = \mathbf{W}\mathbf{R}\mathbf{W}^T. \quad (15)$$

We decompose $\widetilde{\mathbf{R}}$ as

$$\widetilde{\mathbf{R}} = \sum_{i=1}^{M} \tilde{\lambda}_i(\mathbf{w})\mathbf{v}_i(\mathbf{w})\mathbf{v}_i^H(\mathbf{w}), \quad (16)$$

where $\tilde{\lambda}_i(\mathbf{w})$ and $\mathbf{v}_i(\mathbf{w})$ are functions of $\mathbf{w}$.

Therefore, we build a new MDL objective function of both the number of sources and the whitening vector $\mathbf{w}$ as follows

$$MDL_{new}(k, \mathbf{w}) = L_{new}(k, \mathbf{w}) + P_{new}(k), \quad (17)$$

where $L_{new}(k, \mathbf{w})$ is the new log-likelihood function and it is expressed as

$$L_{new}(k, \mathbf{w}) = N(M-k)ln\left[\frac{\frac{1}{M-k}\sum_{i=k+1}^{M}\tilde{\lambda}_i(\mathbf{w})}{\left(\prod_{i=k+1}^{M}\tilde{\lambda}_i(\mathbf{w})\right)^{\frac{1}{M-k}}}\right], \quad (18)$$

where $P_{new}(k)$ is the penalty function considering the nonuniform noise, defined as

$$P_{new}(k) = \frac{1}{2}(k(2M-k)+M)lnN. \quad (19)$$

By minimizing the new MDL objective function with certain constrains on $\mathbf{w}$, we estimate the number of sources as k_{opt} and obtain $\mathbf{w}_{opt}$ below

$$\begin{aligned} (k_{opt}, \mathbf{w}_{opt}) = \min_{k,\mathbf{w}} \; & MDL_{new}(k, \mathbf{w}) \\ subject\ to \quad & \mathbf{w}^T\mathbf{w} = 1 \\ 0 < \mathrm{w}(m) < 1, \; & m = 1, \cdots, M. \\ k \in & \{0, 1, \cdots, M-1\}. \end{aligned} \quad (20)$$

where $\mathrm{w}(m)$ is the m-th element of the vector $\mathbf{w}$, and $\mathbf{w}^H\mathbf{w} = 1$ ensures that the power of each source signal after whitening remains the same.

According to $Lemma1$ in Section.III and Eq.(15), we observe that the nonunifrom noise is whitened as uniform one and thus the virtual sources in $\widetilde{\mathbf{R}}$ disappear if and only if

$$\mathbf{w} = c[\frac{1}{\sigma_1}, \frac{1}{\sigma_2}, \cdots, \frac{1}{\sigma_M}], \quad (21)$$

where c is a constant. In this case, the estimated source number based on $\widetilde{\mathbf{R}}$ is equal to the true source number K.

Therefore, by using $Lemma1$ and following a similar derivation given in [6], we theoretically prove that the $global\ minimization$ of the new MDL in Eq.(20) gives

$$k_{opt} = K$$
$$\mathbf{w}_{opt} = c_0[\frac{1}{\sigma_1}, \frac{1}{\sigma_2}, \cdots, \frac{1}{\sigma_M}], \quad (22)$$

where $c_0 = \frac{1}{\sum_{m=1}^{M} \frac{1}{\sigma_m^2}}$ to satisfy the constrain $\mathbf{w}^T\mathbf{w} = 1$.

The detailed proof of Eq.(22) is omitted here due to the paper length limitation. The global optimization is implemented by the GA algorithm in this paper.

V. Simulation Results

Assume that the array is composed by 5 elements which are uniformly linear with half-wavelength inter-element spacing. Two far-field and narrowband source signals $s_i(t)$, $i = 1, 2$, imping on the array with DOAs of 10° and 30°, respectively.

The two sources have the same power and the signal-to-noise ratio is defined as $SNR = \frac{\sigma_s^2}{\overline{\sigma}_n^2}$, where σ_s^2 is the source signal power and $\overline{\sigma}_n^2$ is the averaged noise power, defined as $\overline{\sigma}_n^2 = \frac{1}{M}\sum_{m=1}^{M} \sigma_m^2$. The number of samples is 1000. The Worst Noise Power Ratio (WNPR) is defined as WNPR = $\frac{\sigma_{nMax}^2}{\sigma_{nMin}^2}$ [14], where σ_{nMax}^2 and σ_{nMin}^2 are the greatest and smallest noise powers, respectively.

For the GA optimization used in the GM-MDL method, the population includes 50 individuals; the crossover probability is set to be 0.7. We compare the performance of the GM-MDL method with the MDL method [6], the NU-MDL method [17] and the SSM method [18] in a set of simulated experiments. Based on 200 trials, the following results are obtained.

Experiment 1 compares the performance with varying SNRs in the case of uniform noise(that is, WNPR = 1). The result is shown in Fig. 2. From Fig. 2, we observe that in the case of uniform noise, the proposed GM-MDL method, NU-MDL, and MDL perform almost the same. The SSM method can correctly detect the number of source when the SNR is not smaller than 5dB. However, it fails in low SNRs. This is consistent with the simulation analysis in [18].

Experiment 2 compares the performance with varying SNRs in the case of nonuniform noise with WNPR equal to 2. The result is shown in Fig. 3. From Fig. 3, we can see that in the case of nonuniform noise, the MDL method fails regardless of the SNR. This is due to the fact that the MDL method is only suitable for uniform noise. The GM-MDL and NU-MDL methods performs similarly and they are superiors to the SSM method when the SNR is low (not higher than 0dB).

Experiment 3 compares the performance with varying SNRs in the case of nonuniform noise with WNPR equal to 15. The result is shown in Fig. 4. From Fig. 4, we can see that in the case of nonuniform noise with a large WNPR, the proposed GM-MDL method demonstrates a robust performance even in low SNRs. Moreover, the SSM method is also robust to the large WNPR, with the condition that the SNR is moderate or high. On the other hand, the NU-MDL fails to detect the number of sources in a large WNPR. This is because in the NU-MDL method, nonunifrom noise remains in the rest of array elements after array element suppression.

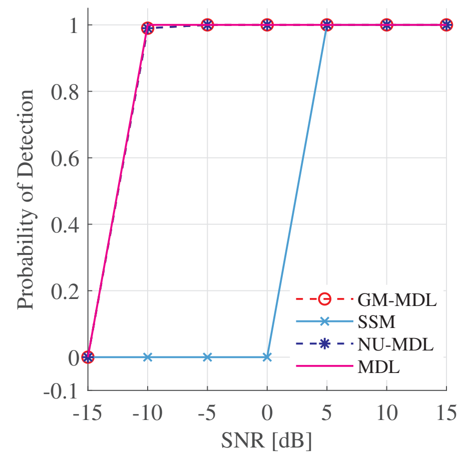

Fig. 2. Probability of detection versus SNR in the case of uniform noise.

Experiment 4 compares the performance with varying snapshots when the WNPR and SNR are set to be 5 and 0dB, respectively. The result is given in Fig. 5. From Fig. 5, it is illustrated that among the aforementioned methods, the proposed GM-MDL method is the most robust to few snapshots and gives a probability of detection close to 1 when the number of snapshots is 20. It is noted that the NU-MDL method and MDL method in the case of few snapshots provide a higher probability of detection than that in large snapshots. This phenomenon for MDL is noticed and analysed in [21] as well. The reason behind this phenomenon is that both the MDL and NU-MDL methods rely on the equality of the smallest eigenvalues. In the case of few snapshots, they might not detect the differences in the smallest eigenvalues. Therefore, they will not detect the virtual sources caused by nonuniform noise, leading to correct detection.

VI. Conclusion

In this paper, we prove that the nonuniform noise leads to virtual sources. Based on this fact, we propose the GM-MDL method to detect the number of sources in the presence of nonuniform noise. The GM-MDL method searches for the global minimization of the new MDL objective function with the whitening vector and the number of sources as arguments. Due to the global minimization, the GM-MDL method is superior to state-of-the-art methods including NU-MDL and SSM methods in the cases of low SNRs, large WNPR, and

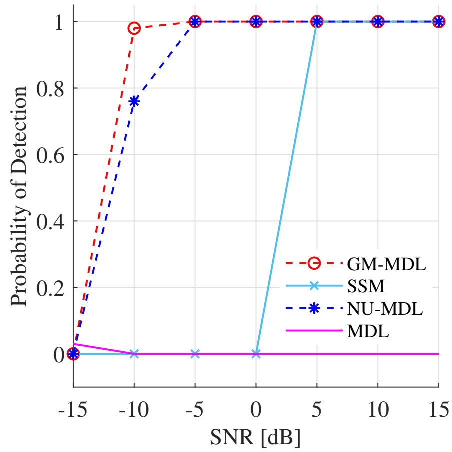

Fig. 3. Probability of detection versus SNR in the case of nonuniform noise with WNPR equal to 2.

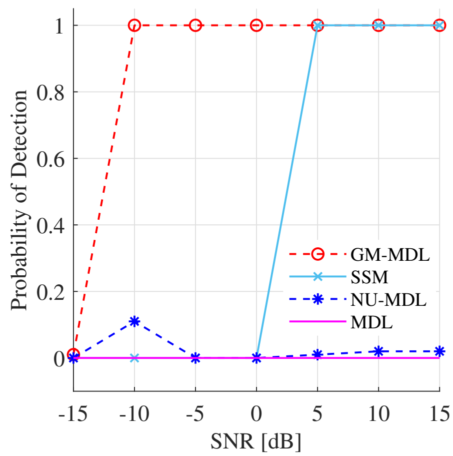

Fig. 4. Probability of detection versus SNR when WNPR = 5

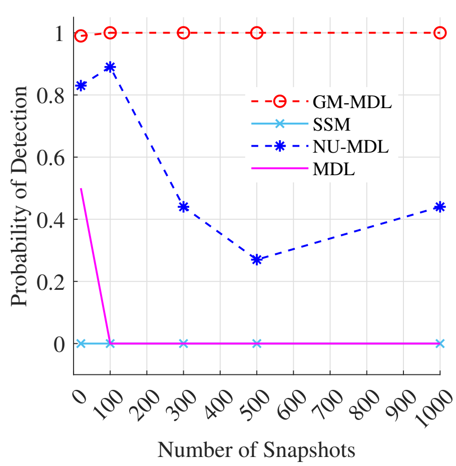

Fig. 5. Probability of detection versus number of snapshots when SNR = 0 dB and WNPR = 5.

few snapshots. In exchange, the GM-MDL method requires higher computational complexity.

REFERENCES

[1] K. Han, A. Nehorai, Improved source number detection and direction estimation with nested arrays and ulas using jackknifing, IEEE Trans. Signal Process. 61 (23) (2013) 6118–6128.

[2] M. S. BartlettK, Tests of significance in factor analysis, British Journal of Statistical Psychology 3 (1950) 77–85.

[3] D. N. LawleyB, Tests of significance for the latent roots of covariance and correlation matrices, Biometrika 43 (1956) 128–136.

[4] S. N. Roy, On a heuristic method of test construction and its use in multivariate analysis, The Annals of Mathematical Statistics 24 (2) (1953) 220–238.

[5] H. Akaike, A new look at the statistical model identification, IEEE Transactions on Automatic Control 19 (6) (1974) 716–723.

[6] M. Wax, T. Kailath, Detection of signals by information theoretic criteria, IEEE Trans. Acoust., Speech, Signal Process. 33 (2) (1985) 387–392.

[7] W. Chen, K. M. Wong, J. P. Reilly, Detection of the number of signals: a predicted eigen-threshold approach, IEEE Trans. Signal Process. 39 (5) (1991) 1088–1098.

[8] G. Schwartz, Estimating the dimension of a model, Annals of Statistics 6 (1978) 461–464.

[9] L. Huang, T. Long, S. J. Wu, Source enumeration for high-resolution array processing using improved gerschgorin radii without eigendecomposition, IEEE Trans. Signal Process. 56 (12) (2008) 5916–5925.

[10] L. Huang, H. C. So, Source enumeration via mdl criterion based on linear shrinkage estimation of noise subspace covariance matrix, IEEE Trans. Signal Process. 61 (19) (2013) 4806–4821.

[11] B. Nadler, Nonparametric detection of signals by information theoretic criteria: Performance analysis and an improved estimator, IEEE Trans. Signal Process. 58 (5) (2010) 2746–2756.

[12] X. Ke, Y. Zhao, L. Huang, On accurate source enumeration: A new bayesian information criterion, IEEE Trans. Signal Process. 69 (2021) 1012–1027.

[13] L. Huang, S. Wu, X. Li, Reduced-rank mdl method for source enumeration in high-resolution array processing, IEEE Trans. Signal Process. 55 (12) (2007) 5658–5667.

[14] M. Pesavento, A. Gershman, Maximum-likelihood direction-of-arrival estimation in the presence of unknown nonuniform noise, IEEE Trans. Signal Process. 49 (7) (2001) 1310–1324.

[15] Hsien-Tsai Wu, Jar-Ferr Yang, Fwu-Kuen Chen, Source number estimators using transformed gerschgorin radii, IEEE Trans. Signal Process. 43 (6) (1995) 1325–1333.

[16] Z. He, A. Cichocki, S. Xie, K. Choi, Detecting the number of clusters in n-way probabilistic clustering, IEEE Trans. Pattern Anal. Mach. Intell. 32 (11) (2010) 2006–2021.

[17] S. Aouada, A. Zoubir, C. M. S. See, Source detection in the presence of nonuniform noise, in: Proceedings of ICASSP, Vol. 2, 2004, pp. ii–165.

[18] M. Wax, A. Adler, Detection of the number of signals by signal subspace matching, IEEE Trans. Signal Process. 69 (2021) 973–985.

[19] M. Wax, A. Adler, Detection of the number of signals in uniform arrays by invariant-signal-subspace matching, IEEE Trans. Signal Process. (2022) 1–1.

[20] E. Fishler, H. Poor, Estimation of the number of sources in unbalanced arrays via information theoretic criteria, IEEE Trans. Signal Process. 53 (9) (2005) 3543–3553.

[21] L. Huang, T. Long, E. Mao, H. C. So, Mmse-based mdl method for robust estimation of number of sources without eigendecomposition, IEEE Trans. Signal Process. 57 (10) (2009) 4135–4142.