3D DDoA-based Self-Positioning of Mobile Devices

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***Abstract*—This paper presents a self-positioning method for mobile devices in three-dimensional (3D) schemes, where the Direction of Arrival (DoA) of an incident signal is mathematically expressed by an azimuth angle and an elevation angle. The method relies on the estimation of the Direction Difference of Arrival (DDoA), which is linked to the DoAs. In this method, the mobile device localizes itself based on DDoAs of the incident signals from the base stations. An algorithm, which computes DDoAs based on the azimuth and elevation angles, is carefully analyzed. Then the position of the mobile device is estimated using these DDoAs. Furthermore, we propose a Maximum Likelihood estimator, which applies iterative procedures, in order to robustify the position estimation. Simulation results show significant performance improvements.**

***Index Terms* -self-positioning, DDoA, Direction Difference of Arrival, 3D localization, Maximum Likelihood.**

I. Introduction

So far, several main traditional positioning techniques have been under researches: Time of Arrival (ToA), Time Difference of Arrival (TDoA), Received Signal Strength (RSS), and Direction of Arrival (DoA) (in some documents, it is also called Angle of Arrival - AoA) [1] . ToA-based [2]–[4] and TDoA-based [5]–[7] positioning require highly accurate clock synchronization among all BSs and mobile device. RSS-based technique [8], [9], on the other hand, is very sensitive to the log normal fadings so it provides rough estimates for localization. DoA-based systems do not require such a synchronization [10].

Generally, positioning methods can be divided into 2 main types, based on where the mobile position estimate is computed.

- **Network-Positioning:** The network of base stations (BSs) computes the coordinates of the mobile device from the signal(s) sent by the mobile device.
- **Self-Positioning:** The mobile device itself computes its coordinates by using signals from the network of BSs.

As for localization based on ToA, TDoA, RSS, the positioning algorithms for network of base stations and mobile device are quite similar. On the other hand, direction-based positioning algorithms are different between network-based and mobile-based localization, because the mathematical expressions of the direction of the incident wave are different between base station and mobile device. In 3D model, each Direction of Arrival (DoA) is expressed by an azimuth angle and an elevation angle, which requires prior knowledge of the x axis and z axis (the verticality in practice). This task is possible at the network of base stations, but likely impossible at the mobile device as its orientation is unknown. Hence, for mobile-based localization, an approach using DDoAs is considered (Fig. 2), because their values do not depend on the orientation of the mobile. In [11], we presented DDoA-based self-positioning algorithm in 2D plane. In this paper, we extend our work into 3D space.

A. Related works

Several papers illustrate their researches and results. In [12], the authors showed a position algorithm using the DDoAs by a gradient iterative procedure. However, they did not show how to get the initial point for the procedure, as well as what to do if the procedure does not converge. The authors in [13] work about DDoA but the tilt of receiver is already given. In [14], a DDoA positioning algorithm is studied. The sensor determines its position by the Visible Light Communications (VLC) emitted from the light-emitting diodes (LEDs) around. Nevertheless, the authors assume that all the LEDs are collinear and the z-coordinate of the sensor is always lower than the common z-coordinate. These assumptions reduce the complexity for the problem, but also lose the generality for the solution. In addition, the authors of [15] suggest using the difference between the azimuths and the difference between the elevations to localize the mobile device. However, these differences are not constants when the orientation of the device changes. In our previous work [16], we tried to give a general solution for this localizing problem. Nonetheless, information about ToAs to estimate the distances from the base stations to the mobile device is required. Consequently, that solution is not feasible when the ToA is not estimated at the mobile device.

B. Our constributions

In this paper, we form an algorithm just using the DDoAs for position estimation. At first, the Least Squares method is studied. This method does not require any additional information. Afterwards, to make the position estimation more accurate, a Maximum Likelihood (ML) estimator is demonstrated.

II. Mobile-based localization by Downlink DoA

A. Problem Formulation

The problem is formulated in [16]. In direction-based localization, the DoA of an incident wave is really challenging

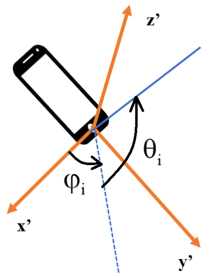

Fig. 1: Azimuth angle and elevation angle of the incident signal from the i-th base station in relative Cartesian coordinate system

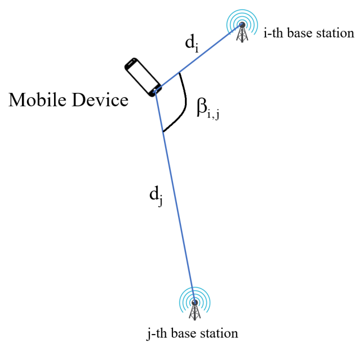

Fig. 2: Localization at mobile device with Direction Difference of Arrival (DDoA)

to be expressed by azimuth and elevation angles in the true Cartesian coordinate system. At the relative coordinate system with regard to the mobile device, the relative azimuth angle is φ_i and and elevation angle is θ_i (Fig. 1). As the tilt of the mobile device is undefined, it is likely impossible to directly localize the mobile device from (φ_i, θ_i). It is essential to find a solution which can estimate the mobile position by (φ_i, θ_i).

We define $\beta_{i,j}$ as the Direction Difference of Arrival (DDoA) between incident waves from i-th base station and j-th base station (Fig. 2), where d_i and d_j are the distance from the mobile device to i-th base station and j-th base station.

B. Linking DDoA and the related DoAs

To calculate $\beta_{i,j}$, we use scalar product of $\overrightarrow{d_i}$ and $\overrightarrow{d_j}$, the vector demonstrating the incident signal from i-th and j-th base station, respectively. We have

$$\overrightarrow{d_i} = (d_i \cos\theta_i \cos\varphi_i,\ d_i \cos\theta_i \sin\varphi_i,\ d_i \sin\theta_i)$$
$$\overrightarrow{d_j} = (d_j \cos\theta_j \cos\varphi_j,\ d_j \cos\theta_j \sin\varphi_j,\ d_j \sin\theta_j)$$

Thus

$$\overrightarrow{d_i}.\overrightarrow{d_j} = d_i d_j(\cos\theta_i\cos\varphi_i\cos\theta_j\cos\varphi_j + \cos\theta_i\sin\varphi_i\cos\theta_j\sin\varphi_j + \sin\theta_i\sin\theta_j) = d_i d_j(\cos\theta_i\cos\theta_j\cos(\varphi_j - \varphi_i) + \sin\theta_i\sin\theta_j) \quad (1)$$

where d_i and d_j are the length of two vectors $\overrightarrow{d_i}$ and $\overrightarrow{d_j}$, respectively

The definition of scalar product of two vectors:

$$\overrightarrow{d_i}.\overrightarrow{d_j} = d_i.d_j.\cos\beta_{i,j} \quad (2)$$

Thus

$$\cos\beta_{i,j} = \cos\theta_i\cos\theta_j\cos(\varphi_j - \varphi_j) + \sin\theta_i\sin\theta_j \quad (3)$$

Considering that $\beta_{i,j} \in [0;\pi]$, we have

$$\beta_{i,j} = \arccos\left(\cos\theta_i\cos\theta_j\cos(\varphi_i - \varphi_j) + \sin\theta_i\sin\theta_j\right) \quad (4)$$

Let $\gamma_{i,j} = \cos\beta_{i,j}$ be the cosine of DDoA.

(4) reveals the relationship between the measured DOAs θ_i, φ_i, θ_j, φ_j and the DDoA $\beta_{i,j}$

Value of $\beta_{i,j}$ always remains unchanged when the mobile device rotates. [17] proves that the DDoA is unchanged no matter which coordinate system is chosen.

In practice, the estimates of φ_i and θ_i can be expressed by:

$$\hat{\varphi}_i = \varphi_i + \tilde{\varphi}_i \quad (5)$$
$$\hat{\theta}_i = \theta_i + \tilde{\theta}_i \quad (6)$$

The authors of [18] illustrate that when there is Gaussian noise in received signal, $\tilde{\varphi}_i$ and $\tilde{\theta}_i$ are asymptotically and independently Gaussian distributed with zero-mean. As a result, we can assume that$\tilde{\varphi}_i$ and $\tilde{\theta}_i$ are independently Gaussian distributed with zero-mean. Their variances are ν_i^2 and μ_i^2, correspondingly.

C. Least Squares position estimation

We have

$\overrightarrow{d_i} = (x_i - x,\ y_i - y,\ z_i - z)$ and $\overrightarrow{d_j} = (x_j - x,\ y_j - y,\ z_j - z)$

The DDoA between the incident waves from i-th base station and j-th base station does not depend on the orientation of the mobile device.

$$\overrightarrow{d_i}.\overrightarrow{d_j} = x^2 - (x_i + x_j)x + x_i x_j + y^2 - (y_i + y_j)y + y_i y_j + z^2 - (z_i + z_j)z + z_i z_j \quad (7)$$

$\overrightarrow{d_i}.\overrightarrow{d_j} = d_i d_j \cos\beta_{i,j}$ Thus

$$x^2 - (x_i + x_j)x + x_i x_j + y^2 - (y_i + y_j)y + y_i y_j + z^2 - (z_i + z_j)z + z_i z_j = d_i d_j \gamma_{i,j} \quad (8)$$

In [16], we use ToAs to estimate d_i and d_j, which makes the positioning problem much easier. However, in this paper, we assume that only DDoAs and their cosines are considered. We recall that

$$d_i = \sqrt{(x_i - x)^2 + (y_i - y)^2 + (z_i - z)^2} \quad (9)$$

As a result

$$x^2 - (x_i + x_j)x + x_i x_j + y^2 - (y_i + y_j)y + y_i y_j + z^2 - (z_i + z_j)z + z_i z_j = \sqrt{(x_i - x)^2 + (y_i - y)^2 + (z_i - z)^2}\sqrt{(x_j - x)^2 + (y_j - y)^2 + (z_j - z)^2}\gamma_{i,j} \quad (10)$$

It is very difficult to solve the equation (10) with 3 variables x, y, z, but we can have an estimation. We take the square of (10). Let

$$\boldsymbol{a}^{(i,j)} = \begin{bmatrix} 1 \\ (x_i + x_j)^2 \\ (y_i + y_j)^2 \\ (z_i + z_j)^2 \\ -2(x_i + x_j) \\ -2(y_i + y_j) \\ -2(z_i + z_j) \\ 2(x_i + x_j)(y_i + y_j) \\ 2(y_i + y_j)(z_i + z_j) \\ 2(z_i + z_j)(x_i + x_j) \\ 2(x_i x_j + y_i y_j + z_i z_j) \\ -2(x_i + x_j)(x_i x_j + y_i y_j + z_i z_j) \\ -2(y_i + y_j)(x_i x_j + y_i y_j + z_i z_j) \\ -2(z_i + z_j)(x_i x_j + y_i y_j + z_i z_j) \end{bmatrix}^T ; \boldsymbol{\omega} = \begin{bmatrix} (x^2 + y^2 + z^2)^2 \\ x^2 \\ y^2 \\ z^2 \\ x(x^2 + y^2 + z^2) \\ y(x^2 + y^2 + z^2) \\ z(x^2 + y^2 + z^2 \\ xy \\ yz \\ zx \\ x^2 + y^2 + z^2 \\ x \\ y \\ z \end{bmatrix}$$

$$
\boldsymbol{b}^{(i,j)} = \begin{bmatrix} 1 \\ 4x_ix_j \\ 4y_iy_j \\ 4z_iz_j \\ -2(x_i+x_j) \\ -2(y_i+y_j) \\ -2(z_i+z_j) \\ 4(x_iy_j+y_ix_j) \\ 4(y_iz_j+z_iy_j) \\ 4(z_ix_j+x_iz_j) \\ x_i^2+y_i^2+z_i^2+x_j^2+y_j^2+z_j^2 \\ -2x_i(x_j^2+y_j^2+z_j^2)-2x_j(x_i^2+y_i^2+z_i^2) \\ -2y_i(x_j^2+y_j^2+z_j^2)-2y_j(x_i^2+y_i^2+z_i^2) \\ -2z_i(x_j^2+y_j^2+z_j^2)-2z_j(x_i^2+y_i^2+z_i^2) \end{bmatrix}^T
$$

Therefore, by taking the square of the left hand side and the right hand side of equation (10), we have:

$$
\begin{aligned} &\boldsymbol{a}^{(i,j)}\omega + (x_ix_j+y_iy_j+z_iz_j)^2 = \\ &\gamma_{i,j}^2\boldsymbol{b}^{(i,j)}\omega + \gamma_{i,j}^2(x_i^2+y_i^2+z_i^2)(x_j^2+y_j^2+z_j^2) \end{aligned} \tag{11}
$$

In matrix formulation, we denote

$$
\hat{\boldsymbol{A}} = \begin{bmatrix} \boldsymbol{a}^{(1,2)} - \hat{\gamma}_{1,2}^2\boldsymbol{b}^{(1,2)} \\ \boldsymbol{a}^{(1,3)} - \hat{\gamma}_{1,3}^2\boldsymbol{b}^{(1,3)} \\ \cdots \\ \boldsymbol{a}^{(i,j)} - \hat{\gamma}_{i,j}^2\boldsymbol{b}^{(i,j)} \end{bmatrix} \tag{12}
$$

$$
\hat{\boldsymbol{h}} = \begin{bmatrix} \hat{\gamma}_{1,2}^2(x_1^2+y_1^2+z_1^2)(x_2^2+y_2^2+z_2^2)-(x_1x_2+y_1y_2+z_1z_2)^2 \\ \hat{\gamma}_{1,3}^2(x_1^2+y_1^2+z_1^2)(x_3^2+y_3^2+z_3^2)-(x_1x_3+y_1y_3+z_1z_3)^2 \\ \cdots \\ \hat{\gamma}_{i,j}^2(x_i^2+y_i^2+z_i^2)(x_j^2+y_j^2+z_j^2)-(x_ix_j+y_iy_j+z_iz_j)^2 \end{bmatrix} \tag{13}
$$

where i from 1 to $N-1$, j from 2 to N, $i<j$. The estimated $\hat{\gamma}_{i,j}^2$ is given in (30) in Appendix A.

We have the equation of approximation

$$
\hat{\boldsymbol{A}}\ \boldsymbol{\omega} = \hat{\boldsymbol{h}} \tag{14}
$$

We have

$$
\hat{\omega} = \min_{\boldsymbol{\omega}} \|\hat{\boldsymbol{A}}\boldsymbol{\omega} - \hat{\boldsymbol{h}}\|^2 \tag{15}
$$

leading to the estimate of $\boldsymbol{\omega}$ being calculated by Least-Square estimation of $\boldsymbol{\omega}$

$$
\hat{\boldsymbol{\omega}} = \boldsymbol{A}^{\dagger}\boldsymbol{h} \tag{16}
$$

where $\boldsymbol{A}^{\dagger} = (\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T$

Estimated coordinate vector of the mobile device are the 3 last elements (the 12-th, 13-th and 14-th elements) of $\hat{\boldsymbol{\omega}}$

$$
\hat{\boldsymbol{x}} = \begin{bmatrix} [\hat{\boldsymbol{\omega}}]_{12} & [\hat{\boldsymbol{\omega}}]_{13} & [\hat{\boldsymbol{\omega}}]_{14} \end{bmatrix}^T \tag{17}
$$

D. Optimizing position by an iterative ML procedure

To optimize $\hat{\boldsymbol{x}}$ obtained in (17), an iterative Maximum Likelihood estimator is applied. In vector form, we denote

$$
\hat{\boldsymbol{\gamma}} = \begin{bmatrix} \hat{\gamma}_{1,2} & \hat{\gamma}_{1,3} & \cdots & \hat{\gamma}_{1,N} \end{bmatrix}^T \tag{18}
$$

$$
\boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} \gamma_{1,2}(\boldsymbol{x}) & \gamma_{1,3}(\boldsymbol{x}) & \cdots & \gamma_{1,N}(\boldsymbol{x}) \end{bmatrix}^T \tag{19}
$$

where

$$
\gamma_{i,j}(\boldsymbol{x}) = \frac{x^2-(x_i+x_j)x+x_ix_j+y^2-(y_i+y_j)y+y_iy_j+z^2-(z_i+z_j)z+z_iz_j}{\sqrt{(x_i-x)^2+(y_i-y)^2+(z_i-z)^2}\sqrt{(x_j-x)^2+(y_j-y)^2+(z_j-z)^2}} \tag{20}
$$

and $\boldsymbol{x} = [x\ \ y\ \ z]^T$.

We have the covariance matrix of $\hat{\boldsymbol{\gamma}}$:

$$
\boldsymbol{C}_{\gamma} = \mathrm{cov}(\hat{\boldsymbol{\gamma}}) = \begin{bmatrix} s_{1,2}^2 & s_{1,2,3}^2 & \cdots & s_{1,2,N}^2 \\ s_{1,2,3}^2 & s_{1,3}^2 & \cdots & s_{1,3,N}^2 \\ \cdots & \cdots & \cdots & \cdots \\ s_{1,2,N}^2 & s_{1,3,N}^2 & \cdots & s_{1,N}^2 \end{bmatrix} \tag{21}
$$

where $s_{i,j}^2$ and $s_{i,j,l}^2$ are expressed by equation (28) in Appendix A and equation (31) in Appendix B, respectively .

The measurement vector $\hat{\boldsymbol{\gamma}}$ is Gaussian distributed with mean vector of $\boldsymbol{f}$ and covariance matrix $\boldsymbol{C}_{\gamma}$, we have the probability density function (pdf):

$$
\mathrm{p}(\hat{\boldsymbol{\gamma}}|\boldsymbol{x}) = \frac{(2\pi)^{-N/2}}{|\boldsymbol{C}_{\gamma}|^{1/2}} \exp\left[\frac{-1}{2}(\hat{\boldsymbol{\gamma}}-\boldsymbol{f})^T\boldsymbol{C}_{\gamma}^{-1}(\hat{\boldsymbol{\gamma}}-\boldsymbol{f})\right] \tag{22}
$$

Maximizing the pdf in (22) is equivalent to finding

$$
\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x}} (\hat{\boldsymbol{\gamma}}-\boldsymbol{f}(\boldsymbol{x}))^T\boldsymbol{C}_{\gamma}(\hat{\boldsymbol{\gamma}}-\boldsymbol{f}(\boldsymbol{x})) \tag{23}
$$

which we shall perform alternatingly.

The possible outcomes of an iterative procedure and how the estimated position of the mobile device is taken from that procedure are carefully analyzed in [11]. We consider Gauss Newton [19] for $\boldsymbol{x}$. At the iteration $(u+1)$:

$$
\hat{\boldsymbol{x}}^{(u+1)} = \hat{\boldsymbol{x}}^{(u)} + \left(\boldsymbol{G}^T\boldsymbol{C}_{\gamma}\boldsymbol{G}\right)^{-1}\boldsymbol{G}^T\boldsymbol{C}_{\gamma}\left(\hat{\boldsymbol{\gamma}}-\boldsymbol{f}\left(\hat{\boldsymbol{x}}^{(u)}\right)\right) \tag{24}
$$

where $\boldsymbol{G}$ is the Jacobian matrix.

$$
\boldsymbol{G} = \frac{\partial \boldsymbol{f}(\boldsymbol{x})}{\partial \boldsymbol{x}^T} \tag{25}
$$

In a nutshell, the Algorithm 1 is proposed for the Gauss-Newton iterative procedure of the proposed ML estimator.

Algorithm 1: Proposed Maximum Likelihood estimator

```
Take the measured Direction of Arrival: azimuth φ̂_i
   and elevation θ̂_i.
Compute γ̂²_{i,j} by (30).
Assign u = 1 and ε sufficiently small.
Assign the coordinate computed by (17) as the first
   estimated coordinate vector x̂^(1) of the mobile device.
repeat
   Compute the estimated DDoA by (20)
   Compute the following estimated coordinate vector
      x̂^(u+1) of the mobile device by (24).
   u = u + 1;
until ||x̂^(u+1) − x̂^(u)||_2 < ε or u > 1000 or
   ||x̂^(u+1)|| = ±∞;
if u > 1000 or ||x̂^(u+1)||_2 = ±∞ then
   x̂^(1) is the estimated position of the mobile device;
else
   x̂^(u) is the estimated position of the mobile device;
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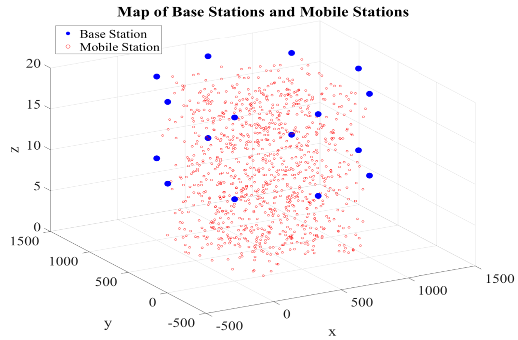

Fig. 3: Map of base stations and random positions of the mobile device

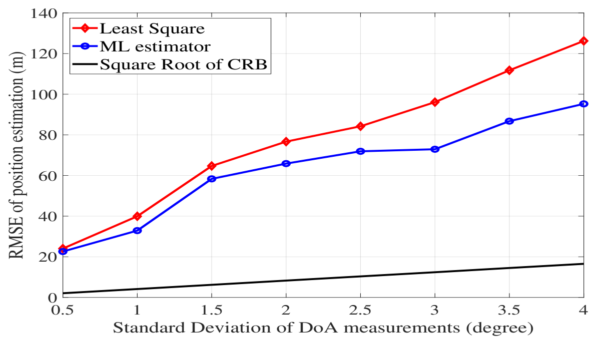

Fig. 4: DDoA-based localization at mobile device: Comparison of RMSE when the standard deviation of DoA measurements (σ) varies from 0.5° to 4°

III. Simulations and Results

A. Simulation Setup

To compare the quality among of algorithms and CRB, we use Root Mean Square Position Error (RMSE) which is defined by

$$\text{RMSE} = \sqrt{\text{E}(\|\hat{\boldsymbol{x}} - \boldsymbol{x}\|^2)} \tag{26}$$

where $\boldsymbol{x}$ is the true position of the mobile device and $\hat{\boldsymbol{x}}$ is its estimate. RMSE averaging is over 1000 mobile positions picked randomly in a cuboid of 1000m x 1000m x 20m (Fig. 3). 8 base stations form a circumscribed circle of the horizontal cross-section of the cuboid.Each base station has 2 antenna arrays at the height of 10m and 20m. Stopping criteria are $\varepsilon = 0.01$. We consider that in each scenario, all the DoA measurements have the same standard deviation: $\mu_1 = \mu_2 = \cdots = \mu_N = \nu_1 = \nu_2 = \cdots = \nu_N = \sigma$.

B. Results

Fig. 4 illustrates the RMSEs of the LS method and ML estimator. It is obvious that the ML estimator improves the accuracy of the LS positioning method by decreasing the RMSE, but still assures the unbiased property of the estimator.

IV. Conclusion

This paper studies direction-based self-positioning problems at mobile devices, which is quite challenging, since the orientation of the mobile device is undefined. Consequently, a DDoA-based positioning algorithm is researched, because only the DDoAs do not change when the mobile device rotates. Analytical solution for Least Squares method is presented. Moreover, we also propose a Maximum Likelihood estimator to optimize the position location. The results show that Least Squares method is feasible and Maximum Likelihood estimator enhance the position estimation.

References

[1] J. A. del Peral-Rosado, R. Raulefs, J. A. López-Salcedo, and G. Seco-Granados, "Survey of cellular mobile radio localization methods: From 1G to 5G," *IEEE Commun. Surveys Tuts.*, vol. 20, pp. 1124–1148, 2018.

[2] M. I. ul Haq and D. Kim, "Improved localization by time of arrival for internet of things in 3D," *22nd International Conference on Applied Electromagnetics and Communications (ICECOM)*, 2016.

[3] D. Ni, O. A. Postolache, C. Mi, M. Zhong, and Y. Wang, "UWB indoor positioning application based on kalman filter and 3-D toa localization algorithm," *11th International Symposium on Advanced Topics in Electrical Engineering (ATEE)*, 2019.

[4] W. Wang, Y. Zhang, and L. Tian, "TOA-based NLOS error mitigation algorithm for 3D indoor localization," *China Communications*, vol. 17, 2020.

[5] Z. Qin, J. Wang, and S. Wei, "A study of 3D sensor array geometry for TDOA based localization," *CIE International Conference on Radar (RADAR)*, 2016.

[6] M. Khalaf-Allah and S. U. Qaisar, "Nuisance parameter exploitation for accurate 3D hyperbolic geolocation - field results," *14th International Radar Symposium (IRS)*, 2013.

[7] H. Nabila, J. Suryana, and A. Izzuddin, "3D multilateration using RF burst," *6th International Conference on Wireless and Telematics (ICWT)*, 2020.

[8] Q. Wang, Z. Duan, and X. R. Li, "Three-dimensional location estimation using biased RSS measurements," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 56, 2020.

[9] C. Zhao, T. Yu, X. Tian, H. Yu, X. Gan, and X. Wang, "A 3-D RSS distribution model based on statistical properties for indoor localization systems," *IEEE/CIC International Conference on Communications in China (ICCC)*, 2015.

[10] D. Munoz, C. Vargas, R. Enriquez-Caldera, and F. B. Lara, *Position Location Techniques and Applications*, 2009.

[11] H. M. Le, D. Slock, and J.-P. Rossi, "2D DDoA-based self-positioning for mobile devices," *EUSIPCO*, 2021.

[12] Y.-C. Cheng, J.-Y. Lin, C.-W. Yi, Y.-C. Tseng, L.-C. Kuo, Y.-J. Yeh, and C.-W. Lin, "AR-based positioning for mobile devices," *40th International Conference on Parallel Processing Workshops*, 2011.

[13] D. Kim, J. K. Park, and J. T. Kim, "Three-dimensional VLC positioning system model and method considering receiver tilt," *IEEE Access*, vol. 7, 2019.

[14] B. Zhu, J. Cheng, Y. Wang, J. Yan, and J. Wang, "Three-dimensional VLC positioning based on angle difference of arrival with arbitrary tilting angle of receiver," *EEE Journal on Selected Areas in Communications*, vol. 36, no. 1, 2018.

[15] S. Sand, A. Dammann, and C. Mensing, *Positioning in Wireless Communication Systems*. Wiley, 2014.

[16] H. M. Le, D. Slock, and J.-P. Rossi, "3d self-positioning algorithm based on joint ddoa-toa," *3rd IEEE Middle East and North Africa COMMunications Conference (MENACOMM)*, 2021.

[17] B. Zhu, J. Cheng, J. Yan, J. Wang, and Y. Wang, "VLC positioning using cameras with unknown tilting angles," *IEEE Global Communications Conference*, 2017.

[18] A. Chan and J. Litva, "MUSIC and maximum likelihood techniques on two-dimensional DOA estimation with uniform circular array," *IEE Proc, Radar Sonar Navig*, vol. 142, no. 3, pp. 105–114, 1995.

[19] A. Bjorck, *Numerical Methods for Least Squares Problems*. SIAM (Society for Industrial and Applied Mathematics), 1996.

[20] J. Gray, "An exact determination of the probability density function under coordinate transformations," *The First IEEE Regional Conference on Aerospace Control Systems,*, 1993.

Appendix

A. Computations of expected value and variance of $\gamma_{i,j}$

In [20], it is proved that if $x \sim \mathcal{N}(x_0, \varsigma^2)$ then

$\mathrm{E}(\sin x) = e^{-\varsigma^2/2} \sin x_0$; $\quad \mathrm{E}(\cos x) = e^{-\varsigma^2/2} \cos x_0$; $\quad \mathrm{var}(\sin x) = \mathrm{var}(\cos x) = \frac{1}{2}\left(1 - e^{-2\varsigma^2}\right)$

$\mathrm{E}(\sin^2 x) = \frac{1}{2} - \frac{1}{2} e^{-2\varsigma^2} + e^{-2\varsigma^2} \sin^2 x_0$; $\quad \mathrm{E}(\cos^2 x) = \frac{1}{2} - \frac{1}{2} e^{-2\varsigma^2} + e^{-2\varsigma^2} \cos^2 x_0$

(4) shows the equation of $\gamma_{i,j} = \cos\beta_{i,j}$ in terms of the related DoAs: $\varphi_i, \varphi_j, \theta_i, \theta_j$. Therefore, the estimated value of the $\gamma_{i,j}$ is:

$$\begin{aligned}\hat{\gamma}_{i,j} &= \mathrm{E}(\gamma_{i,j}) = \mathrm{E}(\cos\beta_{i,j}) = \left(e^{-\mu_i^2/2}\cos\theta_i\right)\left(e^{-\mu_j^2/2}\cos\theta_j\right)\left(e^{-\left(\nu_i^2+\nu_j^2\right)/2}\cos(\varphi_i-\varphi_j)\right) + \left(e^{-\mu_i^2/2}\sin\theta_i\right)\left(e^{-\mu_j^2/2}\sin\theta_j\right) \\ &= e^{-\left(\mu_i^2+\mu_j^2\right)/2}\left(\cos\theta_i\cos\theta_j\right)e^{-\left(\nu_i^2+\nu_j^2\right)/2}\cos\left(\varphi_i-\varphi_j\right) + e^{-\left(\mu_i^2+\mu_j^2\right)/2}\sin\theta_i\sin\theta_j\end{aligned} \tag{27}$$

In addition, the variance of $\gamma_{i,j}$ is:

$$s_{i,j}^2 = \mathrm{var}(\gamma_{i,j}) = \mathrm{var}(\cos\beta_{i,j}) = \mathrm{E}(\cos^2\beta_{i,j}) - (\mathrm{E}(\cos\beta_{i,j}))^2 \tag{28}$$

where $\mathrm{E}(\cos\beta_{i,j})$ is expressed in (27) and

$$\cos^2\beta_{i,j} = (\cos\theta_i\cos\theta_j\cos(\varphi_i-\varphi_j)+\sin\theta_i\sin\theta_j)^2 = \cos^2\theta_i\cos^2\theta_j\cos^2(\varphi_i-\varphi_j)+\sin^2\theta_i\sin^2\theta_j+\frac{1}{4}\sin(2\theta_i)\sin(2\theta_j)\cos(\varphi_i-\varphi_j) \tag{29}$$

As a result,

$$\hat{\gamma}_{i,j}^2 = \mathrm{E}(\cos^2\beta_{i,j}) = h_1h_2h_3 + h_4h_5 + \frac{1}{4}h_6h_7h_8 \tag{30}$$

where

$$h_1 = \mathrm{E}(\cos^2\theta_i) = \frac{1}{2} - \frac{1}{2}e^{-2\mu_i^2} + e^{-2\mu_i^2}\cos^2\theta_i\ ;\ h_2 = \mathrm{E}(\cos^2\theta_j) = \frac{1}{2} - \frac{1}{2}e^{-2\mu_j^2} + e^{-2\mu_j^2}\cos^2\theta_j\ ;$$

$$h_3 = \mathrm{E}(\cos^2(\varphi_i-\varphi_j)) = \frac{1}{2} - \frac{1}{2}e^{-2\left(\nu_i^2+\nu_j^2\right)} + e^{-2\left(\nu_i^2+\nu_j^2\right)}\cos^2(\varphi_i-\varphi_j)\ ;\ h_4 = \mathrm{E}(\sin^2\theta_i) = \frac{1}{2} - \frac{1}{2}e^{-2\mu_i^2} + e^{-2\mu_i^2}\sin^2\theta_i\ ;$$

$$h_5 = \mathrm{E}(\sin^2\theta_j) = \frac{1}{2} - \frac{1}{2}e^{-2\mu_j^2} + e^{-2\mu_j^2}\sin^2\theta_j\ ;\ h_6 = \mathrm{E}(\sin(2\theta_i)) = e^{-2\mu_i^2}\sin(2\theta_i)\ ;$$

$$h_7 = \mathrm{E}(\sin(2\theta_j)) = e^{-2\mu_j^2}\sin(2\theta_j)\ ;\ h_8 = \mathrm{E}(\cos(\varphi_i-\varphi_j)) = e^{-\left(\nu_i^2+\nu_j^2\right)/2}\cos(\varphi_i-\varphi_j)\ ;$$

B. Computations of covariance of $\gamma_{i,j}$ and $\gamma_{i,l}$

The covariance of $\gamma_{i,j}$ and $\gamma_{i,l}$ is

$$s_{i,j,l}^2 = \mathrm{cov}(\gamma_{i,j}, \gamma_{i,l}) = \mathrm{E}(\gamma_{i,j}\gamma_{i,l}) - \mathrm{E}(\gamma_{i,j})\mathrm{E}(\gamma_{i,l}) \tag{31}$$

where $\mathrm{E}(\gamma_{i,j})$ is expressed in (27) and

$$\begin{aligned}&\gamma_{i,j}\gamma_{i,l} = (\cos\theta_i\cos\theta_j\cos(\varphi_i-\varphi_j)+\sin\theta_i\sin\theta_j)(\cos\theta_i\cos\theta_l\cos(\varphi_i-\varphi_l)+\sin\theta_i\sin\theta_l) = \\ &\frac{1}{2}\cos^2\theta_i\cos\theta_j\cos\theta_l(\cos(2\varphi_i-\varphi_j+\varphi_l)+\cos(\varphi_j-\varphi_l)) + \frac{1}{2}\sin(2\theta_i)\sin\theta_j\cos\theta_l\cos(\varphi_i-\varphi_l) + \frac{1}{2}\sin(2\theta_i)\cos\theta_j\sin\theta_l\cos(\varphi_i-\varphi_j) \\ &+\sin^2\theta_i\sin\theta_j\sin\theta_l\end{aligned} \tag{32}$$

As a result,

$$\mathrm{E}(\gamma_{i,j}\gamma_{i,l}) = \frac{1}{2}m_1m_2m_3(m_4+m_5) + \frac{1}{2}m_6m_7m_8m_9 + \frac{1}{2}m_{10}m_{11}m_{12}m_{13} + m_{14}m_{15}m_{16} \tag{33}$$

where

$$m_1 = \mathrm{E}(\cos^2\theta_i) = h_1\ ;\ m_2 = \mathrm{E}(\cos\theta_j) = e^{-\mu_j^2/2}\cos\theta_j\ ;\ m_3 = \mathrm{E}(\cos\theta_l) = e^{-\mu_l^2/2}\cos\theta_l\ ;$$

$$m_4 = \mathrm{E}(\cos(2\varphi_i-\varphi_j+\varphi_l)) = e^{-\left(4\nu_i^2+\nu_j^2+\nu_l^2\right)/2}\cos(2\varphi_i-\varphi_j+\varphi_l)\ ;\ m_5 = \mathrm{E}(\cos(\varphi_j-\varphi_l)) = e^{-\left(\nu_i^2+\nu_j^2\right)/2}\cos(\varphi_j-\varphi_l)\ ;$$

$$m_6 = \mathrm{E}(\sin(2\theta_i)) = h_6\ ;\ m_7 = \mathrm{E}(\cos\theta_l) = e^{-\mu_l^2/2}\cos\theta_l;\ m_8 = \mathrm{E}(\cos\theta_l) = m_3\ ;\ m_9 = \mathrm{E}(\cos(\varphi_i-\varphi_l)) = e^{-\left(\nu_i^2+\nu_l^2\right)/2}\cos(\varphi_i-\varphi_l)\ ;$$

$$m_{10} = \mathrm{E}(\sin(2\theta_j)) = h_7\ ;\ m_{11} = \mathrm{E}(\cos\theta_j) = m_2\ ;\ m_{12} = \mathrm{E}(\sin\theta_l) = e^{-\mu_l^2/2}\sin\theta_l\ ;\ m_{13} = \mathrm{E}(\cos(\varphi_i-\varphi_j)) = h_8;$$

$$m_{14} = \mathrm{E}(\sin^2\theta_i) = h_4\ ;\ m_{15} = \mathrm{E}(\sin\theta_j) = e^{-\mu_j^2/2}\sin\theta_j;\ m_{16} = \mathrm{E}(\sin\theta_l) = m_{12}$$