On the Reconstruction of Multiple Sinusoidal Signals from Compressed Measurements

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***Abstract*—The introduction of compressive sensing in wireless smart transducers can substantially reduce the high impact of sampling rate on their overall power consumption. Such systems are often dealing with signals that can be expressed as a sum of multiple sinusoids, having a frequency-sparse representation. Although the reconstruction of frequency-sparse signals has been widely studied and solutions based on greedy and relaxation methods exist, their performance is degraded in presence of spectral leakage, which affects the sparse representation of the signal and consequently, its estimation accuracy. In this paper, a two-stage optimization approach, named Opti2, is presented for the reconstruction of frequency-sparse signals that can be expressed as a sum of multiple real-valued sinusoidal waveforms. The estimation provided by basis pursuit denoising (BPDN) sparse optimization is computed in the first stage and used as initial guess for the second stage, where a non-linear least squares (NLLS) problem is formulated to improve the estimation of the signal parameters from undersampled data. Simulation results demonstrate that the proposed approach outperforms existing methods in terms of accuracy, showing its robustness to noise and compression rate.**

***Index Terms*—compressive sampling, frequency-sparse signals, multiple sinusoids, recovery algorithm, optimization, spectral leakage.**

I. Introduction

The limited energy resources of wireless smart sensors have been one of the main restrictions for their extensive deployment in monitoring and control applications. In order to minimize the energy consumption in the sensor node, a sampling strategy based on compressive sensing (CS) can be used. This can reduce the sampling rate while preserving the information content of the signal when it has a sparse expansion. In many applications, the sparse representation is in the frequency-domain and the energy efficiency of wireless sensors acquiring such signals can be significantly improved by using a non-uniform sampler (NUS) [1].

Instead of acquiring N samples of the signal, in a CS framework a set of $M \ll N$ measurements is generated by a linear dimensionality reduction. The sparse representation can be in terms of a frame or dictionary, meaning that the signal can be represented in terms of its largest κ coefficients without significant loss. Stable recovery of sparse signals can be guaranteed under certain conditions from just $M = \mathcal{O}(\kappa \log(N/\kappa))$ measurements via convex optimization or iterative greedy algorithms [2]–[4].

Frequency or spectrally sparse signals are sparse with respect to the discrete Fourier transform (DFT) only if they can be expressed as a superposition of sinusoids with frequencies appearing in the lattice of those in the DFT. In practice, such signals are rarely encountered and a DFT frame of redundancy should be introduced [5], [6]. Several recovery algorithms have been developed [5]–[9] to improve the performance of the existing approaches and to extend the recovery guarantees to redundant and coherent dictionaries. In [5], a coherence inhibition model is used resulting in the spectral iterative hard thresholding (SIHT) algorithm that avoids dictionary elements with high coherence. Alternatively, band-excluded local optimization orthogonal matching pursuit (BLOOMP) [7], [8] employs a similar principle to deal with the coherence. The potential of ℓ_1-*synthesis* for recovery of signals from undersampled data, which are sparse in a redundant dictionary, is studied in [6]. In [9], the band-excluded interpolating subspace pursuit (BISP) algorithm is proposed. It combines the band exclusion and polar interpolation functions in a greedy approach to improve the limitations due to the coherence and the discretization of the frequency parameter space. The polar interpolation function is based on the continuous basis pursuit (CBP) technique, proposed in [10], which is combined with orthogonal matching pursuit (OMP) in [11] to account for continuous-valued frequency estimates.

Most of the previous techniques have been developed to address the frequency estimation problem. However, some applications in communication, power line measurements, condition monitoring, speech and audio processing also require an accurate estimation of the signal itself, as well as its amplitude and phase parameters. In vibration monitoring for example, which allows to prevent equipment failures, the estimation accuracy of vibrating signal parameters is essential to identify structural defects before the system reaches a critical state.

The estimation of the parameters of multiple sinusoids have been widely studied [12]. However this problem requires a fresh look when the signal should be reconstructed from un-

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dersampled data. A two-stage recovery approach, called Opti2, is presented in this work to improve the signal estimation accuracy through the estimation of its parameters. A two-stage approach has been previously applied in [13] for the reconstruction of periodic signals, whose spectral content is harmonically related. In this paper, we focus on the reconstruction of signals that can be expressed as a sum of multiple sinusoids with arbitrary frequencies. In the first stage, one of the sparse reconstruction techniques is employed and used as input to the second stage, where a nonlinear least squares optimization problem is formulated to improve the estimation of the signal's parameters from compressed measurements. Experimental results show that the approach here presented outperforms the previously proposed algorithms for spectrally sparse signal recovery with a relatively low computational effort.

II. Compressive Sensing for Multiple Sinusoidal Signals

Many applications deal with signal models that can be expressed as a superposition of K real-valued sinusoidal waveforms, with continuous time representation given by

$$s(t) = \sum_{k=1}^{K} a_k \sin(2\pi f_k t + \theta_k) \tag{1}$$

where a_k, f_k and θ_k are the amplitude, frequency and phase of each sinusoid, respectively. Let us consider $\mathbf{s} \in \mathbb{R}^N$ a finite length discrete representation of the signal (1). Such signals have a κ-sparse representation in the DFT domain, with $\kappa = 2K$, only when the sinusoids have integral frequencies, i.e. they can be expressed as integer multiples of the frequency step size $\delta = f_s/N$ in the DFT basis $\mathbf{\Psi} \in \mathbb{C}^N$, for a given sampling rate f_s. Unfortunately, in the general case of non-integral frequencies, the DFT coefficients do not present the same sparsity properties due to the spectral leakage. A popular solution to this problem is to introduce a redundant DFT frame or dictionary [5], [6], [8]. Such dictionary corresponds to a finer discretization of the Fourier representation, which can be seen as sampling at more closely spaced intervals. The DFT frame $\mathbf{\Psi}_p$ with redundancy factor $p \in \mathbb{N}$ contains $N_p = p \cdot N$ vectors and is defined as

$$\mathbf{\Psi}_p = \begin{bmatrix} \mathbf{e}(\omega_1) & \mathbf{e}(\omega_2) & \dots & \mathbf{e}(\omega_{N_p}) \end{bmatrix}, \tag{2}$$

where each column vector $\mathbf{e}(\omega) \in \mathbb{C}^N$ has elements $e_n(\omega) = \frac{1}{\sqrt{N}} e^{j\omega n}, 0 \le n \le N-1$ and $\omega \in [0, 2\pi]$.

Taking advantage of the sparsity property, a CS framework is used to acquire the signal by a reduced set of $M \ll N$ linear measurements of the form

$$\mathbf{y} = \mathbf{\Phi s} + \mathbf{v} = \mathbf{\Phi \Psi}_p \mathbf{x} + \mathbf{v} = \mathbf{Ax} + \mathbf{v}, \tag{3}$$

where $\mathbf{\Phi}$ is a $[M \times N]$ measurement matrix, $\mathbf{A}$ is the sensing matrix and $\mathbf{v}$ accounts for additive white noise in the measurement process with zero-mean and variance σ_v^2. The vector $\mathbf{x} \in \mathbb{C}^{N_p}$ describes the κ-sparse representation of $\mathbf{s}$ in the redundant dictionary $\mathbf{\Psi}_p$.

Sparse reconstruction methods can recover $\mathbf{x}$, which has a minimum number of non-zero elements (i.e. $\|\mathbf{x}\|_0 \le \kappa$), via convex optimization or greedy algorithms, when the sensing matrix $\mathbf{A}$ obeys the Restricted Isometry Property (RIP) and maximal incoherence between the pairs $(\mathbf{\Phi}, \mathbf{\Psi})$ is achieved. One of the most popular reconstruction techniques based on iterative greedy solutions is OMP [14] due to its low complexity and easy implementation. On the other hand, BPDN [15] and linear Bregman iterations (LBI) [16] are some of the commonly used convex optimization techniques. Those are based on the ℓ_1-norm regularized optimization problem and variations, for which efficient solvers are available [17]. Although the DFT frame in (2) violates the incoherence requirements, the recovery guarantees from the compressed measurements $\mathbf{y}$ have been extended to redundant and coherence dictionaries [6]. In addition, the previously mentioned algorithms for recovering frequency-sparse signals have been developed to address this issue [5]–[9]. The estimation of the signal is then feasible from the recovery of its κ-sparse approximation.

III. Opti2: Signal recovery approach

The reconstruction of the signal $\mathbf{s}$ can be addressed as a parametric estimation problem, where the parameters of each sinusoid a_k, f_k and θ_k should be estimated from the reduced set of measurements $\mathbf{y}$. Even though most of the existing algorithms for sparse reconstruction can find a good estimation of the signal from its κ-sparse approximation $\mathbf{x}$, a more refined estimation of the signal's parameters is needed in some applications. This is why a two-stage method is here introduced, that allows for an accurate recovery of the discrete representation of the signal in (1).

A. Sparse recovery approach

In the first stage, one of the approaches based on the relaxation of the ℓ_0-norm optimization problem is employed. Although the ℓ_1-norm as relaxation of the ℓ_0-norm is weaker in ensuring sparsity, ℓ_1-regularized optimization is a convex problem and admits efficient solution via linear programming techniques. The ℓ_1-regularized optimization is equivalent to the least absolute shrinkage and selection operator (LASSO) [18] problem, also referred to as BPDN by the signal processing community. Thus, $\mathbf{x}$ can be recovered solving the following minimization problem

$$\hat{\mathbf{x}}_1 = \min_{\mathbf{x}} \|\mathbf{y} - \mathbf{Ax}\|_2^2 + \lambda \|\mathbf{x}\|_1, \tag{4}$$

where $\mathbf{A} = \mathbf{\Phi \Psi}_p$ is the sensing matrix and $\lambda \in [0, \infty]$ is the regularization parameter that controls the solution path. A first estimation of the signal is found by taking the real part of $\hat{\mathbf{s}}_1 = \mathbf{\Psi}_p \hat{\mathbf{x}}_1$. Assuming that the number of frequencies K is known, the support of $\hat{\mathbf{x}}_1$, i.e. the set of indexes associated to its first K non-zero coefficients $S_{\hat{\mathbf{x}}_1} = \{l : |x_l| \neq 0, l \in [1, \dots, N_p]\}$, can be used to obtain a first estimation of the frequencies $\hat{f}_{1k} = l_k \cdot f_s / N_p$, that will be used as initial guess in the second stage.

It is worthy to note that in the first stage it is possible to use either a greedy iterative algorithm or convex optimization approach. The performance of the proposed approach is determined by the accuracy of the first estimation. If the first estimation of the frequencies completely fails, the second stage will also deliver poor results. Nevertheless, the existing sparse recovery techniques can be successfully applied to multiple frequency estimation problems, as shown in previous studies.

B. NLLS parameters estimation

In the second stage, we aim to optimize the estimation of f_k and the parameters a_k and θ_k by solving a non-linear least squares optimization problem. Let us consider that signal $\mathbf{s}$ can be expressed as

$$\mathbf{s} = \sum_{k=1}^{K} a_k \sin\left(2\pi f_k \mathbf{n} + \theta_k\right) \tag{5}$$

$$\mathbf{s} = \sum_{k=1}^{K} \alpha_{1k} \sin(2\pi f_k \mathbf{n}) + \alpha_{2k} \cos(2\pi f_k \mathbf{n})$$

$$\mathbf{s} = \begin{bmatrix} \sin(2\pi f_1 \mathbf{n}) \\ \cos(2\pi f_1 \mathbf{n}) \\ \vdots \\ \sin(2\pi f_K \mathbf{n}) \\ \cos(2\pi f_K \mathbf{n}) \end{bmatrix}^T \begin{bmatrix} \alpha_{11} \\ \alpha_{21} \\ \vdots \\ \alpha_{1K} \\ \alpha_{2K} \end{bmatrix} = \mathbf{H}(\mathbf{f})\boldsymbol{\alpha},$$

where $\mathbf{f} = [f_1\ f_2\ \dots\ f_K]$, $\mathbf{n} = [0\ \dots\ N-1]\,/f_s$ is the sampling time vector and the elements in $\boldsymbol{\alpha}$ are given by

$$\alpha_{1k} = a_k \cos\left(\theta_k\right), \qquad \alpha_{2k} = a_k \sin\left(\theta_k\right). \tag{6}$$

To recover $\mathbf{s}$, $\mathbf{f}$ and $\boldsymbol{\alpha}$ are to be estimated from the reduced set of measurements $\mathbf{y}$. This can be done by solving the following non-linear optimization problem

$$\underset{\mathbf{f},\boldsymbol{\alpha}}{\arg\min} \left\|\mathbf{y} - \mathbf{H}_{\boldsymbol{\Phi}}(\mathbf{f})\boldsymbol{\alpha}\right\|_2^2 \tag{7}$$

with $\mathbf{H}_{\boldsymbol{\Phi}}(\mathbf{f}) = \boldsymbol{\Phi}\mathbf{H}(\mathbf{f})$ representing the $[M \times 2K]$ matrix that contains only M out of N samples. The problem of determining $\boldsymbol{\alpha}$ can be reduced to a linear least-squares fit once the estimation of the frequencies $\hat{\mathbf{f}}$ are given, which means that

$$\hat{\boldsymbol{\alpha}}(\mathbf{f}) = \mathbf{H}_{\boldsymbol{\Phi}}(\mathbf{f})^{\dagger}\mathbf{y}, \tag{8}$$

where $\mathbf{H}_{\boldsymbol{\Phi}}(\mathbf{f})^{\dagger}$ is the Moore-Penrose pseudoinverse. The problem (7) can be reformulated to obtain a reduced problem involving only the non-linear parameters $\mathbf{f}$. Then, a second estimation of the frequencies $\hat{f}_{2k}$ can be calculated by solving

$$\hat{\mathbf{f}}_2 = \underset{\mathbf{f}}{\arg\min} \left\|\mathbf{y} - \mathbf{H}_{\boldsymbol{\Phi}}(\mathbf{f})\mathbf{H}_{\boldsymbol{\Phi}}(\mathbf{f})^{\dagger}\mathbf{y}\right\|_2^2. \tag{9}$$

The estimation of $\boldsymbol{\alpha}$ is then found using (8) for the given $\hat{\mathbf{f}}_2$ and the second estimation of the signal is computed as $\hat{\mathbf{s}}_2 = \mathbf{H}(\hat{\mathbf{f}}_2)\hat{\boldsymbol{\alpha}}(\hat{\mathbf{f}}_2)$. The parameters $\hat{a}_k$ and $\hat{\theta}_k$ can be estimated as

$$\hat{a}_k = \sqrt{{\alpha_{1k}}^2 + {\alpha_{2k}}^2}, \qquad \hat{\theta}_k = \arctan\frac{\alpha_{2k}}{\alpha_{2k}}. \tag{10}$$

Note that the problem in (9) is equivalent to the variable projection (VP) functional and can be modeled as a separable NLLS [19]. Then, a VP optimization can be also considered to solve (9) [20]. The proposed approach is summarized in Algorithm 1.

Algorithm 1: Opti2

Input: Compressed measurements $\mathbf{y}$, measurement matrix $\boldsymbol{\Phi}$ and redundant frame or dictionary $\boldsymbol{\Psi}_p$

Output: Reconstructed signal $\hat{\mathbf{s}}$

$\mathbf{A} = \boldsymbol{\Phi}\boldsymbol{\Psi}_p$

Find $\hat{\mathbf{x}}_1$ using (4) $\rightarrow \hat{\mathbf{s}}_1 = \boldsymbol{\Psi}_p\hat{\mathbf{x}}_1$, $\hat{\mathbf{f}}_1 = S_{\hat{\mathbf{x}}_1} \cdot f_s/N_p$

$\mathbf{H}_{\boldsymbol{\Phi}}(\mathbf{f}) = \boldsymbol{\Phi}\mathbf{H}(\mathbf{f})$ where $\mathbf{H}(\mathbf{f})$ is a $[N \times 2K]$ matrix of the form given in (5)

Find $\hat{\mathbf{f}}_2$ solving (9)

$\hat{\boldsymbol{\alpha}}(\hat{\mathbf{f}}_2) = \mathbf{H}_{\boldsymbol{\Phi}}(\hat{\mathbf{f}}_2)^{\dagger}\mathbf{y}$

$\hat{\mathbf{s}} = \mathbf{H}(\hat{\mathbf{f}}_2)\hat{\boldsymbol{\alpha}}(\hat{\mathbf{f}}_2)$ or $\hat{\mathbf{s}} = \sum_{k=1}^{K} \hat{a}_k \sin\left(2\pi\hat{\mathbf{f}}_2\mathbf{n} + \hat{\theta}_k\right)$ with $\hat{a}_k$ and $\hat{\theta}_k$ given by (10)

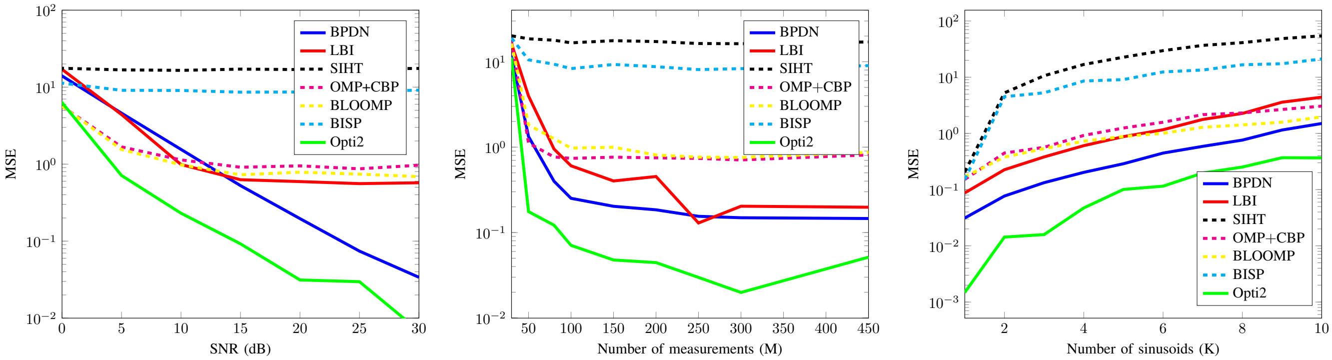
IV. Simulation Results

In this section, simulation results are presented to evaluate the proposed approach. A set of numerical experiments, carried out in Matlab programming environment, have been performed to compare Opti2 with state-of-the-art approaches. The performance is measured in terms of the mean squared error (MSE) of the estimated signal $\hat{\mathbf{s}}$ via Monte Carlo (MC) experiments and averaged over $n_{MC} = 100$ independent trials.

$$\text{MSE} = \frac{1}{n_{MC}} \sum_{i=1}^{n_{MC}} \frac{1}{N} \left\|\hat{\mathbf{s}} - \mathbf{s}\right\|_2^2. \tag{11}$$

An observation interval of $50\,\text{ms}$ and a sampling rate $f_s = 10\,\text{kHz}$ are considered for the simulated data. Discrete signals of length $N = 500$ containing K sinusoidal waveforms are generated. The amplitudes and frequencies are selected uniformly at random at each experiment between $2\,\text{V}$-$5\,\text{V}$ and $200\,\text{Hz}$-$2\,\text{kHz}$ respectively, while the phases are assumed to be in the interval $[0, \pi]$. A DFT frame with redundancy factor $p = 3$ is considered. The measurement matrix $\boldsymbol{\Phi} \in \mathbb{R}^{M \times N}$ contains M rows selected at random from the order N identity matrix to resemble the NUS. This is equivalent to randomly choosing rows of the redundant Fourier frame as sensing matrix $\mathbf{A}$, which guarantees that $\mathbf{A}$ has small restricted isometry constants [6], [21]. The reduced set of measurements $\mathbf{y}$ is generated using (3).

Opti2 is compared with state-of-the-art methods to reconstruct sparse signals: BPDN, LBI, SIHT, OMP+CBP, BLOOMP and BISP. The evaluated approaches aim to recover the sparse representation $\hat{\mathbf{x}}$, from which the signal is obtained by $\hat{\mathbf{s}} = \boldsymbol{\Psi}_p\hat{\mathbf{x}}$. The non-linear optimization problem in (9) is solved using the built-in Matlab function *fminunc*, which uses a Quasi-Newton algorithm. The estimate $\hat{\mathbf{f}}_1$ is used as initial guess. The toolbox [22] is employed for solving (4) and [23], for the convex problem in CBP. On the other hand, the codes for the implementation of LBI and BLOOMP are available in



(a) MSE vs. SNR (b) MSE vs. M (c) MSE vs. K

Fig. 1: Performance comparison of different sparse reconstruction algorithms in terms of the MSE of the estimated signal.

[24] and [25] respectively, while SIHT and BISP can be found in [26].

Fig. 1 shows the results of a set of experiments, where the MSE is evaluated in different scenarios. The solid lines depict algorithms based on convex optimization while the dashed lines depict the ones based on greedy approaches. Noisy observations from the measurement model described in (3) are simulated, where the signal-to-noise ratio (SNR) is defined as the ratio between the power of the compressed noiseless measurements and the noise variance σ_v^2. A fixed number of measurements $M = 150$ and number of sinusoids $K = 4$ are considered for the results in Fig. 1a. The SNR is varied from 0 to $30\,\mathrm{dB}$. It can be observed, that BLOOMP, OMP+CBP and LBI have comparable performance for SNR values over $10\,\mathrm{dB}$, where the latter slightly outperforms the other two. Opti2 outperforms all the evaluated approaches for the considered SNR region, reducing the MSE of the estimated signal in the second optimization step and overcoming the effects caused by the spectral leakage.

For the next experiment, the MSE is evaluated in terms of the number of measurements M. We set $M = \beta N$, where $\beta \in (0, 1]$ and a range of subsampling ratios β is explored to verify the compression level that allows for a successful estimation with noisy measurement (SNR $=$ $20\,\mathrm{dB}$) and signals containing $K = 4$ sinusoids. The results are shown in Fig. 1b. Most of the approaches converge to a MSE with about 20% of compression rate. Opti2 tends to improve its performance as the number of measurements increases till approximately 60% of compression rate.

The performance of the recovery techniques in terms of the number of sinusoids K contained in the signal is studied and presented in Fig. 1c. A fixed number of measurements $M = 150$ and SNR $= 20\,\mathrm{dB}$ are considered. From the results in Fig. 1c, it can be noticed that the accuracy of the reconstructed signal is reduced as the number of sinusoids in s increases, where the sparsity of the signal is given by $\kappa = 2K$. The improvement in the signal reconstruction achieved by Opti2 appears to be higher when only few sinusoids are presented. However, it enhances the signal estimation accuracy for the considered values of K.

The averaged computation time of the evaluated approaches in terms of the number of sinusoids is shown in Fig. 2. Unlike the greedy algorithms, the average runtime of the convex optimization methods remains fairly constant with increasing K. In the greedy approaches, the number of required iterations depends on the sparsity of the signal and consequently, their execution time increases with K. On the other hand, Opti2's average computation time slightly grows with increasing K. This is due to the increasing dimension of the matrix $\mathbf{H}_{\mathbf{\Phi}}(\mathbf{f})$ involved in the optimization problem, where K parameters should be estimated. The computations were performed on a high-performance computer featuring an ADM Ryzen 9 5950X 16 Core processor.

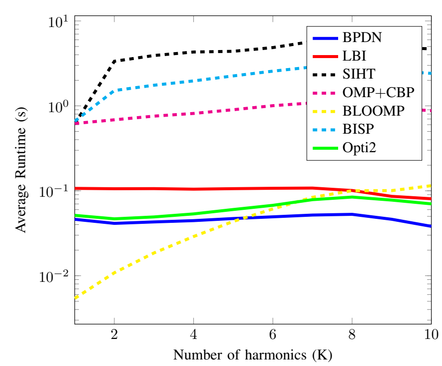


Fig. 2: Performance comparison of different reconstruction algorithms in terms of the average computation time vs K.

A scenario with SNR $= 20\,\mathrm{dB}$, $M = 150$ measurements and $K = 5$ is considered, where the amplitudes and phases of the sinusoids are set to $1\,\mathrm{V}$ and $0\,\mathrm{rad}$, respectively. The MSE of the estimated signal and corresponding frequencies are listed in Table I. The MSE of the estimated frequencies is computed by $\mathrm{MSE}_f = \frac{1}{n_{MC}} \sum_{i=1}^{n_{MC}} \frac{1}{K f_s} ||\hat{\mathbf{f}} - \mathbf{f}||_2^2$. For the multiple frequency estimation problem, Opti2 is outperformed by LBI, BLOOMP and OMP+CBP. The latter accounts for continuous frequency values, leading to the best estimation of the frequencies. On the other hand, Opti2 gives the minimum MSE for the signal estimation problem.

TABLE I: MSE of the estimated frequencies and the estimated signal for each evaluated approach, considering SNR $= 20\,\text{dB}$

Algorithm	MSE_f	MSE
BPDN	9.9839	0.0232
LBI	1.9967	0.0456
SIHT	88.6154	1.9634
OMP+CBP	0.1602	0.4018
BLOOMP	1.6566	0.0710
BISP	349.6665	0.1460
Opti2	9.9623	0.0110

The experiments in Fig. 3 test the capacity of the evaluated algorithms to estimate a signal containing two sinusoids, the frequencies of which lie at a distance $\triangle_f$. The scenario considers $M = 150$, SNR $= 20\,\text{dB}$, $f_1 = 1435.824\,\text{Hz}$ and $f_2 = f_1 + \triangle_f$, with $a_1 = a_2 = 1\,\text{V}$ and $\theta_1 = \theta_2 = 0\,\text{rad}$. Opti2 can correctly recover signals, containing frequencies with a minimum separation of $16.6\,\text{Hz}$, which is equivalent to $\triangle_f = 2.5 \cdot \delta_p$, where $\delta_p = f_s/N_p$ is the frequency step size of the redundant DFT. For closely spaced frequencies, the MSE is low due to the short duration of the frame ($50\,\text{ms}$). With increasing $\triangle_f$, the non-resolved frequencies lead to an increase of the MSE. Opti2 is the first that is capable to resolve the frequencies and thus can reduce the MSE significantly compared to the other algorithms.

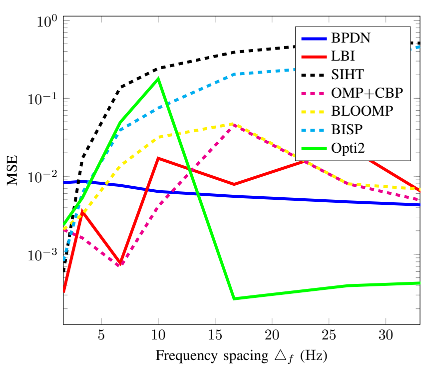


Fig. 3: Performance comparison of different reconstruction algorithms in terms of the MSE for signals containing $K = 2$ sinusoids with $\triangle_f$ spaced frequencies.

V. Conclusions

In this paper, a two-stage reconstruction approach, referred to as Opti2, is proposed to improve the estimation of signals that can be expressed as a sum of real-valued sinusoids with arbitrary frequencies. The estimation provided by one of the well-established recovery techniques for compressed measurements is used as first estimation, which gets refined in a second stage by solving a non-linear least squares problem. Simulation results show that Opti2 outperforms reported techniques used in spectral compressive sensing, achieving accurate results in terms of the MSE, without increasing the computation time considerably.

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