Foreign Exchange Multivariate Multifractal Analysis

Patrice Abry
Université de Lyon, ENS de Lyon
CNRS, Lab. de Physique, Lyon (FR)
patrice.abry@ens-lyon.fr

Yannick Malevergne
Université Paris 1 Panthéon-Sorbonne
PRISM Sorbonne, Paris (FR)
yannick.malevergne@univ-paris1.fr

Herwig Wendt
Université de Toulouse
CNRS, IRIT, Toulouse (FR)
herwig.wendt@irit.fr

Stéphane Jaffard
Université Paris-Est,
CNRS, LAMA, Créteil (FR)
stephane.jaffard@u-pec.fr

Marc Senneret
Vivienne Investissement, Lyon (FR)
msenneret@vivienne-im.com

Laurent Jaffrès
Vivienne Investissement, Lyon (FR)
ljaffres@vivienne-im.com

***Abstract*—After Mandelbrot's seminal work, scale-free and multifractal temporal dynamics have been recognized as classical stylized facts for financial time series and massively documented. Multifractal analysis in finance has however mainly remained univariate (one time series at a time) when multivariate (or basket) properties are critical for financial applications. This is mostly due to a lack of theoretical foundations and practical tools for multivariate multifractal analysis. Expanding on a theoretically-grounded recently proposed multivariate multifractal formalism, the present work performs an original multivariate analysis for a basket of six Foreign Exchange rate time series. Beyond confirming multifractality for each component independently, the definition of cross-multifractalities amongst components is introduced, assessing cross-dependencies in temporal dynamics not already accounted for by cross-correlations. The key practical outcome is to show that, essentially, one same multifractal time governs jointly the temporal dynamics of all the Foreign Exchange time series studied here.**

***Index Terms*—Financial times series, Foreign exchange, basket properties, multivariate multifractal analysis, wavelet leaders.**

I. Introduction

Context. It has been well-documented that financial time series are characterized by a set of *stylized facts* [1]. Two notable such facts consist of the absence of temporal autocorrelation (a consequence of the efficient market hypothesis) and the scale-free nature of temporal dynamics (all time scales, or frequencies, drive temporal dynamics). This lead Mandelbrot to propose *multifractal* fluctuations to model the temporal dynamics of asset prices [2], [3]. This triggered massive research works on financial time series [4]–[8]. However, multifractal analysis remained essentially univariate, one time series studied at a time [7], [9], while the multivariate, or joint, or basket, properties of financial time series are critical for financial applications. Assessing cross-multifractalities in a basket of assets is thus the crucial issue addressed here.

Related work. In [10], Mandelbrot proposed to modify the Brownian motion $B(t)$ classically used to model financial time series as random walks, via a time warping $B(A(t))$, where A denotes a multifractal cascading process. This added a key intuition to multifractality: that of an internal clock $A(t)$ that fluctuates around an average smooth evolution $A(t) \simeq t$ to model time accelerations (and thus shocks) or decelerations in financial markets. Variations of multifractal models for asset prices where further proposed, such as the Markov-switching multifractal model [11] or the multifractal random walk [12]. Empirically, multifractality has been evidenced in numerous financial time series different in nature, e.g., [13] using the MultiFractal-Detrended Flutuation Analysis [14] or the Wavelet Transform Modulus Maxima [15] frameworks. Often, multifractality has been connected with long memory in volatility, while it was shown to be unrelated to the signs of the returns [9], [11], [16], [17]. Except for rare cases where multifractal models were modified to handle several time series jointly [18], multifractal analysis was performed in univariate settings only, in finance as well as in most applications. This is mostly due to a lack of theoretical foundations and practical tools for multivariate multifractal analysis. However, recently a novel formalism based on wavelet leaders has been devised and shown to reach state of the art univariate analysis performance [19]–[22]. Moreover, in contradistinction with earlier multifractal formalisms, it permits a natural theoretically well-grounded formulation of multivariate multifractal analysis [23]–[25]. This thus permits to envisage the assessments of cross-multifractalities amongst financial time series which, to the best of our knowledge, has not yet been performed.

Goals, contributions and outline. The present contribution aims to perform and show the interest of the joint or multivariate multifractal analysis for the exchange rates of six couples of major currencies. Elaborating on [24], [25], Section II introduces the theory and practice of multivariate multifractal analysis based on p-leader multiscale quantities, a recent extension of the wavelet leader formalism introduced in [21], [22] in univariate settings. Section II further provides understanding of the meaning and importance of the cross-multifractalities parameter and of multifractal cross-correlation. The Foreign exchange rates data used in this work are described in Section III. Their multifractalities and cross-multifractalities are assessed, quantified and discussed in Section IV with the major findings that, essentially, one same and unique clock governs jointly the temporal dynamics of all the currencies under consideration and possibly of the entire Foreign Exchange market.

II. Multivariate multifractal Analysis

A. Multivariate multifractal Analysis: Theory

Local regularity. Multifractal analysis aims to quantify the fluctuations in time of the local regularity in a time series $X(t)$ [19]. Local regularity has classically been measured using the Hölder exponent [19]. It has however been recently proposed that novel measures based on the notion of p-exponents ($p > 0$) could be used, offering a wider versatility to analyze real world data and benefiting from more practical robustness. These p-exponents, referred to as $h(t) \geq 0$, are used here. Their precise definition can be found in [21], [22], where they are thoroughly studied. As for the Hölder exponent, the smaller $h(t)$, the more irregular X is around t.

Multivariate multifractal spectrum. Though based on a local regularity measurement, multifractal analysis provides a global and geometrical information on how local regularities vary along time and, in a multivariate setting, across components. Let $\mathbf{h}(t) \triangleq \{h_1(t), \ldots, h_M(t)\}$ denote the values of the p-exponents taken at time t by each of the M-variate times series $\mathbf{X} = (X_1, \ldots, X_M)$. Inspired from [26] in a bivariate setting, it has been proposed in [23]–[25] to define the *multivariate multifractal spectrum* $\mathcal{D}$ of X as the collection of Hausdorff dimensions $\dim_H$ of the sets of points $t \in \mathbb{R}$ at which $\mathbf{h}(t)$ takes on the values $\mathbf{h}$:

$$\mathcal{D}(\mathbf{h}) \triangleq \dim_H \{t \in \mathbb{R}_+ : \{\mathbf{h}(t)\} = \{\mathbf{h}\}\}. \quad (1)$$

B. Wavelet p-leader multivariate multifractal formalism

The estimation of the multivariate multifractal spectrum $\mathcal{D}(\mathbf{h})$ is performed via the so-called multifractal formalisms. They are often based on multiscale quantities, constructed from wavelet analysis [19], [27]. Let ψ denote an oscillating reference pattern, the mother wavelet, characterized by its number of vanishing moments N_ψ, a positive integer defined as $\psi \in C^{N_\psi - 1}$ and $\forall n = 0, \ldots, N_\psi - 1$, $\int_{\mathbb{R}} t^k \psi(t) dt \equiv 0$ and $\int_{\mathbb{R}} t^{N_\psi} \psi(t) dt \neq 0$. The mother wavelet is designed such that dilated and translated templates $\{\psi_{j,k}(t) = 2^{-j/2}\psi(2^{-j}t - k)\}_{(j,k)\in\mathbb{Z}^2}$ form an orthonormal basis of $\mathcal{L}^2(\mathbb{R})$ [28]. The discrete wavelet transform coefficients $d_X(j,k)$ of X are defined as $d_X(j,k) = 2^{-j/2}\langle \psi_{j,k} | X \rangle$.

The *wavelet p-leaders* of X, $\ell_X(j,k)$ are defined as a local L^p-norm ($p > 0$), taken across all $d_X(j', k')$ located in a neighbourhood of $t = 2^j k$ at finer scales $j' \leq j$. Interested readers are referred to [21], [22] for detailed definitions.

It can be shown that wavelet p-leaders reproduce the p-exponents in the limit of fine scales, $L_X(j,k) \sim C 2^{jh(t)}$ as $2^j \to 0$ for $t = 2^j k$. Consequently, with $\mathbf{q} \triangleq \{q_1, \ldots, q_M\}$,

$$\frac{1}{n_j} \sum_{k=1}^{n_j} \Pi_{m=1}^{M} L_{X_m}(j,k)^{q_m} \sim c_q 2^{j\zeta(\mathbf{q})},\ 2^j \to 0. \quad (2)$$

The so-called *scaling exponents* $\zeta(\mathbf{q})$ in (2) are tightly related to $\mathcal{D}(\mathbf{h})$ via their Legendre transform, defining the *multivariate Legendre spectrum* as:

$$\mathcal{L}(\mathbf{h}) = \inf_{\mathbf{q}} (1 + \langle \mathbf{q}, \mathbf{h} \rangle - \zeta(\mathbf{q})), \quad (3)$$

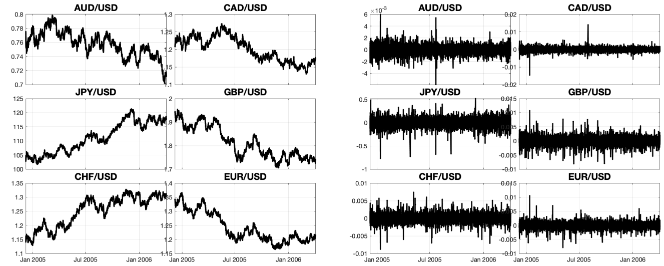

Figure 1. **Foreign Exchange data.** Left: Rates ; Right: Returns

which provides an estimate for $\mathcal{D}(\mathbf{h})$ for large classes of processes, see [24], [25].

Its precise shape thus conveys information regarding the joint or cross fluctuations of local regularities amongst the components of X. For example, in a bivariate setting, a multifractal spectrum $\mathcal{L}(h_1, h_2)$ that would stem directly from the univariate spectra of each time series as $\mathcal{L}_1(h_1) + \mathcal{L}_2(h_2) - 1$ would indicate that multifractalities amongst components are not related. To the converse, a bivariate spectrum whose support would collapse on a line would indicate a total dependence between the multifractalities of each component.

C. Multivariate multifractal analysis: Practice

To measure $\mathcal{L}(\mathbf{h})$, elaborating on [29], one can use the multivariate cumulants of order $s > 0$ of the log-leaders $(\ln L_{X_1}(j,k), \ldots, \ln L_{X_M}(j,k))$ at scale 2^j, denoted $C_{\mathbf{s}}(j)$ with $\mathbf{s} = \{s_1, \ldots s_M\}$, $\forall m, s_m \in \{0, \ldots, M\}$ and $s = \sum_m s_m$ [23]–[25]. For large classes of multivariate multifractal processes, $C_{\mathbf{s}}(j)$ behave as:

$$C_{\mathbf{s}}(j) = c_{\mathbf{s}}^0 + j\, c_{\mathbf{s}} \ln 2, \quad \mathbf{s} \geq 1 \quad (4)$$

where the multifractal coefficients $c_{\mathbf{s}}$ can be related to the $\zeta(\mathbf{q})$

$$\zeta(\mathbf{q}) = \sum_{s} c_{\mathbf{s}} \Pi_{m=1}^{M} q_m^{s_m} / s_m!\ , \quad (5)$$

and hence to the multivariate Legendre spectrum $\mathcal{L}(\mathbf{h})$.

The $c_{\mathbf{s}}$ are estimated by means of linear regressions of $C_{\mathbf{s}}(j)$ versus $\log_2 2^j = j$.

III. Foreign Exchange Rates

The data used here consist of Foreign Exchange rates for six couples of major currencies against the US Dollar (USD): *AUD/USD* (Australian Dollar), *CAD/USD* (canadian Dollar), *JPY/USD* (Japanese Yen), *GBP/USD* (British Pound), *CHF/USD* (Swiss Franc), *EUR/USD* (Euro), from 1999 to 2011. As the years after birth of the Euro shows remarkably stable statistical properties along time, a period of around 18 months, from December 2004 to May 2006 is studied here. Similar conclusions are drawn for the analysis of other 18 month-period, between 2002 and 2008. Data are sampled at an intraday high frequency, of $T_s = 5$min, thus resulting in around 10^5 samples per time series. Rates and returns (increments) for these six couples of currencies and for the chosen period are shown in Fig. 1

For ease of notations, the couple *XXX/USD* will hereafter be referred as *XXX*, thus omitting */USD*. Because we are interested in cross-dependencies, the bivariate analysis corresponding to the pair of couples *XXX/USD* and *YYY/USD* will thus simply be referred to as the pair *XXX-YYY*.

IV. Multivariate multifractal Foreign Exchange

A. Second-order cross-statistics analysis: cross-selfsimilarity

Prior to any practical multifractal analysis, the inspection of the joint second order statistics of the temporal dynamics of X are mandatory. To that end, the multivariate wavelet spectrum $S(j)$ must be computed, defined as the collection of $M \times M$ matrix $S(j)$ as functions of the analysis scales 2^j, whose entries consists of the cross correlations of the wavelet coefficients at scale 2^j: $S_{m,m'}(j) = 1/n_j \sum_k^{n_j} d_{X_m}(j,k) d_{X'_m}(j,k)$. In the present work, the wavelet coefficients are computed using least asymmetric orthornormal wavelets, with $N_\psi = 3$ [28]. For processes with scale-free temporal dynamics, such as multifractal processes, each entry of $S(j)$ behave as a power-law with respect to the analysis scale:

$$S_{m,m'}(j) \simeq K_{m,m'} 2^{2jH_{m,m'}} \quad (6)$$

The scaling exponents $H_{m,m'}$ quantify the auto- and cross-correlations. Notably, $H_{m,m}$ corresponds to the Hurst (or selfsimilarity) exponent of component m.

To further quantify cross-temporal dynamics, one usually use the wavelet-coherence function

$$\rho_{m,m'}(j) = S_{m,m'}(j) / \sqrt{S_{m,m}(j) S_{m',m'}(j)} \quad (7)$$

that consists of scale-dependent correlation coefficients [30].

Fig. 2(left) displays the log wavelet-spectrum $\log_2 S_{m,m}(j)$ for each component as a function of the log of the analysis scales $j = \log_2 2^j$. Linear behaviours in these log-log plots indicate, for all couples of currencies, scale-free dynamics ranging from $j = 2$ to $j = 9$, hence across 7 octaves. These scale-free dynamics are also observed for all pairs $\log_2 S_{m,m'}(j)$ across the same range of scales (cf. Fig. 2 (middle)). Given the sampling period of $T_s = 5$min, this indicates that scale-free dynamics range remarkably from $\simeq$ 10min to $\simeq$ 100hours, which, in terms of trading time, means around 2 weeks, i.e., *for more than two decades of time scales*. These scale-free dynamics also imply that time scales of, e.g., one working day or one working week can not be identified in the data, which is actually consistent with the global and decentralized organization of the foreign exchange market, which operates 24 hours a day, 7 days a week. However, when estimated from linear regressions of $\log_2 S_{m,m'}(j)$ vs. j, none of the estimated scaling exponents $H_{m,m'}$ are found significantly different[1] from $H = 0.5$, see Table I (upper triangle). This clearly indicates the absence of any temporal autocorrelation for all components and of any temporal cross-correlation amongst components, in clear agreement with the efficient market hypothesis.

[1]Throughout this work, statistical significance has been assessed by means of wavelet domain block bootstrap procedures, as devised in [20] and here extended to multivariate analysis; p-values are not reported for space reasons.

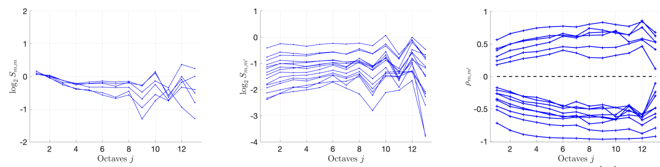

Figure 2. **Scale-free dynamics.** Log wavelet-spectrum $\log_2 S_{m,m}(j)$ (left), $\log_2 S_{m,m'}(j)$ (center), and $\rho_{m,m'}(j)$ (right) as functions of the log of the analysis scales $j = \log_2 2^j$ and for the 6 time series in Fig. 1.

Fig. 2(right) shows that the wavelet coherence functions $\rho_{m,m'}(j)$ for all pairs of components are essentially independent of scales, in accordance with the efficient market hypothesis. It moreover indicates that cross-correlations are significantly non-zero for all but the last analysis scale. This is quantified by computing the average across scales of $\rho_{m,m'}(j)$, highly reminiscent of the pairwise Pearson correlation coefficient. These pairwise cross-correlations (Table I upper triangle) identify two groups of currencies, with positive intra-group correlations, and negative inter-group correlations. The first group gathers EUR, GBP and AUD, with strong intra-group correlations ; the second group (JPY, CHF, CAD) shows less intra-group correlations.

	AUD	CAD	JPY	GBP	CHF	EUR
AUD	0.48	−0.45	−0.50	0.61	−0.61	0.62
CAD	0.49	0.47	0.26	−0.36	0.40	−0.44
JPY	0.49	0.48	0.48	−0.56	0.59	−0.59
GBP	0.50	0.49	0.49	0.49	−0.76	0.76
CHF	0.50	0.49	0.49	0.50	0.49	−0.93
EUR	0.50	0.49	0.48	0.50	0.50	0.50

Table I

Correlation and selfsimilarity analysis. Estimated scaling exponents $H_{m,m'}$ (diagonal and lower triangle, blue). Average across scales Coherence function (upper triangle, red).

B. Beyond second-order cross-statistics: cross-multifractality

Multifractal analysis is here performed with 2-leaders ($p = 2$), shown to yield the best estimation performance [21], [22].

Often, in practice, to perform multivariate multifractal analysis, use is made only of the cumulants of orders $s = 1$ and $s = 2$, thus restricting the M-variate analysis to a collection of bivariate (or pairwise) analyses. For ease of notations, the M cumulants of order 1 (hence univariate) are hereafter labelled $C_m(j)$ and the corresponding multifractal coefficients $c_{\mathbf{s}}$ in Eq. 4 as $c_1(m)$. The $M \times (M+1)/2$ cross-second order cumulants (hence bivariate) and corresponding multifractal coefficients are written $C_{m,m'}(j)$ and $c_2(m,m')$.

Implicitly, this amounts to perform a parabolic approximation of the pairwise scaling exponents $\zeta_{m,m'}(\mathbf{q})$, which by Eq. 3 also implies a pairwise parabolic approximation of $\mathcal{L}_{m,m'}$ around its maximum [23]:

$$\zeta_{m,m'}(q_1,q_2) \approx c_1(m)q_1 + c_1(m')q_2 + \frac{c_2(m,m)}{2}q_1^2 + \frac{c_2(m',m')}{2}q^2 + c_2(m,m')q_1q_2 \quad (8)$$

$$\mathcal{L}_{m,m'}(h_1, h_2) \approx 1 + \frac{c_2(m,m)b}{2}\left(\frac{h_1 - c_1(m)}{b}\right)^2 + \frac{c_2(m',m')b}{2}\left(\frac{h_2 - c_1(m')}{b}\right)^2 - c_2(m,m')b\left(\frac{h_1 - c_1(m)}{b}\right)\left(\frac{h_2 - c_1(m')}{b}\right), \quad (9)$$

where $b \triangleq c_2(m,m)c_2(m',m') - c_2(m,m')^2 \geq 0$. This shows that the position of the maximum of $\mathcal{L}_{m,m'}$ is given by $(c_m, c_{m'})$, and that $c_{m,m}$ and $c_{m',m'}$ quantify the widths of the fluctuations independently for each component. Finally, $c_{m,m'}$ quantify the departure from an isotropic parabola, thus providing a characterization across components of the joint fluctuations of the regularities of each component.

Inspired from $\rho_{m,m'}(j)$, this leads to define,

$$\rho^{MF}_{m,m'}(j) = \frac{C_{m,m'}(j)}{\sqrt{C_{m,m}(j)C_{m',m'}(j)}}, \quad (10)$$

that quantifies a scale-free based higher order statistics dependence coefficient, and is hence hereafter referred to as *multifractal coherence function*, slightly abusively yet with an intuitive meaning. This further prompts for the use of the *multifractal correlation coefficient* defined as:

$$\rho_{MF} = -c_{m,m'}/\sqrt{c_{m,m}c_{m',m'}}.$$

Univariate multifractality. Let us first study the multifractal property of each couple of currencies independently (in a univariate setting). Fig. 3(left column) reports the M first order cumulants $C_m(j)$ (top left) and the M second order cumulants $C_{m,m}(j)$ (bottom left) computed independently for each component. These univariate cumulants show remarkable scaling across a broad range of octaves, consistent with that observed in Fig. 2. The $C_m(j)$ are superimposing for all components, and their corresponding slopes $c_1(m)$ are all found not to depart from 0.5 (see Table II), thus further confirming the absence of autocorrelation in the temporal dynamics of each component independently and the consistency with the efficient market hypothesis. However, the $C_{m,m}(j)$ also display remarkable scaling, with decreasing amplitude along scales, and corresponding slopes $c_2(m,m)$ all significantly negatively departing from 0 (see the diagonal entries of Table III). This is a clear evidence that each of these Foreign Exchange time series displays a clear and robust multifractality, over a broad range of time scales from 10min to above 2 weeks. This indicates that each of these time series possesses independently subtle temporal dynamics in the form of local or transient structures and burstiness, that cannot be captured by the classical correlation and second order statistics. Beyond being significant, the multifractality parameters $c_2(m,m)$ of all components are interestingly found to be of close values: $-0.038 \leq c_2(m,m) \leq -0.024$.

Cross-multifractality. Let us further study cross multifractalities. Fig. 3(central top plots) reports the $M(M-1)/2$ cross second cumulants $C_{m,m'}(j)$ ($m \neq m'$), with scaling across ranges of scales, consistent with that of other plots. The corresponding slopes ($c_2(m,m')$) are all significantly negative

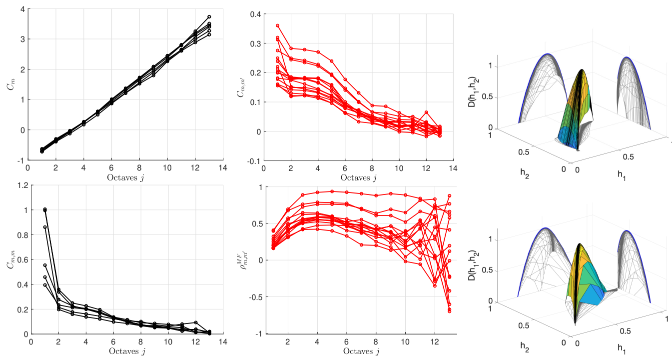

Figure 3. **Foreign Exchange multivariate multifractal analysis.** $C_m(j)$ (top left), $C_{m,m}(j)$ (bottom left), $C_{m,m'}(j)$, $m' \neq m$ (top center), and $\rho^{MF}_{m,m'}(j)$ (bottom center) as functions of the log of the analysis scales j and for the 6 time series in Fig. 1. Bivariate multifractal spectrum for the (GBP, EUR) (top right) and (CHF, EUR) (bottom right) pairs of currencies.

(see Table III lower triangle). In addition, Fig. 3(central bottom plots) shows the $M(M-1)/2$ multifractal coherence functions $\rho^{MF}_{m,m'}$, observed to be weakly dependent on scales (except at coarse scales where estimation is of poor quality because of observation finite duration) and all significantly positive. Further, the *multifractal correlation coefficients* $\rho_{MF}(m,m')$ are found to all positively depart from 0, with values ranging from $0.47 \leq \rho_{MF} \leq 0.95$ and often large (see Table III upper triangle). Fig. 3(right columns) finally illustrates for two pairs of couples of currencies (GBP-EUR) and (CHF-EUR) the estimated bivariate multifractal spectra $\mathcal{L}(h_m, h_{m'})$. The fact that they both consist of a parabola stretched along the first diagonal, hence with large aspect ratios clearly departing from 1, provides intuition to the meaning of large values for ρ_{MF}. These multifractal spectra clearly differ from a spectrum that would correspond to unrelated multifractalities $\mathcal{L}_m(h_m)+\mathcal{L}_{m'}(h_{m'})-1$, where the univariate spectra of each component $\mathcal{L}_m(h_m)$ and $\mathcal{L}_{m'}(h_{m'})$ are also shown.

These plots and results clearly indicate that the cross-temporal dynamics of the exchange rates for all pairs of couples of currencies studied here show related scale-free dynamics, with dependencies that cannot be measured by cross-correlation or cross-Fourier spectrum, but are evidenced and quantified by the cross-multifractality parameters $c_2(m,m')$ and $\rho_{MF}(m,m')$. Positive $\rho_{MF}(m,m')$ indicate that the occurrences of burstiness or of local transient structures are colocalized in time between components.

Discussion. As mentioned in the introduction, after Mandelbrot, financial asset prices are often modelled as compound Brownian motions $B(A(t))$ where the time warping function $A(t)$ entails the multifractal properties. The present analysis suggest that the collection of Foreign exchange rates studied here can be modelled by a multivariate extension of that model $\{B_m(A_m(t)), m = 1, \ldots, M\}$ [31]. The B_m are correlated Brownian motions, with a cross correlation matrix showing significant non-zero entries, both positive or negative. They however do not possess any auto- or cross-correlation in their auto- and cross-temporal dynamics. The A_m are multifractal processes, e.g., multiplicative cascades as proposed by Mandelbrot, each with multifractality controlled by $c_2(m,m)$. In addition, they also possess a cross sectional correlation, with correlation matrix ρ_{MF}, characterized by positive only and significantly departing from 0 entries. This is the correlation amongst those time warping functions A_m that induces cross multifractalities and hence transient dependencies in the cross-temporal dynamics of the $B_m(A_m(t))$ and thus amongst Foreign exchange times series.

Finally, returning to the original intuition of multifractality understood as a time warping, the fact that the entries of ρ_{MF} are all positive and large, indicates that all time warping functions A_m are very similar for all components. In financial terms, this can be interpreted as the fact that the currency rates studied here are governed by one unique and same clock across the world. This provides a different understanding of the Foreign Exchange market compared to the one suggested by classical second-order statistics with two groups of currencies.

To the best of our knowledge, this unique clock timing the the dynamics of the majors currencies had never been evidenced before, because a theoretically-grounded and practically efficient multivariate multifractal formalism was lacking.

In an effort toward reproducible research and open science, Matlab codes implementing multivariate multifractal analysis are made publicly available[2].

	AUD	CAD	JPY	GBP	CHF	EUR
$c_1(m)$	0.48	0.49	0.49	0.51	0.52	0.52

Table II

Location of the maximum of the multifractal spectrum: $c_1(m)$.

ρ_{MF}	AUD	CAD	JPY	GBP	CHF	EUR
AUD	−0.024	0.47	0.70	0.61	0.64	0.68
CAD	−0.014	−0.039	0.59	0.66	0.67	0.68
JPY	−0.017	−0.018	−0.024	0.71	0.84	0.95
GBP	−0.018	−0.025	−0.021	−0.038	0.78	0.86
CHF	−0.020	−0.026	−0.025	−0.030	−0.038	0.92
EUR	−0.020	−0.026	−0.028	−0.032	−0.035	−0.037

Table III

Multivariate multifractal analysis and multifractal parameters.
MULTIFRACTALITY PARAMETERS $c_2(m,m')$ (DIAGONAL AND LOWER TRIANGLE, BLUE). MULTIFRACTAL CORRELATION COEFFICIENTS $\rho_{MF}(m,m')$ (UPPER TRIANGLE, RED).

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