A System Theoretic Approach for the Reduction of Large-Scale Room Acoustic Models

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***Abstract*—Efficiently simulating sound density in room acoustic models poses a significant challenge since it involves the solution of large-scale systems of equations, which can result in unreasonably/unacceptably long computation times. However, in many cases, sound density measurements only need to be taken at certain points in the room rather than every point, which allows the use of Model Order Reduction (MOR) techniques. System theoretic techniques like balanced truncation (BT) are well-established and can be applied to the sound diffusion equation, offering reliable error bounds. This paper presents a low-rank BT algorithm in order to generate compact models, which can be efficiently and accurately simulated over many timesteps. The experimental results show that this method can provide extreme order reduction percentages of 99.99% and thus accelerate simulations by up to $59\times$ while maintaining a relative error of less than 0.75%.**

***Index Terms*—Room Acoustic Models, Sound Diffusion Equation, Finite Difference Method, Balanced Truncation, Model Order Reduction**

I. Introduction

There has been significant interest in room acoustic modeling and prediction in recent years. Modeling sound density in an enclosed space can be very complex, as it requires taking into account various geometrical properties, absorption features, and boundary conditions. To address this challenge, a diffusion equation method has been developed that can effectively model these parameters [1], [2]. In order to compute the sound density, recent methodologies have focused on solving the linear system of equations that result from modeling approaches such as Finite Elements Method (FEM) [3] and Finite Difference Methods (FDM) [4], [5].

All of the aforementioned systems of the room acoustic modeling methods are enormous and require solving large-scale systems of equations at multiple timesteps [6], [7]. However, in certain scenarios, it may only be necessary to compute the sound density at specific points to assess the performance of individual sound devices in the room. In such cases, a Model Order Reduction (MOR) process can be used to replace the large room acoustic model with a much smaller model that behaves similarly at the points of interest [8].

MOR methods are categorized into two types. Moment-Matching (MM) techniques are computationally efficient in producing reduced-order models but they exhibit some drawbacks, the most prominent of which is that they do not offer any a-priori error bound [9]. This can lead to reduced models that may not be accurate enough or of sufficient small order. Contrary, system theoretic techniques, such as Balanced Truncation (BT), are based on rigorous mathematical concepts from system theory to provide an overall bound on the error without having to compute the reduced-order model first [10]. However, they require the solution of the expensive Lyapunov matrix equations and the storage of the corresponding dense matrices, even if the system matrices are sparse.

This paper introduces an innovative and efficient approach for reducing large-scale acoustic models. Our proposed method combines the power of the low-rank BT method with the Extended Krylov Subspace (EKS) technique to solve the Lyapunov matrix equations [11], leading to remarkable computational efficiency. To validate the effectiveness of our approach, we employed it on several room benchmarks with varying sizes, discretization points and sound sources. Experimental results demonstrate a significant reduction in simulation time compared to the original models, with negligible errors, which proves the suggested method to be a significant step forward in the field of room acoustics modeling.

The rest of the paper is organized as follows. Section II outlines the diffusion equation along with the appropriate boundary conditions. Section III describes the discretization process for the FDM-based room acoustics model. Section IV describes the BT application with the EKS method for the Lyapunov equations. Finally, Section V presents the results and a discussion of the method's advantages, followed by the conclusions in Section VI.

II. Theoretical Background

The diffusion equation is a type of parabolic partial differential equation (PDE) that can be utilized in various physical problems, such as heat conduction in solids, population dispersion, and other similar physics problems. As a result, a diffusion-type model has been established for sound modeling and prediction, which was initially presented in [1] and later examined for numerical simulations in [2]. Subsequently, in [12], the authors extended the traditional geometrical room acoustic model by drawing an analogy with the radiative transfer theory for scattering and absorbing radiation. More specifically, the PDE that describes the diffusion equation model for sound energy is the following:

$$\frac{\partial w(\mathbf{r},t)}{\partial t} = D\nabla^2 w(\mathbf{r},t) + cmw(\mathbf{r},t) + P(\mathbf{t})\delta(\mathbf{r}-\mathbf{r}_s) \quad (1)$$

with term $w(\mathbf{r},t)$ being the sound energy at the position r defined on a domain V and time t, $P(\mathbf{t})$ the sound source

term, which is located at the position $\mathbf{r}$, and $mw(\mathbf{r},t)$ the atmospheric attenuation within the room term, where m is the absorption coefficient of air [13]. The above equation is considered as an inhomogeneous parabolic PDE where ∇^2 is the Laplace operator term, and $D = \lambda c/3$ is the diffusion coefficient term, with c being a constant that describes the speed of sound. Finally, Eq. 1 must be accompanied by a set of mixed boundary conditions [14], [15]:

$$D\frac{\partial w(\mathbf{r},t)}{\partial \mathbf{n}} + A_X(\mathbf{r},\alpha)cw(\mathbf{r},t) = 0 \tag{2}$$

in order to incorporate the varying absorption levels of the sound field's surfaces in an efficient manner. In the above equation, $\mathbf{n}$ is the unity vector normal to the boundary surface and $A_X(\mathbf{r},\alpha)$ is the factor that correlates the distribution of the surface absorption given an absorption coefficient α.

III. Problem Formulation

In this section, we discretize Eq. 1 and 2 over the space coordinates, in order to transform the PDE into a system of Ordinary Differential Equations (ODEs). For this purpose, we apply an FDM discretization methodology with Δx, Δy and Δz being the step at each spatial coordinate in the cartesian axis and n_x, n_y and n_z the total number of discretization nodes per coordinate for different levels of accuracy. Consequently, the dependent variable w is transformed to:

$$w(\mathbf{r},t) = w(i\Delta x, j\Delta y, k\Delta z, t) = w_{i,j,k}(t) \tag{3}$$

Using a second-order difference approximation on the second derivative, the partial differential equation (PDE) of Equation 1 for an internal node located at the position (i,j,k) can be expressed as:

$$\begin{aligned}\dot{w}_{i,j,k}(t) = &\frac{D}{\Delta x^2}\left(w_{i-1,j,k}(t) - 2w_{i,j,k}(t) + w_{i+1,j,k}(t)\right)\\ &+ \frac{D}{\Delta y^2}\left(w_{i,j-1,k}(t) - 2w_{i,j,k}(t) + w_{i,j+1,k}(t)\right)\\ &+ \frac{D}{\Delta z^2}\left(w_{i,j,k-1}(t) - 2w_{i,j,k}(t) + w_{i,j,k+1}(t)\right)\\ &- cmw_{i,j,k}(t)\end{aligned} \tag{4}$$

Then, assuming a uniform discretization step for all spatial coordinates, the discretized equation for an internal node at position (i,j,k) can be simplified to:

$$\begin{aligned}\dot{w}_{i,j,k}(t) = \frac{D}{\Delta x^2}&(w_{i-1,j,k}(t) + w_{i+1,j,k}(t)\\ &+w_{i,j-1,k}(t) + w_{i,j+1,k}(t)\\ &+w_{i,j,k-1}(t) + w_{i,j,k+1}(t)\\ &-(6 + \frac{\Delta x^2}{D}cm)w_{i,j,k}(t))\end{aligned} \tag{5}$$

For the surface nodes, we discretize Eq. 2. For the sake of simplicity, we demonstrate the derived equations on x-axis. Let the dimensions of the room be L_x, L_y, and L_z, then the surface nodes located at $x = 0$ and $x = L_x$ can be written as:

$$\begin{aligned}D\frac{\partial w(x,t)}{\partial x} + cA_X(x,\alpha)w(x,t) = 0 \quad &\text{at } x = 0\\ -D\frac{\partial w(x,t)}{\partial x} + cA_X(x,\alpha)w(x,t) = 0 \quad &\text{at } x = L_x\end{aligned} \tag{6}$$

Note that the spatial derivatives are approximated in the direction of increasing coordinate x, hence the minus sign at $x = L_x$. Although first-order approximations can be applied on Eq. 6, it is proven to be less accurate than the second-order approximation of the first derivative [16]. So, by applying the second-order three-point backward and forward formulae of the first derivative at the surface points of x-axis, Eq. 6 becomes:

$$\begin{aligned}\frac{D}{2\Delta x}(3w_{0,j,k}(t) - 4w_{1,j,k}(t) + w_{2,j,k}(t))\\ +A_{X_{0,j,k}}w_{0,j,k}(t) = 0\\ \frac{D}{2\Delta x}(3w_{L_{x-2},j,k}(t) - 4w_{L_{x-1},j,k}(t) + w_{L_x,j,k}(t))\\ +A_{X_{L_x,j,k}}w_{L_x,j,k}(t) = 0\end{aligned} \tag{7}$$

Then, we express Eq. 7 with respect to $w_{0,j,k}(t)$ and $w_{L_x,j,k}(t)$ respectively and replace them in Eq. 5. This actually means that the surface nodes are eliminated from the system and their acoustical density energy is included in the neighboring nodes. Surface nodes on dimensions y and z are handled in a similar manner. Therefore, we result to $n = n_x n_y n_z - 2(n_x + n_y + n_z)$ equations which are described by Eq. 5 for $i = 1,\ldots,L_{x-1}$, $j = 1,\ldots,L_{y-1}$ and $k = 1,\ldots,L_{z-1}$.

In case a node is also a sound source, the discretized sound source density term $P_{i,j,k}(t)$ is added to Eq. 5. So, the resulting equation is:

$$\dot{\mathbf{w}}(t) = \mathbf{G}\,\mathbf{w}(t) + \mathbf{B}\mathbf{u}(t) \tag{8}$$

where N is the number of the internal nodes and p is the number of sound sources in the room; $\mathbf{G} \in \mathbb{R}^{n\times n}$ is the coefficient matrix, $\mathbf{B} \in \mathbb{R}^{n\times p}$ is the input-to-state matrix that corresponds the sound sources to the discretization nodes and $\mathbf{u}(t) \in \mathbb{R}^p$ is the input vector containing the sound sources densities.

In numerous instances, it is not necessary to model the diffusion of sound throughout all internal nodes in a space, but only at certain points of interest. This leads to the requirement for a secondary equation that filters out the internal nodes and monitors sound densities solely at the specific points of interest. Thus, we can conclude at the following LTI ODE system:

$$\begin{aligned}\dot{\mathbf{w}}(t) &= \mathbf{G}\,\mathbf{w}(t) + \mathbf{B}\mathbf{u}(\mathbf{t}),\\ \mathbf{y}(t) &= \mathbf{L}\,\mathbf{w}(t)\end{aligned} \tag{9}$$

with $y \in \mathbb{R}^q$ being an output vector with the sound energy density at the specific points that we want to simulate. $\mathbf{L}$ is the state-to-output matrix that selects q out of n internal states and corresponds them to the output points. Finally, in case we want to monitor the sound density on a surface node, the corresponding row of matrix $\mathbf{L}$ instead of ones, it contains the

coefficients of Eq. 7 with respect to the index of the surface node whose value needs to be monitored.

IV. MOR BY BT

Bearing in mind the LTI system of Eq. 9, the objective of MOR is to produce a reduced model as:

$$\begin{aligned} \dot{\tilde{\mathbf{w}}}(t) &= \tilde{\mathbf{G}}\,\tilde{\mathbf{w}}(t) + \tilde{\mathbf{B}}\mathbf{u}(\mathbf{t}), \\ \mathbf{y}(t) &= \tilde{\mathbf{L}}\,\tilde{\mathbf{w}}(t) \end{aligned} \tag{10}$$

with $\tilde{\mathbf{G}} \in \mathbb{R}^{r\times r}$, $\tilde{\mathbf{B}} \in \mathbb{R}^{r\times p}$, $\tilde{\mathbf{L}} \in \mathbb{R}^{q\times r}$ and $r \ll n$. The reduced system is a good and robust approximation of the original system as the output error is bounded, i.e., $||\tilde{\mathbf{y}}(t) - \mathbf{y}(t)||_2 < \varepsilon||\mathbf{u}||_2$ for a given input $\mathbf{u}(t)$ and an error bound ε. Considering Plancherel's theorem [17], the bounded output error can be analogously formulated in the frequency domain as $||\tilde{\mathbf{y}}(s) - \mathbf{y}(s)||_2 < \varepsilon||\mathbf{u}||_2$. By taking into account the transfer functions of the original and the reduced model:

$$\begin{aligned} \mathbf{H}(s) &= \mathbf{L}(s\mathbf{I} - \mathbf{G})^{-1}\mathbf{B}, \\ \tilde{\mathbf{H}}(s) &= \tilde{\mathbf{L}}(s\tilde{\mathbf{I}} - \tilde{\mathbf{G}})^{-1}\tilde{\mathbf{B}} \end{aligned} \tag{11}$$

then the output error bound in the frequency domain becomes:

$$\begin{aligned} ||\tilde{\mathbf{y}}(s) - \mathbf{y}(s)||_2 &= ||\tilde{\mathbf{H}}(s)\mathbf{u}(s) - \mathbf{H}(s)\mathbf{u}(s)||_2 \\ &\leq ||\tilde{\mathbf{H}}(s) - \mathbf{H}(s)||_\infty ||\mathbf{u}(s)||_2 \end{aligned} \tag{12}$$

where $||\,.\,||_\infty$ denotes the $\mathcal{L}_\infty$-norm, which for rational transfer functions is the $\mathcal{H}_\infty$-norm. Consequently, by bounding the distance between the two transfer functions, the output error can also be bounded as $||\tilde{\mathbf{H}}(s) - \mathbf{H}(s)||_\infty < \varepsilon$.

MOR methods such as BT take advantage of the controllability and observability Gramian matrices $\mathbf{P}$ and $\mathbf{Q}$ which can be computed by solving the Lyapunov matrix equations [18]:

$$\begin{aligned} \mathbf{G}\mathbf{P} + \mathbf{P}\mathbf{G}^T &= -\mathbf{B}\mathbf{B}^T, \\ \mathbf{G}^T\mathbf{Q} + \mathbf{Q}\mathbf{G} &= -\mathbf{L}^T\mathbf{L} \end{aligned} \tag{13}$$

The controllability Gramian $\mathbf{P}$ represents the degree to which the states are reachable by the inputs while the observability matrix $\mathbf{Q}$ depicts how observable are the states from the outputs. The reduced model is constructed by truncating the states that are either difficult to reach or to observe. However, since these are two independent metrics, there may be states that are problematic regarding only one of them, hence the process *balancing*. This transformation produces a balanced realization of the system, which means the system is described by a degree that combines both the controllability and observability degrees. Thus, there exists a transformation matrix $\mathbf{T}$ that transforms the model into:

$$\begin{aligned} \mathbf{T}\dot{\mathbf{w}}(t) &= \mathbf{T}\mathbf{G}\mathbf{T}^{-\mathbf{1}}(\mathbf{T}\mathbf{w}(t)) + \mathbf{T}\mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{L}\mathbf{T}^{-\mathbf{1}}(\mathbf{T}\mathbf{w}(t)) \end{aligned} \tag{14}$$

by preserving the transfer function $\mathbf{H}(s)$ and making [18]:

$$\mathbf{P} = \mathbf{Q} = diag(\sigma_1, \sigma_2, ..., \sigma_n) \tag{15}$$

where $\sigma_i, i = 1, \ldots, n$ are the Hankel singular values (HSVs) of the model, which correspond to the square root of the eigenvalues of the multiplication of the Gramians, i.e., $\sigma_i = \sqrt{\lambda_i(\mathbf{PQ})}, i = 1, \ldots, n$. The states of the balanced system of Eq. 14 that are more observable or reachable correspond to the largest HSVs. So, if only r of these states are maintained and the rest $n - r$ are truncated, then the distance between the transfer functions of the original and the reduced system is limited to:

$$||\mathbf{H}(s) - \tilde{\mathbf{H}}(s)||_\infty \leq 2(\sigma_{r+1} + \sigma_{r+2} + \cdots + \sigma_n) \tag{16}$$

Eq. 16 is an a-priori error bound for the order selection of the reduced model. The complete BT procedure is exhibited in Alg. 1.

Algorithm 1 Balanced Truncation

1: Compute the Gramians $\mathbf{P}$ and $\mathbf{Q}$ by solving the Lyapunov Eq. 13
2: Obtain the Cholesky factors $\mathbf{P} = \mathbf{Z}_P\mathbf{Z}_P^T$ and $\mathbf{Q} = \mathbf{Z}_Q\mathbf{Z}_Q^T$
3: Calculate the singular value decomposition of $\mathbf{Z}_P^T\mathbf{Z}_Q = \mathbf{U}\mathbf{\Sigma}\mathbf{V}$ with $\Sigma = diag(\sigma_1, ..., \sigma_n)$
4: Compute the truncated part of the balancing transformations $\mathbf{T}_{(r\times n)} = \mathbf{\Sigma}_{(r\times r)}^{-1/2}\mathbf{V}_{(r\times n)}\mathbf{Z}_Q^T$ and $\mathbf{T}_{(n\times r)}^{-1} = \mathbf{Z}_P\mathbf{U}_{(n\times r)}\mathbf{\Sigma}_{(r\times r)}^{-1/2}$
5: Obtain the reduced matrices as $\tilde{\mathbf{C}} = \mathbf{T}_{(r\times N)}\mathbf{C}\mathbf{T}_{(N\times r)}^{-1}$, $\tilde{\mathbf{G}} = \mathbf{T}_{(r\times n)}\mathbf{G}\mathbf{T}_{(n\times r)}^{-1}$, $\tilde{\mathbf{B}} = \mathbf{T}_{(r\times n)}\mathbf{B}$, $\tilde{\mathbf{L}} = \mathbf{L}\mathbf{T}_{(n\times r)}^{-1}$

Although BT offers great accuracy, as it provides an a-priori error that depends on the reduced order, this method has a noteworthy drawback - the large computational and storage demands. The first three steps of Alg. 1 are very expensive tasks of complexity $O(n^3)$ and need the storage of the dense Gramian matrices, even if the system matrices are sparse.

However, in most cases, the number of sound sources and the output points where we want to monitor the sound density are much smaller than the states, i.e., $p, q \ll n$. Consequently, the products $\mathbf{B}\mathbf{B}^T$ and $\mathbf{L}^T\mathbf{L}$ have a much smaller rank than the original system. This is also the case for the corresponding Gramians [18], which can be approximated by low-rank products $\mathbf{P} \approx \mathbf{Z}_P\mathbf{Z}_P^T$ and $\mathbf{Q} \approx \mathbf{Z}_Q\mathbf{Z}_Q^T$ with $\mathbf{Z}_P, \mathbf{Z}_Q \in \mathcal{R}^{n\times k}$ $(k \ll n)$, instead of the full Cholesky factorizations. This reduces the complexity and the storage demands of the SVD in step 3 of Alg. 1 to size k.

For the solution of the Lyapunov equations in low-rank format, Krylov subspace methods are an efficient and straightforward solution. In particular, the extended Krylov subspace (EKS) method [11] employs both the standard Krylov subspace as well as the inverted Krylov subspace, which radically aids in the system's convergence. Therefore, we propose the employment of the EKS method in order to solve the Lyapunov equations in low-rank format and thus, render this method suitable for large-scale room acoustics simulation systems.

A. Low-Rank BT using EKS

Krylov subspace methods are a powerful tool to project the system onto a subspace of lower dimension, then solve

TABLE I: Comparison between the reduced model and the original system at $t = 10$s

Bench-mark	Room Size (m)	Discr. Step (m)	#Sound Sources	Original Model		**Reduced Model**				Reduction Percentage (%)	Speedup (×)	Rel. Error (%)
				Order	Exec. Time(s)	Order	Reduction Time (s)	Exec. Time (s)	Time-step (s)			
room1	$8 \times 8 \times 4$	1	1	405	0.033	5	0.029	0.003	0.1	98.765	11.00	0.001
room2	$8 \times 8 \times 4$	0.4	2	4851	1.437	9	0.451	0.027	0.2	99.814	53.22	0.027
room3	$16 \times 8 \times 2$	0.2	5	36531	5.944	14	1.863	0.312	0.4	99.962	19.05	0.043
room4	$10 \times 10 \times 10$	0.1	10	704969	105.382	37	13.820	1.836	0.4	99.995	57.40	0.205
room5	$12 \times 12 \times 12$	0.1	20	1367631	474.385	58	53.496	7.927	0.6	99.996	59.84	0.755

the small-scale Lyapunov equations and directly obtain the low-rank approximations of the Gramians. If we set $\mathbf{A}_E \equiv \mathbf{G}, \mathbf{B}_E \equiv \mathbf{B}$ and denote $\mathbf{K} \in \mathcal{R}^{n \times k}$ as a projection matrix that corresponds to the k-dimensional Krylov subspace $\mathcal{K}_k(\mathbf{A}_E, \mathbf{B}_E)$, then the low-rank solution $\mathbf{X} \in \mathbb{R}^{n \times k}$ of the controllability Gramian $\mathbf{P}$ can be obtained by solving the small-scale Lyapunov equation:

$$(\mathbf{K}^T \mathbf{A}_E \mathbf{K})\mathbf{X} + \mathbf{X}(\mathbf{K}^T \mathbf{A}_E \mathbf{K})^T = -\mathbf{K}^T \mathbf{B}_E \mathbf{B}_E^T \mathbf{K} \quad (17)$$

Then, $\mathbf{X}$ can be back-projected onto the original n-dimensional space in order to provide a low-rank factor $\mathbf{Z} \in \mathbb{R}^{\mathbf{n} \times \mathbf{k}}$ of $\mathbf{P}$ as $\mathbf{Z} = \mathbf{KS}$, where $\mathbf{S}$ results from the Cholesky factorization of $\mathbf{X}$. Similarly for the observability Gramian $\mathbf{Q}$, having $\mathbf{A}_E^T, \mathbf{L}^T$ in place of $\mathbf{A}_E$ and $\mathbf{B}_E$ respectively, we obtain $\mathbf{Y}$, the low-rank approximation of $\mathbf{Q}$.

The subspace selection has a fundamental effect on the quality of the reduction, so, numerous iterations are required in order for this method to converge. This issue can be resolved by the EKS method [19], which enriches the standard Krylov subspace $\mathcal{K}_k(\mathbf{A}, \mathbf{B})$ with the inverted Krylov subspace $\mathcal{K}_k(\mathbf{A}^{-1}, \mathbf{B})$, producing the EKS:

$$\mathcal{K}_k^E(\mathbf{A}_E, \mathbf{B}_E) = \mathcal{K}_k(\mathbf{A}_E, \mathbf{B}_E) + \mathcal{K}_k(\mathbf{A}_E^{-1}, \mathbf{B}_E) = span\{\mathbf{B}_E, \mathbf{A}_E^{-1}\mathbf{B}_E, \mathbf{A}_E\mathbf{B}_E, \mathbf{A}_E^{-2}\mathbf{B}_E, \mathbf{A}_E^2\mathbf{B}_E, \ldots, \mathbf{A}_E^{-(k-1)}\mathbf{B}_E, \mathbf{A}_E^{k-1}\mathbf{B}_E\}$$

and initializing it with the pair $\{\mathbf{B}, \mathbf{A}_E^{-1}\mathbf{B}\}$. The projection matrix $\mathbf{K}$ is constructed iteratively, and the small-scale solutions are solved until a sufficient level of accuracy is achieved.

V. Experimental Results

To experimentally assess the accuracy and scalability of the proposed methodology, we conducted room acoustic analyses on five volumes of varying sizes with an average sound absorption coefficient of $a = 1/6$ and diverse sound source densities, as seen in the first four columns of Table I. *Room Size* describes the three-dimensional size of each benchmark, *Discr. Size* depicts the spatial discretization step of the room of the FDM method, while *#Sound Sources* corresponds to the number of sound sources for each analysis. Finally, all experiments were executed on a Windows machine with 32GB memory and a 4.7GHz Intel Core i9 processor with 16 threads while the proposed methodology was implemented in Python 3.9 using the numerical packages from the Scipy library.

We performed sound density analysis on both the original and the proposed reduced model at $t = 10s$ and compared the

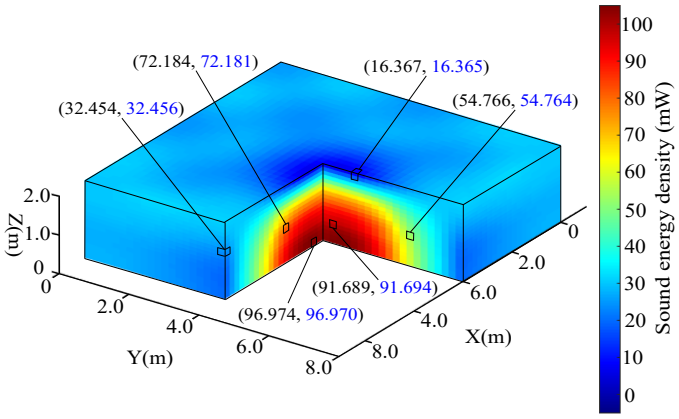

Fig. 1: Comparison of the sound energy density for *room1* at $t = 10$s. The sound energy value at each cubic node is shown as a tuple, with the reduced model solution in black and the original system solution in blue text.

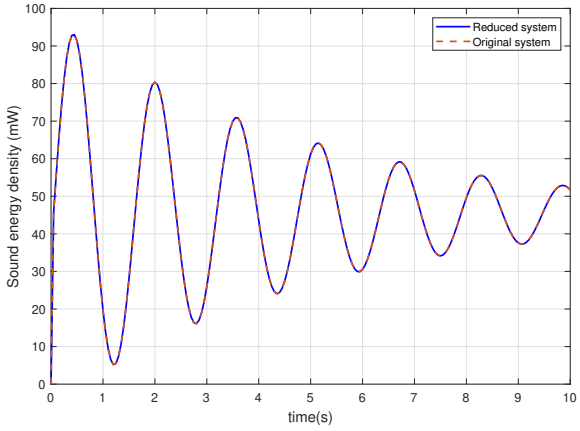

Fig. 2: Transient sound energy response for an internal node at room1 in the range $[0, 10]$ seconds.

accuracy and the execution time of both transient analyses. The results of our experiments are illustrated in the remaining columns of Table I, where the fifth and sixth columns describe the original system, i.e., its order and transient analysis execution time. The next four columns refer to the proposed reduced model. In particular, *Order* is the resulting reduced order of the model and *Reduction time* is the time needed for the reduction procedure, *Exec. time* refers to the total time of the transient analysis while *Timestep Time* is the execution time of each

timestep. Finally, the last three columns correspond to the percentage of the order reduction between the two models, the execution speedup achieved by our approach in comparison to the original system and the relative error between the two methods. As can be observed, our method can achieve extreme reduction percentages of 99.996% with a very low relative error of less than 0.8% while achieving a speedup of the transient analysis of $60\times$.

To further illustrate the reliability of the proposed method, we provide graphical comparisons of the two methods. In Fig. 1, we provide the spatial distribution of sound density for *room1* benchmark at $t = 10$s. As can be easily seen, in almost all monitoring points, our method offers very reliable accuracy. Furthermore, in Fig. 2, we provide a comparison between the transient analyses of the original and the reduced model on an internal node at $t = 10$s. As expected, the two responses exhibit an exceedingly high degree of concordance, as they are almost indistinguishable from each other. The above results demonstrate that system theoretic techniques like BT can achieve very high reduction percentages, of about 99%, and thus can lead to very compact reduced-order models fro the efficient capture of the sound density effects in room acoustic models.

VI. Conclusions

In this paper, a system theoretic methodology for efficiently simulating sound density in very large room acoustic models was introduced. The proposed approach capitalizes on the fact that sound density measurements are often only necessary at specific locations, and an efficient BT method to reduce the system's order and speed up the transient analysis. According to experimental results, the reduced model produced by this method can achieve an order reduction percentage of about 99.9% resulting in a speedup of up to $59\times$ with respect to the original system, with a relative error that is below 0.75%.

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