Overdetermined systems of polynomial equations: tensor-based solution and application

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***Abstract*—Overdetermined systems of polynomial equations are the natural extension of overdetermined systems of linear equations. While the latter are solved systematically through well-established numerical linear algebra techniques, we contribute to the development of tensor-based tools to handle the more general polynomial case, specifically for applications in signal processing and related areas. The method involves computing the nullspace of the so-called Macaulay matrix and determining the Canonical Polyadic Decomposition (CPD) of a third-order tensor. The practical utility of this method is demonstrated for a blind multi-source localization problem.**

***Index Terms*—Macaulay matrix, polynomial system, root solving, tensor**

I. Introduction

Solving systems of linear equations is an essential practice in many scientific disciplines. A natural extension of linear equations is polynomial equations, which unsurprisingly are also ubiquitous. Multivariate polynomials form an essential modelling tool in computational biology, chemistry, robotics, optimization, economics, and statistics (see, e.g., [1]–[6]).

Methods for solving sets of (multivariate) polynomials have been studied extensively in the field of computational algebraic geometry [7], [8]. These methods are roughly divided into two categories: (i) homotopy continuation methods, which retrieve the solution of a desired system by continuous deformation of a starting system with known roots (see, e.g., [9], [10]), and, (ii) algebraic methods, which reduce the root-solving problem to an eigenvalue problem by either symbolic (e.g., Gröbner basis) and/or numerical means (see, e.g., [11]–[14])

So far, the literature has primarily focused on the noiseless square case, where the coefficients of the polynomials are known with full precision and the number of equations equals the number of unknowns. On the other hand, a critical component of engineering applications is the estimation of system parameters from overcomplete sets of noisy equations. For the linear case, numerical linear algebra provides effective methods and well-established theory [15]. Analogous methods to treat the more general polynomial case are, however, far fewer [16], and relatively underdeveloped. Nonetheless, the transition from systems of linear to polynomial equations fits well in the larger evolution from informed (matrix-based) to blind (tensor-based) signal separation [17].

This paper contributes to the development of tensor-based tools to solve overdetermined systems of noisy polynomial equations. We show that, while overdetermined linear systems are effectively handled using matrix techniques, their polynomial counterparts can be solved using tensor techniques. The method involves computing the nullspace of the so-called Macaulay matrix from resultant theory [18] and exploits the shift-invariant structure in the nullspace to formulate the root recovery as a multidimensional harmonic retrieval (MHR) problem. Critical to mitigating noise is expressing the MHR problem as a canonical polyadic decomposition (CPD) of a third-order tensor. This makes the estimation more robust by taking into account all slices of the tensor at once.

The paper builds on the foundations laid in our earlier work [19], [20] but capitalizes on sets of equations that are overdetermined and solved in a total-least-squares sense. Numerical experiments are conducted to highlight the relevant properties. Furthermore, the method is applied to an interesting case study in which the positions of two transmitters are estimated from the power received at an arbitrary configuration of antennas in their vicinity. Apart from its technical value, this paper is meant to have some tutorial value for researchers in signal processing who are new to sets of polynomial equations.

The remainder of this paper is outlined as follows. Section II introduces systems of polynomial equations with a focus on the differences with linear systems. The Macaulay tensor-based method is explained in Section III. The numerical experiments (including the case study) are covered in Section IV. Conclusions are provided in Section V.

Notation

We write scalars, vectors, matrices and tensors as lower-case, bold lower-case, bold capital and calligraphic letters respectively, i.e., a, $\mathbf{a}$, $\mathbf{A}$, $\mathcal{A}$ respectively. Our N unknown variables are denoted as w, x, y and z or $x_1, x_2, \ldots, x_N$. The homogenization variable is denoted t (see Section II-B). The polynomial coefficients are always denoted c. We write the third-order canonical polyadic decomposition (CPD) of tensor

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$\mathcal{A} = [\![\mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_3]\!] = \sum_{r=1}^{R} \mathbf{U}_1(:,r) \otimes \mathbf{U}_2(:,r) \otimes \mathbf{U}_3(:,r)$ with $\otimes$ symbolizing the outer product. The binomial coefficient is denoted $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

II. Systems of Polynomial Equations

In this section, we introduce some relevant notions on systems of polynomial equations. Section II-A and Section II-B introduce general definitions and homogenized systems, respectively. The latter is required to characterize the number of roots of a square system, explained in Section II-C.

A. Polynomial equations

A *monomial* in N variables is written as $x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_N^{\alpha_N}$ or in vectorized notation $\mathbf{x}^{\boldsymbol{\alpha}}$ with $\mathbf{x} = \begin{bmatrix} x_1 & \ldots & x_N \end{bmatrix}^\top$ the unknown variables and $\boldsymbol{\alpha} = \begin{bmatrix} \alpha_1 & \ldots & \alpha_N \end{bmatrix}^\top$ the nonnegative integer powers. Its degree d refers to the *total degree*, i.e. $d = \sum_{n=1}^{N} \alpha_n$. Throughout this paper, monomials are ordered from smallest to largest degree, and, in case of a tie, sorted lexicographically on $\boldsymbol{\alpha}$. A Vandermonde vector $\mathbf{v}_d$ stores all monomials up to degree d in the aforementioned ordering.

A *polynomial* $p_d(\mathbf{x})$ is a linear combination of monomials with coefficients $c_{\boldsymbol{\alpha}}$. Its degree is defined as the largest degree of any of its monomials with nonzero coefficients.

Σ_d denotes a *system* of S polynomials of degree (at most) d. This will usually be written as $\mathbf{C}_d \mathbf{v}_d = \mathbf{0}$, where the $(s,j)^{\text{th}}$ entry of matrix $\mathbf{C}_d$ contains the coefficient associated with the j^{th} monomial in $\mathbf{v}_d$ of the s^{th} polynomial in the system. A linear system, classically $\mathbf{A}\mathbf{x} = \mathbf{b}$, is thus denoted as $\Sigma_1 : \mathbf{C}_1 \mathbf{v}_1 = \mathbf{0}$ instead, where $\mathbf{C}_1 = \begin{bmatrix} -\mathbf{b} & \mathbf{A} \end{bmatrix}$ and $\mathbf{v}_1 = \begin{bmatrix} 1 & \mathbf{x}^\top \end{bmatrix}^\top$. A quadratic system is denoted as $\Sigma_2 : \mathbf{C}_2 \mathbf{v}_2 = \mathbf{0}$ with $\mathbf{v}_2 = \begin{bmatrix} \mathbf{v}_1^\top & x_1^2 & \ldots & x_1 x_N & x_2^2 & \ldots & x_2 x_N & \ldots & x_N^2 \end{bmatrix}^\top$.

B. Homogenization and roots at infinity

The method to be discussed in Section III computes the roots of homogeneous systems, i.e., where all monomials have the exact same degree. Therefore, to solve a system, its equations first need to be homogenized.

The homogenization is done by adding a new variable t and multiplying all monomials by a power of t such that all monomials have the same degree. This yields the polynomial $p_{h,d}(t, x_1, \ldots, x_N) = t^d p_d(\frac{x_1}{t}, \ldots, \frac{x_N}{t})$. Similarly, the homogenized Vandermonde vector is denoted $\mathbf{v}_{h,d}$.

Any root $(t, x_1, \ldots, x_N)$ of the homogenized system Σ_d^h is scale invariant, i.e., $(\beta t, \beta x_1, \ldots, \beta x_N)$ for any β is also a root. To counteract this, we normalize the roots through scaling such that $t = 1$, which yields the affine roots. Some roots of the homogenized system cannot be normalized this way as $t = 0$; these are so-called roots at infinity.

The affine roots correspond with the original roots of systems, but the roots at infinity are "artificial roots" that are introduced by the homogenization. The system $\Sigma_1 : \{x + y = 1; \;\; x + y = 2\}$ for example searches for the crossing of two parallel lines. It has no affine roots, but its homogenized system has a root at infinity instead, namely $(t, x, y) = (0, 1, -1)$. This can be interpreted geometrically as lines crossing at infinity in direction (1,-1).

C. Number of solutions: linear vs. polynomial systems

Let us assume systems have a finite number of roots throughout this section.

Square systems of linear equations will always have exactly $L = 1$ root, including roots at infinity. If the system is overdetermined (more equations than variables) it will have at most one root. For bivariate linear systems, the two equations correspond to straight lines and will intersect at exactly one point, unless the lines are parallel, implying a root at infinity.

Bézout's theorem indicates that a square system of polynomial equations (with a finite number of roots) will have exactly $L = \prod_{n=1}^{N} d_n$ (potentially complex) roots with N the number of variables and d_n the degree of polynomial n, counting multiplicity and roots at infinity (see, e.g., [21, Theorem 7.7]). In engineering applications, there are often only a few relevant solutions (for example equilibria), but many more equations than in a square system. The overdeterminedness may, besides mitigating noise, rule out some candidate solutions that are physically meaningless, allowing $L < \prod_{n=1}^{N} d_n$. (In the use-case of Section IV-B for instance, adding a fifth equation reduces the number of affine roots from 24 to 2, the expected number of roots.) Figure 1 shows an example of such an overdetermined system ($S = 3$, $N = 2$, $d = 2$) with only one root, while any 2 equations always yield $d^N = 4$ roots.

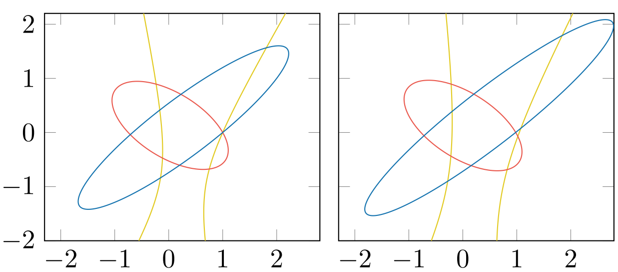

Fig. 1. Zero lines of the polynomials of $\Sigma_2^{(a)}$ in Eq. (1), see Section III-B. Left without noise, right with a signal-to-noise ratio (SNR) on the coefficients of 20dB. The noise makes the root only approximate.

III. Macaulay Tensor-Based Root Solving

We introduce a methodology analogous to total least squares (TLS) for linear systems that can be utilized to solve polynomial systems with a finite number of roots. In the Macaulay tensor-based method, the coefficient matrix $\mathbf{C}_{d_\Sigma}$ is extended to a Macaulay matrix $\mathbf{M}_{d_\mathbf{M}}$ (see Section III-B) of sufficient size (see Section III-C). From its nullspace, the roots are then extracted by solving an MHR problem (see Section III-D) through a CPD (see Section III-E). If needed, the roots at infinity can be removed before the MHR (see Section III-F). The steps are summarized in Algorithm 1.

A. Generalizing the total least squares method

TLS solves a linear system $\Sigma_1 : \mathbf{C}_1 \mathbf{v}_1 = \mathbf{0}$ with a vector in the nullspace of $\mathbf{C}_1$ [22]. Similarly, we would like to solve a polynomial system $\Sigma_{d_\Sigma} : \mathbf{C}_{d_\Sigma} \mathbf{v}_{d_\Sigma} = \mathbf{0}$ via the nullspace of $\mathbf{C}_{d_\Sigma}$. The nullspace estimation linearizes the problem; it forgoes the monomial structure in $\mathbf{v}_{d_\Sigma}$. To ensure that this nullspace is spanned by the Vandermonde vectors generated by the roots, i.e., to avoid extra vectors in the nullspace, we need to add more equations, which is why the matrix $\mathbf{C}_{d_\Sigma}$ is extended to a Macaulay matrix $\mathbf{M}_{d_\mathbf{M}}$ of sufficient size.

B. The Macaulay matrix

The Macaulay matrix $\mathbf{M}_{d_\mathbf{M}}$ of degree $d_\mathbf{M}$ of a system Σ_{d_Σ} of degree d_Σ of S polynomials $p_s(\mathbf{x})$, for $s = 1, \ldots, S$, has in its rows the coefficients of all polynomials $\mathbf{x}^{\boldsymbol{\alpha}} p_s$ of at most degree $d_\mathbf{M}$ [18]. The coefficients in column i correspond to the coefficients of the i^{th} monomial of $\mathbf{v}_{d_\mathbf{M}}$. The roots of the system Σ_{d_Σ} satisfy $\mathbf{M}_{d_\mathbf{M}} \mathbf{v}_{d_\mathbf{M}} = \mathbf{0}$. Notice that $\mathbf{C}_{d_\Sigma}$ equals $\mathbf{M}_{d_\Sigma}$ and that $\mathbf{M}_d$ is a subblock of $\mathbf{M}_{d+1}$.

As an example let us look at the system shown on Fig. 1,

$$\Sigma_2^{(a)} : \begin{cases} -3- \ x-2y+4x^2+6xy+7y^2 = 0 \\ -2- \ x+ \ y+3x^2-7xy+5y^2 = 0 \\ \ 1+7x+ \ y-8x^2+3xy+ \ y^2 = 0 \end{cases} . \tag{1}$$

For $d_\mathbf{M} = 3$ (and not for $d_\mathbf{M} = 2$), we find that $\mathbf{v}_3$ evaluated at the only solution ($\mathbf{x}^{(1)} = \begin{bmatrix} 1 & 0 \end{bmatrix}^\top$) is the only vector $\mathbf{z}$ (up to scaling) satisfying $\mathbf{M}_3^{(a)} \mathbf{z} = \mathbf{0}$ with $\mathbf{v}_3 = \left[\begin{array}{c|cc|ccc|cccc} 1 & x & y & x^2 & xy & y^2 & x^3 & x^2y & xy^2 & y^3 \end{array}\right]^\top$ and

$$\mathbf{M}_3^{(a)} = \left[\begin{array}{c|cc|ccc|cccc} -3 & -1 & -2 & 4 & 6 & 7 & 0 & 0 & 0 & 0 \\ -2 & -1 & 1 & 3 & -7 & 5 & 0 & 0 & 0 & 0 \\ 1 & 7 & 1 & -8 & 3 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & -3 & 0 & -1 & -2 & 0 & 4 & 6 & 7 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 & 3 & -7 & 5 & 0 \\ 0 & 1 & 0 & 7 & 1 & 0 & -8 & 3 & 1 & 0 \\ \hline 0 & 0 & -3 & 0 & -1 & -2 & 0 & 4 & 6 & 7 \\ 0 & 0 & -2 & 0 & -1 & 1 & 0 & 3 & -7 & 5 \\ 0 & 0 & 1 & 0 & 7 & 1 & 0 & -8 & 3 & 1 \end{array}\right].$$

C. Degree of regularity

Starting at some degree $d^*_\mathbf{M}$, the nullity of the Macaulay matrix stabilizes if the associated system has a finite number of roots. This $d^*_\mathbf{M}$ is the degree of regularity, the lowest degree for which the nullspace is spanned by only vectors corresponding with roots. An upper bound on the degree of regularity $d^*_\mathbf{M}$ for a square system with $S = N$ polynomials of degree $d_1, \ldots, d_N$ respectively is given by $d^*_\mathbf{M} \leq \sum_{n=1}^{N} d_n - N$ [7, Section 3.4]. For overdetermined systems ($S > N$), the degree of regularity will typically be lower than that of a square system with N of its S equations, but may exceptionally be slightly higher, even above the bound (see, e.g., system $\Sigma_2^{(a)}$ from Eq. (1)).

D. Recovery of roots from the nullspace

For simplicity of exposition, assume that all roots of Σ_d are distinct. For the more general case, see [20].

Once the Macaulay matrix is formed of degree $d_\mathbf{M} \geq d^*_\mathbf{M}$, the roots can be extracted from its nullspace $\mathbf{N}_{d_\mathbf{M}}$. This amounts to an MHR problem [23], which we will write as

$$\mathbf{N}_{d_\mathbf{M}} = \mathbf{V}_{h,d_\mathbf{M}} \mathbf{G} \tag{2}$$

with $\mathbf{V}_{h,d_\mathbf{M}}[:, l] = \mathbf{v}^{(l)}_{h,d_\mathbf{M}}$ (the homogenized Vandermonde vector $\mathbf{v}_{h,d_\mathbf{M}}$ evaluated in the l^{th} root) and $\mathbf{G}$ an a priori unknown square invertible mixing matrix.

One way of solving this, exploiting the structure in $\mathbf{V}_{d_\mathbf{M}}$, is to require

$$\mathbf{S}_t \mathbf{V}_{h,d_\mathbf{M}} = \mathbf{V}_{h,d_\mathbf{M}-1} \mathbf{D}_t, \tag{3}$$

$$\mathbf{S}_{x_n} \mathbf{V}_{h,d_\mathbf{M}} = \mathbf{V}_{h,d_\mathbf{M}-1} \mathbf{D}_{x_n}, \tag{4}$$

for $n = 1, \ldots, N$,where $\mathbf{S}_{x_n}$ ($\mathbf{S}_t$) selects all rows of $\mathbf{V}_{h,d_\mathbf{M}}$ corresponding with monomials where x_n (t) is raised to the power one or higher and $\mathbf{D}_{x_n}$ ($\mathbf{D}_t$) is a diagonal matrix with $\begin{bmatrix} {x_n}^{(1)} & \ldots & {x_n}^{(L)} \end{bmatrix}$ (the n^{th} coordinate of each of the L solutions) on its diagonal. For a quadratic bivariate system with $L = 4$ roots and $d_\mathbf{M} = 2$, Eq. (4) for x becomes

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}^\top \begin{bmatrix} \left(t^{(1)}\right)^2 & \left(t^{(2)}\right)^2 & \left(t^{(3)}\right)^2 & \left(t^{(4)}\right)^2 \\ t^{(1)}x^{(1)} & t^{(2)}x^{(2)} & t^{(3)}x^{(3)} & t^{(4)}x^{(4)} \\ t^{(1)}y^{(1)} & t^{(2)}y^{(2)} & t^{(3)}y^{(3)} & t^{(4)}y^{(4)} \\ \left(x^{(1)}\right)^2 & \left(x^{(2)}\right)^2 & \left(x^{(3)}\right)^2 & \left(x^{(4)}\right)^2 \\ x^{(1)}y^{(1)} & x^{(2)}y^{(2)} & x^{(3)}y^{(3)} & x^{(4)}y^{(4)} \\ \left(y^{(1)}\right)^2 & \left(y^{(2)}\right)^2 & \left(y^{(3)}\right)^2 & \left(y^{(4)}\right)^2 \end{bmatrix} =$$

$$\begin{bmatrix} t^{(1)} & t^{(2)} & t^{(3)} & t^{(4)} \\ x^{(1)} & x^{(2)} & x^{(3)} & x^{(4)} \\ y^{(1)} & y^{(2)} & y^{(3)} & y^{(4)} \end{bmatrix} \begin{bmatrix} x^{(1)} & 0 & 0 & 0 \\ 0 & x^{(2)} & 0 & 0 \\ 0 & 0 & x^{(3)} & 0 \\ 0 & 0 & 0 & x^{(4)} \end{bmatrix}. \tag{5}$$

If we now combine Eq. (2) with Eqs. (3) and (4), and name $\mathbf{C}_{x_n} = \mathbf{S}_{x_n} \mathbf{N}_{d_\mathbf{M}}$, we obtain

$$\mathbf{C}_t = \mathbf{V}_{h,d_\mathbf{M}-1} \mathbf{D}_t \mathbf{G}, \tag{6}$$

$$\mathbf{C}_{x_n} = \mathbf{V}_{h,d_\mathbf{M}-1} \mathbf{D}_{x_n} \mathbf{G}, \tag{7}$$

with only matrices $\mathbf{C}_t, \mathbf{C}_{x_1}, \ldots, \mathbf{C}_{x_N}$ known. This is a joint diagonalization problem. In tensor terms, we search for a CPD of tensor $\mathcal{C}$ with slices $\mathbf{C}_t, \mathbf{C}_{x_1}, \ldots, \mathbf{C}_{x_N}$ [19], namely

$$\mathcal{C} = [\![\mathbf{V}_{h,d_\mathbf{M}-1}, \mathbf{G}, \mathbf{X}]\!] \ \text{ with } \mathbf{X} = \begin{bmatrix} t^{(1)} & \ldots & t^{(L)} \\ x_1^{(1)} & \ldots & x_1^{(L)} \\ \vdots & \ddots & \vdots \\ x_N^{(1)} & \ldots & x_N^{(L)} \end{bmatrix}. \tag{8}$$

The rank of the CPD is thus equal to the number of roots. Except for a few rare cases, this CPD is expected to be essentially unique (see, e.g., [19] for technical details).

E. Computing the CPD

From an algebraic perspective, two slices of tensor $\mathcal{C}$ suffice, allowing one to solve the CPD problem with a generalized eigenvalue decomposition (GEVD). However, taking into account the whole tensor generally yields better numerical results, especially in the presence of noise. To this end, the GEVD can be replaced by a generalized eigenspace decomposition (GESD) [24], which takes into account more slices when eigenvalues are not well separated. Using optimization in a second step, more specifically for instance non-linear least squares (NLS) [25], can significantly further improve these results. Both of these improvements come at a computational cost, which is often negligible compared to the cost of computing the nullspace of the Macaulay matrix.

F. Removing roots at infinity

Roots at infinity are an artefact of the homogenisation process and are irrelevant in many signal processing applications. There can even be an infinite amount of roots at infinity, prohibiting any root extraction, for instance in the use case of Section IV-B. Notice that the homogenized Vandermonde vector $\mathbf{v}_{h,d_{\mathbf{M}}}$ corresponding to a root at infinity of multiplicity ξ will have nonzero elements only in its entries corresponding with monomials of degree at least $d_{\mathbf{M}-\xi+1}$. By computing the nullspace $\mathbf{N}_{d_{\mathbf{M}}}$ of degree $d_{\mathbf{M}}$, and keeping the rows corresponding with degree $d_{\not\infty} = d_{\mathbf{M}} - \xi$ at most, we will be left with a matrix spanned only by $\mathbf{v}_{d_{\not\infty}}$ corresponding with the affine roots [13].

Algorithm 1 Macaulay-based solution of polynomial systems.

Input a system Σ_d which has a finite number of roots.
Output matrix $\mathbf{X}$ which contains the roots of Σ_{d_Σ}.

1: Create the Macaulay matrix $\mathbf{M}_{d_{\mathbf{M}}}$ associated with Σ_{d_Σ} of sufficiently large degree $d_{\mathbf{M}}$.
2: Compute $\mathbf{N}_{d_{\mathbf{M}}} = \text{null}(\mathbf{M}_{d_{\mathbf{M}}})$.
3: Optional: remove roots at infinity.
4: Form $\mathcal{C}$ by determining and stacking $\mathbf{C}_t, \mathbf{C}_{x_1}, \ldots, \mathbf{C}_{x_N}$.
5: Compute the CPD of $\mathcal{C} = [\![\mathbf{V}_{d_{\mathbf{M}}-1}, \mathbf{G}, \mathbf{X}]\!]$.

G. Computational complexity

The size of the Macaulay matrix is $R \times C$ with $R = S\binom{N+d_{\mathbf{M}}-d_{\mathbf{\Sigma}}-1}{N-1}$ and $C = \binom{N+d_{\mathbf{M}}-1}{N-1}$. Computing the null space in step 2 of Algorithm 1 takes $O(RC^2)$ flops with a standard SVD solver. Computing the CPD (with any method) in step 5 is preceded by a standard orthogonal compression to a $L \times L \times (N+1)$ tensor, requiring $O(N^2L^2C)$ flops, which is the largest cost of this step (if the system has few roots, i.e., L is small). The other steps of Algorithm 1 are negligible in comparison, making step 2 typically the most expensive. The complexity of step 2 could be further improved with better adapted SVD solvers and taking advantage of the structure (see, e.g., [26] for promising initial results).

IV. Experiments

In this section, the effectiveness of the Macaulay tensor-based method is shown, especially its leveraging of overdeterminedness to mitigate noise. First, Section IV-A shows a synthetic problem to get a grasp on these problems and compare solution methods and then Section IV-B expands to a practical use case: multi-source localization in the near field.

A. Synthetic problem

To test the method, square bivariate cubic systems are generated with random standard normal complex coefficients. The systems are expanded to S equations by adding random standard normal complex linear combinations of the first two polynomials. Scaled random normal noise is added to obtain a fixed signal-to-noise ratio (SNR). The error is measured as the Frobenius norm between the exact and obtained roots. The exact roots are obtained by solving the noiseless problem. Results are shown in Fig. 2.

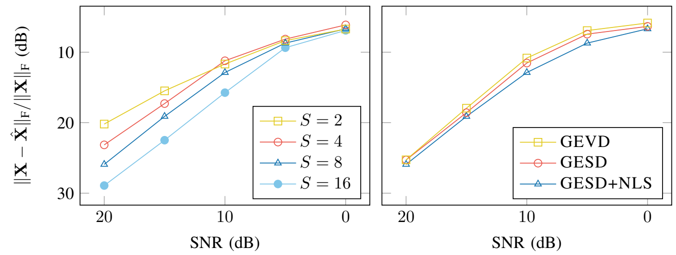

Fig. 2. On the left, we see that more equations yield lower errors, similar to linear systems. On the right, we see that each methodological improvement suggested in Section III-E results in a lower error. On the left side, GESD+NLS was used as a method, and on the right side $S = 8$. Data points are the median normalized Frobenius error on roots over 200 randomly generated problems for different numbers of equations S (left) and different methods (right) at different noise levels (given as SNR in dB).

B. Case study: multi-source localization in the near-field

Two users are transmitting signals to S surrounding antennas as shown in Fig. 3. Using the Friis transmission equation,

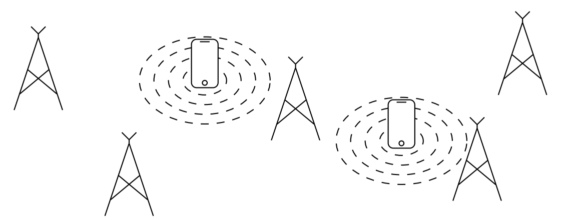

Fig. 3. Graphical depiction of the problem in Section IV-B. We see two source signals retrieved by $S = 5$ antennas in the near field.

the power measured at the antennas can be characterized as

$$P_{r,i} = \frac{A_{r,i}A_{t,1}}{R^2_{r,i;t,1}\lambda^2}P_{t,1} + \frac{A_{r,i}A_{t,2}}{R^2_{r,i;t,2}\lambda^2}P_{t,2}, \tag{9}$$

where $P_{r,i}$ is the power received by antenna i, $P_{t,j}$ is the power transmitted by user j, $A_{r,i}$ and $A_{t,j}$ are the effective aperture area of the antennas of receiver i and transmitter j, respectively, λ is the propagation wavelength and $R_{r,i;t,j}$ is the distance between receiver i and transmitter j. Assume that all transmission is isotropic, i.e., $A_{r,i} = A_{t,j} = 1$ for all i and j and that $\frac{P_{t,1}}{\lambda^2} = \frac{P_{t,2}}{\lambda^2} = b$, a known constant. If these are not identical, the problem becomes easier as the asymmetry ensures that there is only one solution. This yields the equation

$$P_{r,i} = \frac{b}{R^2_{r,i;t,1}} + \frac{b}{R^2_{r,i;t,2}}. \tag{10}$$

Assuming this problem is in 2D, we write the distance based on the x and y coordinate of the transmitters and receivers as $R^2_{r,i;t,j} = (x_{r,i} - x_{t,j})^2 + (y_{r,i} - y_{t,j})^2$. If $x_{r,i}$, $y_{r,i}$ and $P_{r,i}$ are known for all i, one can determine the position of the users. Each antenna yields a polynomial equation of degree $d_\Sigma = 4$ and in $N = 4$ unknowns ($x_{t,1}$, $y_{t,1}$, $x_{t,2}$ and $y_{t,2}$) of the type

$$R^2_{r,i;t,2}R^2_{r,i;t,1}P_{r,i} = bR^2_{r,i;t,2} + bR^2_{r,i;t,1}. \tag{11}$$

One issue is that there is an infinite number of roots at infinity, namely all points satisfying $(x^2_{t,1}+y^2_{t,1})(x^2_{t,2}+y^2_{t,2}) =$

0 and $t = 0$, irrespective of the number of antennas S. (This equation is found by homogenizing the equations and setting the homogenization variable t to zero.) We thus need optional step 3 of truncating the nullspace, see Section III-F.

The position of both users and all S antennas is generated uniformly random between [0,1]. Noise is then added to the power received at each antenna as $P_{r,i,\text{noisy}} = (1+\epsilon_{r,i})P_{r,i,\text{exact}}$ with $\epsilon_{r,i}$ standard normal noise with standard deviation chosen to obtain the desired SNR. Results are shown in Fig. 4.

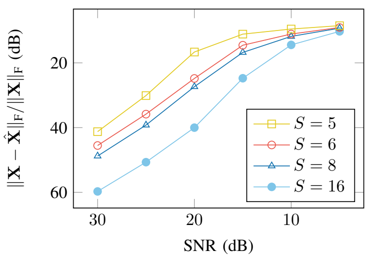

Fig. 4. The position of the users can effectively be estimated from the antennas. More antennas yield better accuracy, as expected. Data points are the normalized median Frobenius error on the coordinates of both users over 200 randomly generated problems for different numbers of antennas S and different levels of noise (given as SNR in dB).

As we have $N = 4$ unknowns, we need at least $S = 4$ equations. From our observations, this leads to 24 affine roots, meaning the positions cannot be determined uniquely. By adding an equation for a fifth receiver antenna, the system becomes overdetermined, having only the two physically-relevant affine roots (due to symmetry as both users are interchangeable). Adding even more equations (or thus antennas) increases the accuracy in the presence of noise.

The hyperparameters were determined from a few tests. The Macaulay matrix $\mathbf{M}$ was formed of degree $d_{\mathbf{M}} = 8$. Afterwards, its nullspace $\mathbf{N}$ was computed with a fixed number of columns, namely 186 (194 for $S = 5$, 190 for $S = 6$, 186 otherwise). It was noticed that the roots at infinity had at most multiplicity $\xi = 4$, meaning that the nullspace had to be truncated to degree $d_{\not\infty} = 4$. After a rank-2 compression, two vectors were used to form the tensor from which the $L = 2$ roots were obtained.

V. Conclusions

The Macaulay tensor-based method forms a powerful tool for the solution of systems of polynomial equations in practical engineering problems. It leverages overdeterminedness to mitigate the impact of noise. A relevant use case was studied in which this method allowed the retrieval of two transmitter locations based solely on the power received at five or more antennas.

Future work will be geared towards the development of more systematic approaches for determining correct hyperparameters, namely the nullity, the degree of regularity and the associated degree with only affine roots. The method's time and memory complexity can still be improved by exploiting the structure of the Macaulay matrix in the nullspace computation; see, e.g., [26] for promising initial results.

References

[1] L. Pachter and B. Sturmfels, *Algebraic statistics for computational biology*. Cambridge university press, 2005, vol. 13.

[2] A. Morgan, *Solving polynomial systems using continuation for engineering and scientific problems*. SIAM, 2009.

[3] I. Z. Emiris, "Sparse elimination and applications in kinematics," Ph.D. dissertation, University of California, Berkeley, 1994.

[4] J. B. Lasserre, "Global optimization with polynomials and the problem of moments," *SIAM J. Optim.*, vol. 11, no. 3, pp. 796–817, 2001.

[5] B. Sturmfels, *Solving systems of polynomial equations*, ser. CBMS Regional Conference Series in Mathematics. AMS, 2002, no. 97.

[6] M. Drton, B. Sturmfels, and S. Sullivant, *Lectures on algebraic statistics*. Springer Science & Business Media, 2008, vol. 39.

[7] D. A. Cox, J. Little, and D. O'Shea, *Using algebraic geometry*, 1st ed., ser. Graduate Texts in Mathematics. Springer, 1998.

[8] ——, *Ideals, varieties, and algorithms: an introduction to computational algebraic geometry and commutative algebra*, 3rd ed., ser. Undergraduate Texts in Mathematics. Springer, 2007.

[9] T. Y. Li, "Numerical solution of multivariate polynomial systems by homotopy continuation methods," *Acta Numer.*, vol. 6, p. 399–436, 1997.

[10] J. Verschelde, P. Verlinden, and R. Cools, "Homotopies exploiting Newton polytopes for solving sparse polynomial systems," *SIAM J. Numer. Anal.*, vol. 31, no. 3, pp. 915–930, 1994.

[11] D. Lazard, "Resolution des systemes d'equations algebriques," *Theoret. Comput. Sci.*, vol. 15, no. 1, pp. 77–110, 1981.

[12] H. J. Stetter, *Numerical polynomial Algebra*. SIAM, 2004.

[13] P. Dreesen, "Back to the roots: Polynomial system solving using linear algebra," Ph.D. dissertation, KU Leuven, 2013.

[14] K. Batselier, "A numerical linear algebra framework for solving problems with multivariate polynomials," Ph.D. dissertation, KU Leuven, 2013.

[15] G. Golub and C. V. Loan, *Matrix Computations*, 4th ed. Johns Hopkins University Press, 2012.

[16] H. J. Stetter, "Matrix eigenproblems are at the heart of polynomial system solving," *SIGSAM Bull.*, vol. 30, no. 4, p. 22–25, 1996.

[17] N. D. Sidiropoulos, L. De Lathauwer, X. Fu, K. Huang, E. E. Papalexakis, and C. Faloutsos, "Tensor decomposition for signal processing and machine learning," *IEEE Trans. Signal Process.*, vol. 65, no. 13, pp. 3551–3582, 2017.

[18] F. S. Macaulay, "Some formulae in elimination," *Proc. Lond. Math. Soc.*, vol. 1, no. 1, pp. 3–27, 1902.

[19] J. Vanderstukken and L. De Lathauwer, "Systems of polynomial equations, higher-order tensor decompositions and multidimensional harmonic retrieval: A unifying framework. Part I: The canonical polyadic decomposition," *SIAM J. Matrix Anal. Appl.*, vol. 42, no. 2, pp. 883–912, 2021.

[20] J. Vanderstukken, P. Kürschner, I. Domanov, and L. De Lathauwer, "Systems of polynomial equations, higher-order tensor decompositions, and multidimensional harmonic retrieval: A unifying framework. Part II: The block term decomposition," *SIAM J. Matrix Anal. Appl.*, vol. 42, no. 2, pp. 913–953, 2021.

[21] R. Hartshorne, *Algebraic geometry*. Springer Science & Business Media, 2013, vol. 52.

[22] S. Van Huffel and J. Vandewalle, *The total least squares problem: computational aspects and analysis*, ser. Frontiers in Applied Mathematics. SIAM, 1991.

[23] M. Haardt and J. Nossek, "Simultaneous Schur decomposition of several nonsymmetric matrices to achieve automatic pairing in multidimensional harmonic retrieval problems," *IEEE Trans. Signal Process.*, vol. 46, no. 1, pp. 161–169, 1998.

[24] E. Evert, M. Vandecappelle, and L. De Lathauwer, "A recursive eigenspace computation for the canonical polyadic decomposition," *SIAM J. Matrix Anal. Appl.*, vol. 43, no. 1, pp. 274–300, 2022.

[25] L. Sorber, M. Van Barel, and L. De Lathauwer, "Optimization-based algorithms for tensor decompositions: Canonical polyadic decomposition, decomposition in rank-$(L_r, L_r, 1)$ terms, and a new generalization," *SIAM J. Optim.*, vol. 23, no. 2, pp. 695–720, 2013.

[26] N. Govindarajan, R. Widdershoven, S. Chandrasekaran, and L. De Lathauwer, "A fast algorithm for computing macaulay nullspaces of bivariate polynomial systems," Technical Report 23-16, ESAT-STADIUS, KU Leuven (Leuven, Belgium), 2023.