Robust M-Type Error-State Kalman Filters for Attitude Estimation

Andrea Bellés
German Aerospace Center
Neustrelitz, Germany

Daniel Medina
German Aerospace Center
Neustrelitz, Germany

Paul Chauchat
IETR, CentraleSupélec
Cesson-Sévigné, France

Samy Labsir
IPSA/TéSA
Toulouse, France

Jordi Vilà-Valls
ISAE-SUPAERO
Toulouse, France

***Abstract*—State estimation techniques appear in a plethora of engineering fields. Both standard Kalman filter (KF) and its nonlinear extensions, as well as particle filters, consider a known system model (i.e., functions and noise statistics), an assumption which may not hold in practice. A problem of particular interest is how to deal with outliers in the observation model. A possible solution is to resort to the framework of robust statistics, where a robust score function is used to mitigate the impact of outlying measurements, leading to robust M-type KFs. In this contribution, some of these robust filtering results are extended to the case where states may live on a manifold (unit norm quaternion), and propose robust iterated error-state M-type KF solutions. An illustrative example is provided to show the performance of the proposed filter and support the discussion.**

I. INTRODUCTION

Attitude estimation refers to finding the relative orientation between two Cartesian frames or, in other words, to the determination of the spatial orientation for a platform. Orientation information is fundamental to aid vehicles with large inertia, such as airplanes or ships, and to operate systems equipped with pointing devices, such as satellites. The attitude problem also arises in computer vision applications, being used to relate keyframes or register point clouds.

Within the context of navigation, recursive attitude estimation is addressed with nonlinear extensions of the original Kalman Filter (KF), such as the Error State KF (ESKF) [1], [2] or the Invariant KF (IKF) [3]–[5]. The aforementioned filters preserve the geometrical properties for the attitude estimates or, conversely, assure that the sequence of estimates remain in the corresponding manifold. The underlying concept relates to estimating a vector of perturbations around the state on the algebra associated with the target manifold. For three-dimensional attitude systems, for instance, that manifold is SO(3), while for the rotation-translation pair is SE(3). When it comes to sensor integration, attitude systems typically include *relative* (i.e., angular rate) and *absolute* (i.e., observation models for the orientation) attitude information. Relevant works on recursive attitude estimation include the use of star-trackers and magnetometers for space applications [6]–[8], or visual sensors and Global Navigation Satellite Systems (GNSS) antenna arrays for vehicular platforms [9]–[11].

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Unfortunately, outlying measurements are often captured by attitude-related sensor modalities. This is related to signal reflection and multipath in GNSS, wrong data association in visual systems or unexpected magnetic fields in magnetometers. Moreover, gyroscope systems may experiment scale issues or misaligments, with the subsequent deterioration of the angular rate integration. Under the presence of contaminated data, the performance of conventional attitude filters rapidly degrades and the estimates become unreliable. On the positive side, a number of resilient alternatives exist which may be categorized in: i) robust statistics-based filters, which are *nearly-optimal* under nominal Gaussian conditions, while outlier insensitive under *normal mixture* noise distributions [12]–[14]; ii) variational inference-based filters, designed for state estimation under heavy-tailed parametric distributions [15]–[17]; iii) fault detection and exclusion algorithms, which identify the faulty measurements based on statistical tests [18], [19]; and iv) recent contributions on linearly constrained filtering also provide a new paradigm for protecting against model mismatch on prediction and observation models [20], [21], whose extension on manifold-based estimation is also available [22].

With the exception of a few examples [23]–[25], the topic of robust filtering has not been applied to the attitude topic. More specifically, a connection between robust estimation and the most widespread attitude filters (i.e., the ESKF and IKF) has not been drawn. In this work, we lay the focus on robust statistics-based filters, which make use of a robust score function to mitigate the impact of outlying measurements. This leads to a family of so-called (linear or nonlinear) M-type KFs. In this contribution, we propose two new nonlinear robust iterated ESKFs for states living on a manifold (unit norm quaternion), which overcome some of the limitations of existing solutions. An illustrative example is provided to show the performance gain with respect to standard techniques.

The rest of the paper is structured as follows. Section II introduces the discrete state space, with Section III showcasing the basics for ESKF and its iterated extension. The main contribution of this work is introduced in Section IV-B, with a list of robust M-type filters for the attitude problem. Finally, Sections V and VI present the Monte Carlo (MC) experimentation and the concluding remarks, respectively.

II. System Model

The following state-space model (SSM) is considered,

$$\mathbf{x}_k = \mathbf{f}_{k-1}\left(\mathbf{x}_{k-1}, \boldsymbol{\omega}_{k-1}, \mathbf{v}_{k-1}\right), \tag{1}$$
$$\mathbf{y}_k = \mathbf{h}_k\left(\mathbf{x}_k\right) + \boldsymbol{\eta}_k, \tag{2}$$

with known process and observation functions, $\mathbf{f}_{k-1}(\cdot)$ and $\mathbf{h}_k(\cdot)$; $\mathbf{v}_{k-1} \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_{k-1})$ and $\boldsymbol{\eta}_k \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_k)$ the process and (nominal) measurement noise sequences; and $\boldsymbol{\omega}_{k-1}$ a system input (e.g., gyroscope measurements). The state is

$$\mathbf{x}_k^\top = \left[\mathbf{q}_k^\top, \mathbf{b}_{\omega,k}^\top\right], \ \mathbf{x}_k \in \mathcal{S}^3 \times \mathbb{R}^3, \tag{3}$$

where $\mathbf{q}_k$ denotes a unit norm quaternion rotation and $\mathbf{b}_{\omega,k}$ an unknown input bias/disturbance. The goal is to recursively estimate $\mathbf{x}_k$ from measurements up to time k, $\mathbf{y}_{1:k}$. If the system is linear, the optimal solution in the mean square error (MSE) sense is provided by the well-known KF, but for nonlinear models as in (1)-(2) one must resort to nonlinear KF-type solutions (i.e., the linearized extended KF (EKF) being the simplest one) or sequential MC methods. In the particular case considered in this contribution, the recursive estimation must preserve the inherent nonlinear geometric constraint, i.e., the unit norm of the quaternion [26], [27].

III. On the Error-State KF

The problem stated above is typically addressed with geometric tools such as Lie group theory [28]. A possible solution is given by the so-called error-state KF (ESKF) or indirect KF [1], where $\mathbf{x}_k$ belongs to a manifold and its perturbations $\delta\mathbf{x}_k \in \mathbb{R}^6$ "live" in the tangent space of that manifold (i.e., the algebra of the manifold), which allows to formulate the unknown state as the composition $\mathbf{x}_k = \widehat{\mathbf{x}}_k \oplus \delta\mathbf{x}_k$,

$$\mathbf{x}_k = \widehat{\mathbf{x}}_k \oplus \delta\mathbf{x}_k = \left\{ \begin{array}{c} \widehat{\mathbf{q}}_k \circ \delta\mathbf{q}_k \\ \widehat{\mathbf{b}}_{\omega,k} + \delta\mathbf{b}_{\omega,k} \end{array} \right., \ \delta\mathbf{x}_k = \mathbf{x}_k \ominus \widehat{\mathbf{x}}_k, \tag{4}$$

with $\circ$ the quaternion product, and $\delta\mathbf{q}_k$ the quaternion obtained from the rotation vector for the attitude errors $\delta\boldsymbol{\psi}_k$ ($\delta\mathbf{x}_k^\top = [\delta\boldsymbol{\psi}_k^\top, \delta\mathbf{b}_{\omega,k}^\top]$). Indeed, the Euclidean space for $\delta\boldsymbol{\psi}_k$ connects to the Lie algebra $\mathbf{u}\varphi \in \mathfrak{s}^3$ (with $\mathbf{u}$ an unit vector of rotation and φ the rotated angle) with the isomorphism $(\cdot)^\wedge : \mathbb{R}^3 \mapsto \mathfrak{s}^3$. Then, the Lie algebra connects with the 3D unit-sphere $\mathcal{S}^3$ manifold through exponential mapping. The overall procedure is given by

$$\delta\boldsymbol{\psi} \in \mathbb{R}^3 \xrightarrow{(\cdot)^\wedge} \mathbf{u}\varphi \in \mathfrak{s}^3 \xrightarrow{\exp(\cdot)} \delta\mathbf{q} \in \mathcal{S}^3, \tag{5}$$
$$(\delta\boldsymbol{\psi})^\wedge : \left\{ \begin{array}{l} \mathbf{u} = \frac{\delta\boldsymbol{\psi}}{\|\delta\boldsymbol{\psi}\|^2} \\ \varphi = \|\delta\boldsymbol{\psi}\|^2 \end{array} \right., \exp(\mathbf{u}\varphi) : \begin{bmatrix} \cos(\varphi/2) \\ \mathbf{u}\sin(\varphi/2) \end{bmatrix},$$

then $\mathbf{q}\{\delta\boldsymbol{\psi}\}$ corresponds to the mapping between the Euclidean and unit quaternion spaces via the relationships in (5),

$$\mathbf{q}\{\delta\boldsymbol{\psi}\} \triangleq e^{\mathbf{u}\varphi/2} = \cos\frac{\varphi}{2} + \mathbf{u}\sin\frac{\varphi}{2} = \begin{bmatrix} \cos(\varphi/2) \\ \mathbf{u}\sin(\varphi/2) \end{bmatrix}.$$

With respect to (4), and considering the relationships above, $\delta\mathbf{q}_k = \mathbf{q}\{\delta\boldsymbol{\psi}_k\}$. Refer to [28]–[30] for a detailed discussion on Lie group theory.

A. Standard Error-State (Extended) KF Formulation

For the problem at hand, in order to preserve the unit norm quaternion constraint, the ESKF uses the $\oplus$ operator in (4) (i.e., instead of the standard addition) to linearize and update the system. This ensures that the state estimate stays on the smooth (usually Riemannian) manifold. If $\widehat{\mathbf{x}}_{k-1|k-1}$ denotes the state estimate at discrete time $k-1$, and $\mathbf{P}_{k-1|k-1}$ the corresponding estimation error covariance, then the ESKF is

$$\widehat{\mathbf{x}}_{k|k-1} = \mathbf{f}_{k-1}\left(\widehat{\mathbf{x}}_{k-1|k-1}, \boldsymbol{\omega}_{k-1}\right) \tag{6a}$$
$$\mathbf{P}_{k|k-1} = \mathbf{F}_{k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}_{k-1}^\top + \mathbf{Q}_{k-1}, \tag{6b}$$
$$\mathbf{K}_k = \mathbf{P}_{k|k-1}\mathbf{H}_k^\top\left(\mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k^\top + \boldsymbol{\Sigma}_k\right)^{-1}, \tag{6c}$$
$$\widehat{\mathbf{x}}_{k|k} = \widehat{\mathbf{x}}_{k|k-1} \oplus \mathbf{K}_k(\mathbf{y}_k - \mathbf{h}(\widehat{\mathbf{x}}_{k|k-1})), \tag{6d}$$
$$\mathbf{P}_{k|k} = (\mathbf{I} - \mathbf{K}_k\mathbf{H}_k)\mathbf{P}_{k|k-1} \tag{6e}$$

where $\mathbf{F}_{k-1}, \mathbf{H}_k$, represent the Jacobians of the process and observation models, $\mathbf{f}_{k-1}, \mathbf{h}_k$ (i.e., w.r.t. $\mathbf{x}_k = \widehat{\mathbf{x}}_k \oplus \delta\mathbf{x}_k$).

B. Iterated Error-State (Extended) KF

It is known that the EKF loses the optimality guarantees of the linear KF mainly due to linearization errors. A possible solution to such problem is to use the iterated EKF, which aims at finding the best linearization point for the measurement Jacobian at each update using a Gauss-Newton scheme [31]. Therefore, iterated schemes are particularly useful when the linearization errors of the measurement function are the main source of instability. In the ESKF context and considering states living on manifolds, the iterative sequence $(\widehat{\mathbf{x}}_{k|k}^{(i)})_{i\geq 0}$ to compute $\widehat{\mathbf{x}}_{k|k}$ leading to an iterated ESKF (I-ESKF) is (using the associated $\oplus, \ominus$ operators and $\widehat{\mathbf{x}}_{k|k}^{(0)} = \widehat{\mathbf{x}}_{k|k-1}$) [32], [33],

$$\widehat{\mathbf{x}}_{k|k}^{(i+1)} = \widehat{\mathbf{x}}_{k|k-1} \oplus \mathbf{K}_k^{(i)}\Big(\mathbf{y}_k - \mathbf{h}_k(\widehat{\mathbf{x}}_{k|k}^{(i)}) + \mathbf{H}_k^{(i)}(\widehat{\mathbf{x}}_{k|k}^{(i)} \ominus \widehat{\mathbf{x}}_{k|k-1})\Big), \tag{7}$$

where $\mathbf{H}_k^{(i)}$ is the Jacobian computed at $\widehat{\mathbf{x}}_{k|k}^{(i)}$, and $\mathbf{K}_k^{(i)}$ the Kalman gain computed from $\mathbf{P}_{k|k-1}$ and $\mathbf{H}_k^{(i)}$. Such iterated correction step is sketched in Algorithm 1.

Algorithm 1: Iterated ESKF Correction Step

Input : $\widehat{\mathbf{x}}_{k|k-1}$, $\mathbf{P}_{k|k-1}$, $\mathbf{y}_k, \boldsymbol{\Sigma}_k$

Initialize $\widehat{\mathbf{x}}_{k|k}^{(0)} = \widehat{\mathbf{x}}_{k|k-1}$

for $i = 0, 1, 2, \ldots$ *until convergence* **do**

- Compute Jacobian $\mathbf{H}_k^{(i)}$ from $\widehat{\mathbf{x}}_{k|k}^{(i)}$
- Compute Kalman gain $\mathbf{K}_k^{(i)} = \mathbf{P}_{k|k-1}{\mathbf{H}^{(i)}}_k^\top(\mathbf{H}_k^{(i)}\mathbf{P}_{k|k-1}{\mathbf{H}^{(i)}}_k^\top + \boldsymbol{\Sigma}_k)^{-1}$
- Update $\widehat{\mathbf{x}}_{k|k}^{(i+1)}$ from (7)

Return: $\widehat{\mathbf{x}}_{k|k} = \widehat{\mathbf{x}}_{k|k}^{(i)}$, $\mathbf{P}_{k|k} = (\mathbf{I} - \mathbf{K}_k^{(i)}\mathbf{H}_k^{(i)})\mathbf{P}_{k|k-1}$

IV. Robust M-Type Error-State KFs

A. Robust Filtering: M-Estimation Approach

The KF correction step for a general nonlinear problem can be expressed as a maximum a posteriori (MAP) optimization

$$\widehat{\mathbf{x}}_{k|k} = \arg\min_{\mathbf{x}_k} \left(\|\mathbf{x}_k - \widehat{\mathbf{x}}_{k|k-1}\|^2_{\mathbf{P}_{k|k-1}} + \|\mathbf{h}_k(\mathbf{x}_k) - \mathbf{y}_k\|^2_{\boldsymbol{\Sigma}_k} \right)$$

and, for states living on a manifold, as

$$\widehat{\mathbf{x}}_{k|k} = \arg\min_{\mathbf{x}_k} \left(\|\mathbf{x}_k \ominus \widehat{\mathbf{x}}_{k|k-1}\|^2_{\mathbf{P}_{k|k-1}} + \|\mathbf{h}_k(\mathbf{x}_k) - \mathbf{y}_k\|^2_{\boldsymbol{\Sigma}_k} \right).$$

Alternatively, filters based on the robust statistics framework provide a solution to the following minimization,

$$\widehat{\mathbf{x}}_{k|k} = \arg\min_{\mathbf{x}_k} \left(\|\mathbf{x}_k - \widehat{\mathbf{x}}_{k|k-1}\|^2_{\mathbf{P}_{k|k-1}} + \|\mathbf{h}_k(\mathbf{x}_k) - \mathbf{y}_k\|^2_{\bar{\boldsymbol{\Sigma}}_k} \right)$$

where a modified measurement noise covariance matrix $\bar{\boldsymbol{\Sigma}}_k$ allows to account for outlying observations. Indeed, the use of robust score functions may mitigate the effect of outliers on the final state estimate. Conditional on a set of estimated weights, such covariance is given by

$$\bar{\boldsymbol{\Sigma}}_k = \boldsymbol{\Sigma}_k^{1/2} \mathbf{W}_k^{-1} \boldsymbol{\Sigma}_k^{\top/2}, \tag{8}$$

where $\boldsymbol{\Sigma}_k^{1/2}$ is the Cholesky factorization of $\boldsymbol{\Sigma}_k$ and $\mathbf{W}_k$ is a weighting matrix given by

$$\mathbf{W}_k = \operatorname{diag}\left[\mathbf{w}\left(\boldsymbol{\Sigma}_k^{-1/2}(\mathbf{y}_k - \mathbf{h}_k(\mathbf{x}_k))\right)\right], \tag{9}$$

with $\mathbf{w}(\cdot)$ a function derived from a robust score function (e.g., Huber, Tukey or IGG weighting functions in Fig. 1) [12], [34]. The application of robust statistics' weighting functions within conventional KF schemes can be done by exploiting the information KF form [34], or reformulating the KF as a regression problem [12], [13]. The extension of the former when some state elements may live on a manifold (e.g., such as the quaternion) was recently done in [24]. Hereinafter, this work focuses on its regression-based counterpart.

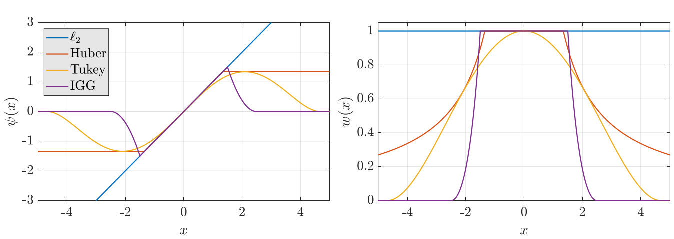

Fig. 1. Score (left) and weighting (right) functions for conventional LS (ℓ_2), Huber, Tukey and IGG functions, with tuning parameters set for 95% efficiency (i.e., to the nominal Gaussian distribution).

B. Regression-Based M-Type KF (M-KF)

The standard Euclidean (extended) KF can be reformulated as a regression problem and solved with a least squares (LS) estimator. In that case, the state prediction $\widehat{\mathbf{x}}_{k|k-1}$ is used as an observation, and included into an augmented observation vector, $\tilde{\mathbf{y}}_k$, with the corresponding measurement noise covariance, $\tilde{\boldsymbol{\Sigma}}_k$, and observation model Jacobian matrix, $\tilde{\mathbf{H}}_k$,

$$\tilde{\mathbf{y}}_k = \begin{bmatrix} \mathbf{y}_k - \mathbf{h}_k(\widehat{\mathbf{x}}_{k|k-1}) + \mathbf{H}_k \widehat{\mathbf{x}}_{k|k-1} \\ \widehat{\mathbf{x}}_{k|k-1} \end{bmatrix}, \tag{10a}$$

$$\tilde{\boldsymbol{\Sigma}}_k = \begin{bmatrix} \boldsymbol{\Sigma}_k & \\ & \mathbf{P}_{k|k-1} \end{bmatrix}, \quad \tilde{\mathbf{H}}_k = \begin{bmatrix} \mathbf{H}_k \\ \mathbf{I} \end{bmatrix}. \tag{10b}$$

Using (10a)-(10b), the (E)KF update step can be reformulated as,

$$\widehat{\mathbf{x}}_{k|k} = \widehat{\mathbf{x}}_{k|k-1} \oplus \left(\tilde{\mathbf{H}}_k^\top \tilde{\boldsymbol{\Sigma}}_k^{-1} \tilde{\mathbf{H}}_k\right)^\dagger \tilde{\mathbf{H}}_k^\top \tilde{\boldsymbol{\Sigma}}_k^{-1} \tilde{\mathbf{y}}_k, \tag{10c}$$

$$\mathbf{P}_{k|k} = \left(\tilde{\mathbf{H}}_k^\top \tilde{\boldsymbol{\Sigma}}_k^{-1} \tilde{\mathbf{H}}_k\right)^\dagger, \tag{10d}$$

where $(\cdot)^\dagger$ is the Moore–Penrose inverse. Considering the regression-based update step in (10c)-(10d) and the robust covariance solution briefly introduced in (8)-(9), the M-KF $\widehat{\mathbf{x}}_{k|k}$ is computed by an iterative process $(\widehat{\mathbf{x}}_{k|k}^{(i)})_{i\geq 0}$ (i.e., over the weights as in the standard M-estimator) until convergence is reached [12], starting at $i=0$ with $\widehat{\mathbf{x}}_{k|k}^{(0)} = \widehat{\mathbf{x}}_{k|k-1}$,

$$\widehat{\mathbf{x}}_{k|k}^{(i+1)} = \widehat{\mathbf{x}}_{k|k-1} \oplus \left(\tilde{\mathbf{H}}_k^\top \boldsymbol{\Omega}_k^{(i)} \tilde{\mathbf{H}}_k\right)^\dagger \tilde{\mathbf{H}}_k^\top \boldsymbol{\Omega}_k^{(i)} \tilde{\mathbf{y}}_k, \tag{11a}$$

$$\boldsymbol{\Omega}_k^{(i)} = \tilde{\boldsymbol{\Sigma}}_k^{-\top/2} \tilde{\mathbf{W}}_k^{(i)} \tilde{\boldsymbol{\Sigma}}_k^{-1/2}, \tag{11b}$$

$$\tilde{\mathbf{W}}_k^{(i)} = \operatorname{diag}\left[\mathbf{w}\left(\tilde{\boldsymbol{\Sigma}}_k^{-1/2}(\tilde{\mathbf{y}}_k)\right)\right]. \tag{11c}$$

Once convergence is reached, the covariance matrix of the associated estimate is,

$$\mathbf{P}_{k|k} = \left(\tilde{\mathbf{H}}_k^\top \boldsymbol{\Omega}_k^{(*)} \tilde{\mathbf{H}}_k\right)^\dagger, \tag{12}$$

with $\boldsymbol{\Omega}_k^{(*)}$ computed with the final weighting matrix.

For on-manifold systems, we can follow a similar reasoning, nonetheless it is actually easier to directly adapt the (iterated) ESKF update formula as in (7).

C. Robust Iterated Error-State Quaternion KFs

Notice that i) for nonlinear systems the previous M-KF formulation assumes that the first order Taylor approximation of the measurement function is good enough (i.e., no iterative procedure over the measurement Jacobian linearization point), and ii) such formulation is not adapted to state elements living on a manifold (e.g., such as the quaternion). An iterated formulation of the (nonlinear) M-KF for the problem at hand is not available in the literature, and its derivation is the main goal of this contribution.

1) Robust I-ESKF (RI-ESKF): The easiest way to merge the iterated linearization, the quaternion state estimation and a robust weighting function is to modify Algorithm 1, which uses (7), as in Algorithm 2. Notice that there is a first iterative process to deal with the linearization, and a second iterative process over the weights as in the standard regression M-estimator.

Algorithm 2: RI-ESKF Correction Step

Input : $\widehat{\mathbf{x}}_{k|k-1}$, $\mathbf{P}_{k|k-1}$, $\mathbf{y}_k$, $\mathbf{\Sigma}_k$

Initialize $\widehat{\mathbf{x}}^{(0)} = \widehat{\mathbf{x}}_{k|k-1}$, $\mathbf{W}^{(0)} = \mathbf{I}$

for $j = 1, 2, \ldots$ *until convergence* **do**

- Update $\bar{\mathbf{\Sigma}}_k = \mathbf{\Sigma}_k^{1/2} \, (\mathbf{W}^{(j-1)})^{-1} \, \mathbf{\Sigma}_k^{\top/2}$
- Obtain $\widehat{\mathbf{x}}^{(j)}$, $\mathbf{P}^{(j)}$ from Alg. 1, with inputs $\widehat{\mathbf{x}}^{(j-1)}$, $\mathbf{P}_{k|k-1}$, $\mathbf{y}_k$, $\bar{\mathbf{\Sigma}}_k$
- Compute $\mathbf{W}^{(j)}$ from (9) with $\widehat{\mathbf{x}}^{(j)}$

Return: $\widehat{\mathbf{x}}_{k|k} = \widehat{\mathbf{x}}^{(j)}$, $\mathbf{P}_{k|k} = \mathbf{P}^{(j)}$

Algorithm 3: R^{2}I-ESKF Correction Step

Input : $\widehat{\mathbf{x}}_{k|k-1}$, $\mathbf{P}_{k|k-1}$, $\mathbf{y}_k$, $\mathbf{\Sigma}_k$

Initialize $\widehat{\mathbf{x}}^{(0)} = \widehat{\mathbf{x}}_{k|k-1}$, $\mathbf{W}_{\mathbf{y}}^{(0)} = \mathbf{I}$, $\mathbf{W}_{\mathbf{x}}^{(0)} = \mathbf{I}$

for $j = 1, 2, \ldots$ *until convergence* **do**

- Update $\bar{\mathbf{\Sigma}}_k = \mathbf{\Sigma}_k^{1/2} \, (\mathbf{W}^{(j-1)})^{-1} \, \mathbf{\Sigma}_k^{\top/2}$
- Update $\bar{\mathbf{P}}_{k|k-1} = \mathbf{P}_{k|k-1}^{1/2} \, (\mathbf{W}_{\mathbf{x}}^{(j-1)})^{-1} \, \mathbf{P}_{k|k-1}^{\top/2}$
- Obtain $\widehat{\mathbf{x}}^{(j)}$, $\mathbf{P}^{(j)}$ from Alg. 1, with inputs $\widehat{\mathbf{x}}^{(j-1)}$, $\bar{\mathbf{P}}_{k|k-1}$, $\mathbf{y}_k$, $\bar{\mathbf{\Sigma}}_k$
- Compute $\mathbf{W}_{\mathbf{y}}^{(j)}$ from (9) with $\widehat{\mathbf{x}}^{(j)}$
- Compute $\mathbf{W}_{\mathbf{x}}^{(j)} = \text{diag}\left[\mathbf{w}\left(\mathbf{P}_{k|k-1}^{-1/2}\left(\widehat{\mathbf{x}}_{k|k-1} \ominus \widehat{\mathbf{x}}^{(j)}\right)\right)\right]$

Return: $\widehat{\mathbf{x}}_{k|k} = \widehat{\mathbf{x}}^{(j)}$, $\mathbf{P}_{k|k} = \mathbf{P}^{(j)}$

2) Outlier-Free Prediction Robust I-ESKF (R^{2}I-ESKF): A further refinement of the previous robust I-ESKF, in order to mitigate possible outliers at the KF prediction step (i.e., doubly robust), is to consider a robust weighting of the prediction error covariance $\mathbf{P}_{k|k-1}$. In that case, Algorithm 2 is udpated as in Algorithm 3. The resulting R^{2}I-ESKF with outlier-free prediction is equivalent to Algorithm 2 but with $\widehat{\mathbf{x}}_{k|k}^{(j+1)}$ computed with Algorithm 3. Notice that such formulation reminds the regression-based M-KF which also weights both measurements and state prediction.

V. Illustrative Example

To evaluate the performance of the different robust M-type KFs presented in this contribution, w.r.t. the standard I-ESKF, the recursive Wahba's problem is considered [35]. Thus, the state estimation problem in Eqs. (1)–(3) is assumed. The dynamical model consists of the integration of gyroscope measurements, while the observation model includes a set of baseline observations collected over $N+1$ antennas on a vehicle, such that $\mathbf{y}_k \sim \mathcal{N}(vec(\mathbf{R}_k\mathbf{L}), \sigma_\rho^2 \mathbf{I}_n)$. Here, $\mathbf{L} = [\mathbf{l}_1, \ldots, \mathbf{l}_N]$ is a set of three-dimensional $N = 4$ baselines on the local frame, $\mathbf{R}_k$ the local-to-inertial frame rotation at kth time instance, $n = 3 \cdot N$ the total number of observations. The number of MC experiments is set to $M = 100$, the total duration of the experiment is 1000 s, with the prediction and correction steps being processed at 1 Hz. For three periods of 150 s, corresponding with the gray areas in Fig. 2, the observations captured by one of the baselines are outliers.

TABLE I
MC simulation parameters.

Initial std. dev.	Attitude: 1 [deg] Gyroscope bias: 0.1 [deg/h]
Process noise std. dev.	Gyroscope: 0.02 [deg / $\sqrt{s^3}$] Bias random walk: $2 \cdot 10^{-5}$ [deg / $\sqrt{s^3}$]
Obs. noise std. dev.	Code zenith-referenced (σ_ρ): 0.1 [m]
Outliers	25%, α = 100, σ = $\alpha\sigma_\rho$

As a metric of performance, the following intrinsic mean squared attitude error (IMSE) at time k is considered

$$\frac{1}{M}\sum_{i=1}^{M} \text{Log}_{SO(3)}^{\vee}\left(\mathbf{R}_k^{-1}\left(\widehat{\mathbf{R}}_k\right)_i\right)^{\top} \text{Log}_{SO(3)}^{\vee}\left(\mathbf{R}_k^{-1}\left(\widehat{\mathbf{R}}_k\right)_i\right)$$

where $(\widehat{\mathbf{R}})_i$ is the i-th realization estimate.

Figure 2 depicts the IMSE of the quaternion for the proposed filters. On the one hand, the performance of the I-ESKF is significantly degraded when outliers are present as the observations are directly affected by them, and the filter is not capable to recover from them when nominal conditions are back. On the other hand, both robust filters achieve much better performance (very close to the ideal I-ESKF), especially during the outliers intervals.

VI. Conclusions

In this paper, we propose two new robust counterparts to the iterated ESKF for states living on a manifold (unit norm quaternion) in order to mitigate the impact of outliers in the observation model, which may otherwise cause a performance breakdown. Hence, we leverage M-estimation methodology to apply it to attitude estimation. To handle the nonlinearities arising from the observation model and the M-estimator, two cascaded iterative processes are implemented. Thus, we proposed to use 1) a robust iterated ESKF (RI-ESKF), whose update uses an inner loop for the robust weighting function, and 2) an outlier-free prediction robust I-ESKF (R^{2}I-ESKF) which not only applies robust weighting to the measurements, but also to the prediction state in order to robustify both prediction and correction steps. The impact of outliers on standard I-ESKF, and the capacity of the proposed filters to mitigate them, were illustrated on numerical examples. In conclusion, the proposed robust filters are much less impacted by the outliers and their performance is highly similar to the ideal one, contrary to standard I-ESKF.

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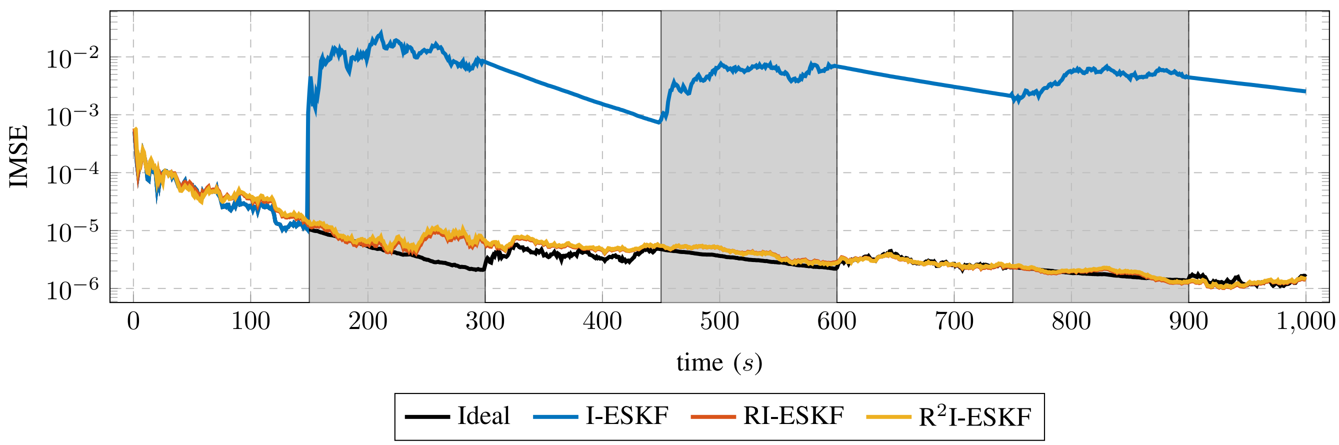

Fig. 2. **Non-nominal case: evolution of the IMSE over time.**

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