Asymptotic Performance Analysis of the Regularized Least Squares Precoding with Restricted Transmit Power in Multi-User Massive MIMO

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***Abstract*—This paper characterizes the regularized least squares (RLS) precoding scheme in multi-user massive multiple-input multiple-output (MU-mMIMO) communication systems. To allow for the use of cheap power amplifiers (PAs) with very limited dynamic ranges, the studied precoder is formulated as a non-closed form solution of a convex problem in which the power at each antenna is constrained below a predefined maximum power. By leveraging the convex Gaussian min-max theorem (CGMT), we characterize the statistics of the precoded symbols and the distortion error at each user under the assumption of Gaussian channels. Based on this, the bit error rate (BER) and a tight lower bound of the signal-to-noise and distortion ratio ($\mathrm{SINAD}_{\mathrm{lb}}$) are asymptotically approximated. As a major outcome of our analysis, we establish that there is an average transmit power that asymptotically optimizes the $\mathrm{SINAD}_{\mathrm{lb}}$ and the BER performance. Such a value can be achieved by properly tuning the power control parameter. Numerical simulations are provided to support the accuracy of our theoretical predictions.**

***Index Terms*—Non-linear precoder, asymptotic analysis, regularized least squares, convex optimization, convex Gaussian min-max theorem (CGMT)**

I. Introduction

Massive multiple-input multiple-output (MIMO) systems, known also as large-scale MIMO techniques, is a key technology for 5G wireless communication that brings important gains in terms of spectral efficiency through the exploitation of extra spatial degrees of freedom provided by the excess of antennas [1], [2]. However, when it comes to implementation, one major bottleneck is the requirement of a dedicated RF chain for each antenna to process the pass-band communication signals in the base band [3], [4], which makes these systems expensive and power-consuming. To reduce the number of RF chains, hybrid precoding is one possible solution that combines several antenna elements into a single one fed by a dedicated RF chain [5], [6]. Another solution is the antenna selection, which encourages the sparsity of the precoded vector by signal processing algorithms [7]. Generally, for the sake of reducing the cost of each RF chain, cheap system modules are employed and the performance imperfections should be compensated for by using advanced signal processing algorithms. From this perspective, the work in [8] considers the regularized least squares (RLS) precoder with a restricted power to realize a limited peak-to-average ratio (PAPR), making it possible to use inexpensive power amplifiers (PAs). To be specific, the precoder in [8] penalizes both the residue sum of squares (RSS) to minimize the received distortion error power and the square of the ℓ_2 norm of the precoded vector to generally reduce the transmit power. It also upper bounds the transmit power to force the PAs to work within a given linear range. Under this setting, our main contribution is to sharply characterize the behavior of the precoded entries and the received distortion error at the user side in a multi-user massive MIMO (MU-mMIMO) communication system where the aforementioned RLS precoder with a restricted transmit power is in operation. Based on this characterization, we provide explicit approximations of the asymptotic bit error rate (BER) and an asymptotically tight- lower bound of the signal-to-noise and distortion ratio ($\mathrm{SINAD}_{\mathrm{lb}}$). The most closed literature in relation to this topic is represented by the work in [9], which provides for the same setting asymptotic expressions for the distortion error power and the average ergodic rate. However, the work in [9] is based on the *non-rigorous* replica method, while the main tool underlying our results is the well-proved convex Gaussian min-max theorem (CGMT). In addition, we characterize the asymptotic distribution of the distortion error, which is instrumental to deriving the asymptotic BER, while the work in [9] is limited to the analysis of the asymptotic distortion error power. Additionally, we provide new insights into the impact of the system parameters on the behavior of the precoder and its performance. An interesting finding shown from our results is that the performance of the precoder does not always improve as the average transmit power increases. The reason lies in that an increase in the transmit power may result in higher distortion error power. As a result, there is an optimal transmit power that achieves the best trade-off between the increase in the transmit power and the optimizes the performance in terms of SINAD and BER.

As aforementioned, underlying our study is the framework of the Convex Gaussian Min-max Theorem (CGMT), first proposed by Stojnic [10] before being formally developed in [11] and [12]. The CGMT has been utilized successfully in the characterization of convex-optimization-based estimators with applications to high-dimensional linear regression [12], binary classification [13], and phase retrieval problems [14]. In the line of MIMO communication system analysis, the CGMT has been applied to characterize the performance of non-explicit decoders under the assumption of real Rayleigh channels with real additive Gaussian noise, for Binary Phase Shift Keying (BPSK) signaling [15], [16] as well as M-ary Pulse Amplitude Modulation (M-PAM) signaling [17]. In this paper we are the first to apply the CGMT to non-linear precoders, thus extending the scope of this framework to the analysis of non-linear algorithms at the transmitting side.

The remainder of this paper is organized as follows: Section II introduces the system model and formulates the problem. Then, in Section III we state our main results characterizing the asymptotic behavior of the studied precoder and its performance metrics. Numerical simulations are finally provided in Section IV to support the accuracy of our results before the paper is concluded in Section V.

II. System model and problem formulation

We consider slow narrow-band downlink transmissions between a base station equipped with n antennas and m single-antenna users. Let $\mathbf{s} = [s_1, s_2, ..., s_m]^T = \{\pm 1\}^m$ denote the BPSK symbol vector intended to the users and $\mathbf{x} = [x_1, x_2, ..., x_n]^T$ the precoded transmitted vector. The received signal at the k-th user writes as

$$\begin{aligned} y_k =& \mathbf{h}_k^T \mathbf{x} + z_k & (1) \\ =& \sqrt{\rho} s_k + \mathbf{h}_k^T \mathbf{x} - \sqrt{\rho} s_k + z_k. & (2) \end{aligned}$$

Stacking the received signals in (1) into a vector $\mathbf{y} = [y_1, \cdots, y_m]^T$ yields

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{z}.$$

The equation in (2) decomposes the received signal into three parts: $\sqrt{\rho} s_k$ is the correct intended symbol, while $d_k = \mathbf{h}_k^T \hat{\mathbf{x}} - \sqrt{\rho} s_k$ and z_k refer to the distortion error and the additive Gaussian noise assumed here to be of mean 0 and variance σ^2. The precoder should be designed to limit the resulting distortion error power caused by the user interference and the noise. To meet this goal, the regularized least squares precoder proposed in [9] is formulated as the solution to the following optimization problem:

$$\hat{\mathbf{x}} = \arg\min_{\mathbf{x} \in \mathbb{X}^n} \|\mathbf{H}\mathbf{x} - \sqrt{\rho}\mathbf{s}\|^2 + \mu \|\mathbf{x}\|^2, \tag{3}$$

where ρ is a positive power control factor, μ is a positive regularization parameter and $\mathbb{X} = [-\sqrt{P}, \sqrt{P}]$ to restrict the transmit power at each antenna. For any given $P < \infty$, the precoder in (3) cannot be obtained in a closed form, which makes the analysis of its behavior a challenging task.

In this paper, the precoder in (3) is characterized, under the following assumptions:

Assumption 1: The number of user equipments m and the n antennas providing service at the base station grow to infinity at a fixed ratio $\delta := \frac{m}{n}$.

Assumption 2: The entries of the channel matrix $\mathbf{H}$ are independent and identically distributed as Gaussian random variables with mean 0 and variance $\frac{1}{n}$.

III. Main results

A. Behaviors of the precoded entries and the distortion error

The performance of the precoder depends on the distribution of the solution to (3) as well as the distortion error received by the users. Both of these quantities will be characterized in the sequel to pave the way for accurate performance analysis of the studied precoder. More specifically, we have the following main result which characterizes the empirical distribution of the precoder vector $\hat{\mathbf{x}}$ as well as the distortion error vector $\mathbf{d} = \mathbf{H}\hat{\mathbf{x}} - \sqrt{\rho}\mathbf{s}$.

Theorem 1: Consider the following max-min optimization problem:

$$\overline{\phi} = \max_{\beta \geq 0} \min_{\tau \geq 0} \frac{\tau \beta \delta}{2} + \frac{\rho \beta}{2\tau} - \frac{\beta^2}{4} + Y(\beta, \tau), \tag{4}$$

where

$$Y(\beta, \tau) = \frac{\beta}{\alpha} \left(\mathbb{E}_{H \sim \mathcal{N}(0,1)} \left[(H - \sqrt{P}\alpha)^2 \mathbf{1}_{\{H \geq \sqrt{P}\alpha\}} \right] - \frac{1}{2} \right),$$

with $\alpha = 1/\tau + 2\mu/\beta$.

(i) Given $\mu > 0$ or $\mu = 0$ and $\delta \geq 1$, the optimization problem in (4) admits a unique finite saddle-point $(\beta^\star, \tau^\star)$.

(ii) Let $\hat{\mathbf{x}}$ be the solution of (3), under Assumption 1 and Assumption 2, the empirical distribution $\frac{1}{n}\sum_{i=1}^n \boldsymbol{\delta}_{[\hat{\mathbf{x}}]_i}$ converges weakly to the distribution of the random variable $f(H)$ defined as:

$$f(H) := \begin{cases} -\sqrt{P} & \text{if } H \leq -\sqrt{P}\alpha^\star \\ \frac{H}{\alpha^\star} & \text{if } -\sqrt{P}\alpha^\star \leq H \leq \sqrt{P}\alpha^\star \\ \sqrt{P} & \text{if } H \geq \sqrt{P}\alpha^\star \end{cases}$$

where $\alpha^\star = 1/\tau^\star + 2\mu/\beta^\star$ and $H \sim \mathcal{N}(0, 1)$.

(iii) For $k = 1, \cdots, m$, define $d_k = [\mathbf{H}\hat{\mathbf{x}}]_k - \sqrt{\rho} s_k$. Then under Assumption 1 and Assumption 2, the empirical distribution $\frac{1}{m}\sum_{i=1}^m \boldsymbol{\delta}_{(d_k, s_k)}$ converges weakly to the distribution of the random variable:

$$\left(\frac{\beta^\star}{2} \frac{\sqrt{(\tau^\star)^2\delta - \rho} H - \sqrt{\rho} S}{\tau^\star \delta}, S \right) \tag{5}$$

where $H \sim \mathcal{N}(0, 1)$ and S is a discrete binary variable taking 1 and -1 with equal probabilities.

Due to the page limits, we refer the interested reader to [18] for the detailed proof.

With this theorem at hand, we derive the asymptotic approximation for the average transmit power:

$$\begin{aligned} P_b :=& \frac{\|\hat{\mathbf{x}}\|^2}{n} & (6) \\ \to& 2P\mathbb{P}\left[H \geq \sqrt{P}\alpha^\star \right] + \frac{\mathbb{E}\left[H^2 \mathbf{1}_{\{-\sqrt{P}\alpha^\star \leq H \leq \sqrt{P}\alpha^\star\}} \right]}{(\alpha^\star)^2} & \\ =& \delta(\tau^\star)^2 - \rho := P_b^\star, & (7) \end{aligned}$$

and the distortion error power

$$
\begin{aligned}
P_d :=& \frac{\|\mathbf{H}\hat{\mathbf{x}} - \sqrt{\rho}\mathbf{s}\|^2}{m} \\
\to& \frac{(\beta^\star)^2}{4\delta} := P_d^\star.
\end{aligned}
\tag{8}
$$

B. Performance analysis

From (2), the received SINAD at user k is

$$
\mathrm{SINAD}_k = \frac{\rho}{\mathbb{E}_{s_k}\left|\mathbf{h}_k^T\hat{\mathbf{x}} - \sqrt{\rho}s_k\right|^2 + \sigma^2}. \tag{9}
$$

and hence, the average per user SINAD of the system is given by:

$$
\mathrm{SINAD} = \mathbb{E}\left[\frac{1}{m}\sum_{k=1}^{m}\mathrm{SINAD}_k\right]
$$

Applying the Jensen's inequality, we may lower-bound the average per-user SINAD by:

$$
\mathrm{SINAD} \geq \mathrm{SINAD}_{\mathrm{lb}} := \frac{\rho}{\frac{1}{m}\sum_{k=1}^{m}\mathbb{E}\left|\mathbf{h}_k^T\hat{\mathbf{x}} - \sqrt{\rho}s_k\right|^2 + \sigma^2}
$$

which according to (5) converges to:

$$
\mathrm{SINAD}_{\mathrm{lb}} \to \frac{\rho}{\frac{(\beta^\star)^2}{4\delta} + \sigma^2} := \mathrm{SINAD}_{\mathrm{lb}}^\star. \tag{10}
$$

We claim that the above lower bound in (10) is a very tight approximation of the average per user SINAD. In fact, all users should experience the same channel statistics according to our Assumption 2, and hence $\frac{1}{m}\sum_{k=1}^{m}\mathbb{E}\left|\mathbf{h}_k^T\hat{\mathbf{x}} - \sqrt{\rho}s_k\right|^2$ should be a good approximation of the per-user distortion error power $\mathbb{E}_{s_k}\left|\mathbf{h}_k^T\hat{\mathbf{x}} - \sqrt{\rho}s_k\right|^2$. This claim will be supported by our numerical results in which we compare our theoretical predictions with their empirical counterparts.

The asymptotic distribution given in (5) allows us to derive the following approximation for the BER as below:

$$
\begin{aligned}
\mathrm{BER} :=& \frac{1}{m}\sum_{i=1}^{m}\mathbf{1}_{\{\mathrm{sign}(y_i)\neq s_i\}} \\
\to& \frac{1}{m}\sum_{k=1}^{m}\mathbb{P}[s_k = 1]\mathbb{P}[\sqrt{\rho}s_k + d_k + z_k < 0|s_k = 1] \\
&+ \mathbb{P}[s_k = -1]\mathbb{P}[\sqrt{\rho}s_k + d_k + z_k > 0|s_k = -1] \\
\to& Q\left(\frac{\sqrt{\rho} - \frac{\beta^\star\sqrt{\rho}}{2\tau^\star\delta}}{\sqrt{\frac{(\beta^\star)^2}{4}\frac{(\tau^\star)^2\delta - \rho}{(\tau^\star)^2\delta^2} + \sigma^2}}\right) := \mathrm{BER}^\star.
\end{aligned}
\tag{11}
$$

It is worth mentioning that while the distortion error power could be derived from a careful analysis of the asymptotic cost, the bit error rate derivation should rely on the expression of the asymptotic distribution given by (5), which is one of our major contributions over the work in [9].

IV. Numerical simulations and remarks

A. Simulation settings

In this section, we verify the accuracy of our proposed results and study the impact of the considered parameters P, μ, ρ and the number of users and transmit antennas m and n on the average transmit power P_b, the received distortion error power P_d, the SINAD and the BER. In all figures, asymptotically predicted values are shown in solid lines and the corresponding simulated results averaged over 50 runs are represented in markers. Very low values of empirical values of the BER will not be represented as they require averaging over a high number of runs. The standard deviation of Gaussian noise z_k is fixed to $\sigma = 0.05$. For $n = 256$ which constitutes a reasonable value for the number of antennas in massive MIMO systems, we can see that the theoretical predictions made in Section III match the simulations very well, although they are in theory valid under the assumption of m and n growing to infinity.

B. Results with remarks

Figure 1 shows the asymptotic analysis in terms of average transmit power P_b, the distortion error power P_d, the SINAD's lower bound and the BER with respect to δ for different values of P, ρ and μ. As can be seen, the performance in terms of BER and SINAD always asymptotically improves when δ goes to zero. This is because δ tending to zero implies that more antennas are employed compared to the number of served users. In this situation, the system is better off using small values for the regularization coefficient μ. However, interestingly, the power P_b is not a monotonous function of δ: for δ taking small values, P_b increases before decreasing beyond moderate values of δ.

To further investigate how the system relies on the power control parameter ρ, we plot in Figure 2 the average transmit power P_b and the distortion error power P_d with respect to ρ for different values of P. As shown, the average transmit power P_b is an increasing function of ρ, tending to zero when $\rho \downarrow 0$ and P as $\rho \to \infty$. This shows that we can always tune ρ to achieve a target average transmit power P_b between $(0, P)$. However, as shown in the right display of Figure 2, using high values for the power control parameter ρ to increase the average transmit power results in high distortion levels. This leads us to anticipate that the performance of the precoder does not necessarily improve with the increase in the average transmit power.

To confirm this claim, we conducted an experiment in which precoders designed for $P = 30$ and $P = 40$ use the same average transmit power $P_b^\star$ by properly adjusting their control power parameter ρ, as shown in Figure 2. Moreover, for both precoders, we assume that the regularization parameter μ is tuned to the value that maximizes the $\mathrm{SINAD}_{\mathrm{lb}}^\star$. Figure 3 shows the performance metrics with respect to $P_b^\star$ for these values of P. We note that for both schemes, there exists a $P_b^\star$ for which the performance in terms of SINAD and BER are optimal.

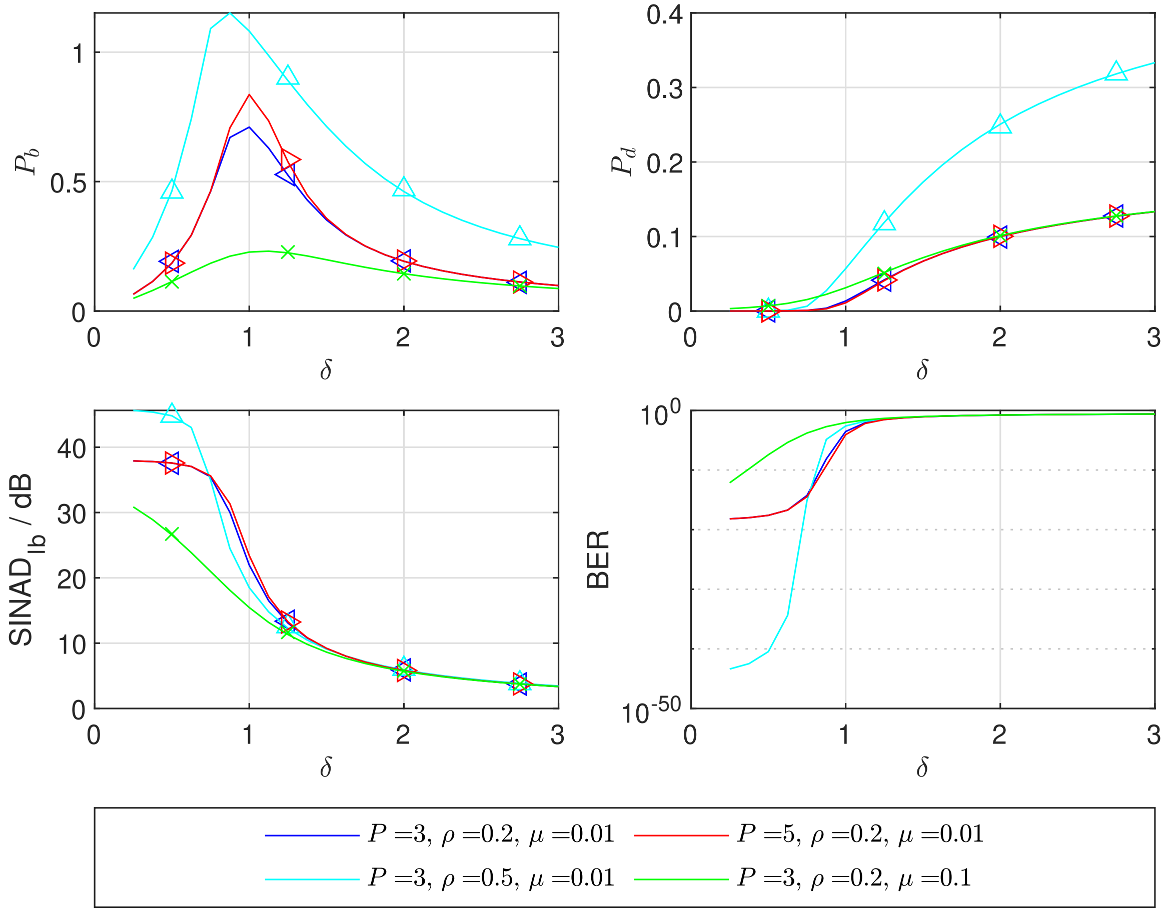

Fig. 1. Asymptotic analysis versus different parameter combinations.

V. Conclusion

In this paper, we study the asymptotic characterization of the RLS precoding scheme with a restricted transmit power of the multi-user massive MIMO communication systems when the number of antennas and the served users grow large at a fixed ratio. Going beyond the previous studies in [9], we rely on the rigorous CGMT framework and present predictions for the distortion error. Based on this, we asymptotically infer the performance in terms of BER and SINAD. As a major insight, our analysis demonstrates that by tuning the power control parameter, an appropriate average transmit power can be realized to asymptotically optimize the $\mathrm{SINAD}_{\mathrm{lb}}$ and the BER performance.

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References

[1] Z. Zhang, M. McGuire, and M. Sima, "Iterative channel estimation for large scale mimo with highly quantized measurements in 5g," in *2020 28th European Signal Processing Conference (EUSIPCO)*, 2021, pp. 1643–1647.

[2] E. Vlachos, C. Mavrokefalidis, and K. Berberidis, "Channel estimation for uav-based mmwave massive mimo communications with beam squint," in *2022 30th European Signal Processing Conference (EUSIPCO)*, 2022, pp. 1696–1700.

[3] D. Perez-Adan, J. P. Gonzalez-Coma, O. Fresnedo, and L. Castedo, "Low-complexity hybrid transceivers for uplink multiuser mmWave MIMO by user clustering," in *2019 27th European Signal Processing Conference (EUSIPCO)*, 2019, pp. 1–5.

[4] Y. Karacora, A. Kariminezhad, and A. Sezgin, "Hybrid beamforming: Where should the analog power amplifiers be placed?" in *ICASSP 2019 - 2019 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, 2019, pp. 4719–4723.

[5] C. B. Barneto, T. Riihonen, S. D. Liyanaarachchi, M. Heino, N. González-Prelcic, and M. Valkama, "Beamformer design and optimization for joint communication and full-duplex sensing at mm-waves," *IEEE Transactions on Communications*, vol. 70, no. 12, pp. 8298–8312, 2022.

[6] R. Magueta, D. Castanheira, A. Silva, R. Dinis, and A. Gameiro, "Two-step hybrid multiuser equalizer for sub-connected mmWave Massive MIMO SC-FDMA systems," in *2018 26th European Signal Processing Conference (EUSIPCO)*, 2018, pp. 822–826.

[7] A. Bereyhi, M. A. Sedaghat, R. R. Müller, and G. Fischer, "GLSE precoders for massive MIMO systems: Analysis and applications," *CoRR*, vol. abs/1808.01880, 2018. [Online]. Available: http://arxiv.org/abs/1808.01880

[8] A. Bereyhi, V. Jamali, R. R. Müller, G. Fischer, R. Schober, and

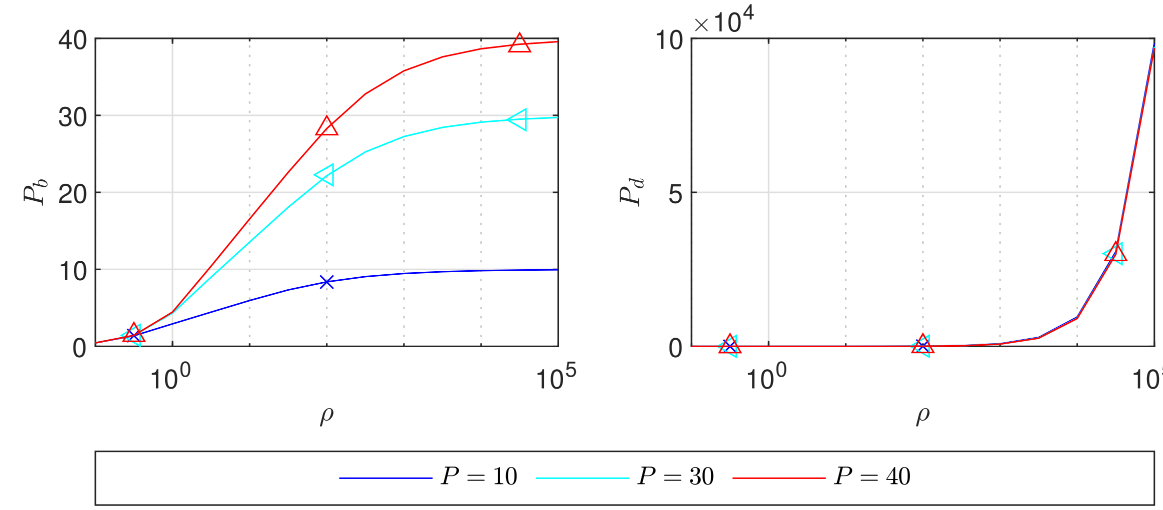

Fig. 2. P_b and P_d versus ρ under different P values. Here $\delta = 1$ and $\mu = 0.01$.

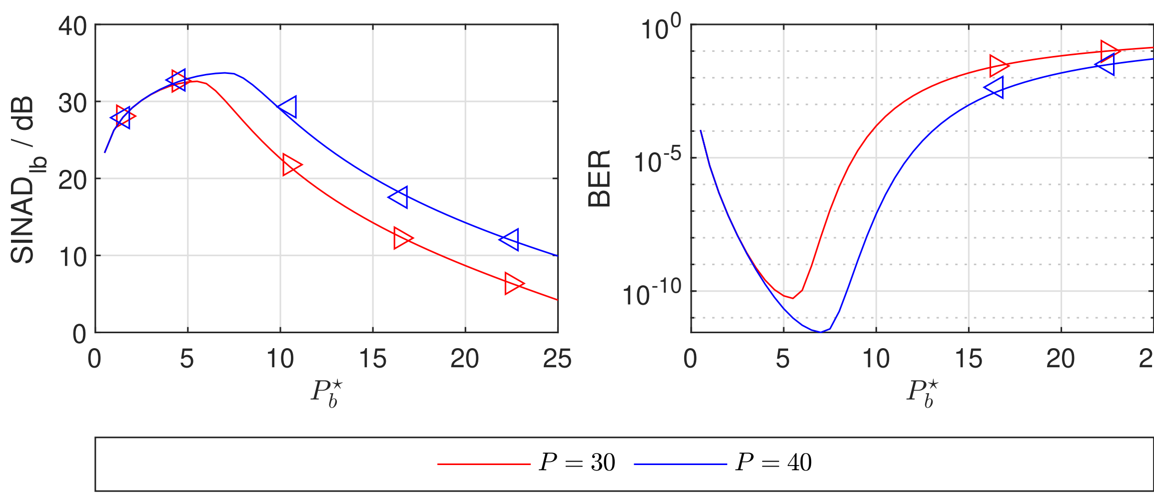

Fig. 3. SINAD and BER versus $P_b^\star$ under different P values. Here $\delta = 1$, ρ is tuned to realize the corresponding $P_b^\star$ and μ is set to the value maximizing $\mathrm{SINAD}_{\mathrm{lb}}^\star$.

A. M. Tulino, "PAPR-limited precoding in massive MIMO systems with reflect- and transmit-array antennas," *CoRR*, vol. abs/1912.00485, 2019. [Online]. Available: http://arxiv.org/abs/1912.00485

[9] M. A. Sedaghat, A. Bereyhi, and R. R. Müller, "Least square error precoders for massive MIMO with signal constraints: Fundamental limits," *IEEE Transactions on Wireless Communications*, vol. 17, no. 1, pp. 667–679, Jan. 2018.

[10] M. Stojnic, "A framework to characterize performance of LASSO algorithms," *CoRR*, vol. abs/1303.7291, 2013. [Online]. Available: http://arxiv.org/abs/1303.7291

[11] C. Thrampoulidis, S. Oymak, and B. Hassibi, "A tight version of the Gaussian min-max theorem in the presence of convexity," *CoRR*, vol. abs/1408.4837, 2014. [Online]. Available: http://arxiv.org/abs/1408.4837

[12] C. Thrampoulidis, E. Abbasi, and B. Hassibi, "Precise error analysis of regularized m-estimators in high dimensions," *IEEE Transactions on Information Theory*, vol. 64, no. 8, pp. 5592–5628, Aug. 2018.

[13] Z. Deng, A. Kammoun, and C. Thrampoulidis, "A model of double descent for high-dimensional binary linear classification," 2019. [Online]. Available: https://arxiv.org/abs/1911.05822

[14] O. Dhifallah, C. Thrampoulidis, and Y. M. Lu, "Phase retrieval via polytope optimization: Geometry, phase transitions, and new algorithms," 2018. [Online]. Available: https://arxiv.org/abs/1805.09555

[15] C. Thrampoulidis, E. Abbasi, W. Xu, and B. Hassibi, "BER analysis of the box relaxation for BPSK signal recovery," in *2016 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, 2016, pp. 3776–3780.

[16] C. Thrampoulidis, W. Xu, and B. Hassibi, "Symbol error rate performance of box-relaxation decoders in massive MIMO," *IEEE Transactions on Signal Processing*, vol. 66, no. 13, p. 3377–3392, Jul. 2018.

[17] C. Thrampoulidis and W. Xu, "The performance of box-relaxation decoding in massive MIMO with low-resolution ADCs," in *2018 IEEE Statistical Signal Processing Workshop (SSP)*, 2018, pp. 821–825.

[18] X. Ma, A. Kammoun, A. M. Alrashdi, T. Ballal, M.-S. Alouini, and T. Y. Al-Naffouri, "Asymptotic analysis of rls-based digital precoder with limited papr in massive mimo," *IEEE Transactions on Signal Processing*, vol. 70, pp. 5488–5503, 2022.