Direction-of-Arrival Estimation for Correlated Sources and Low Sample Size

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***Abstract*—In this paper, we study the problem of recovering the direction-of-arrival in difficult scenarios of highly correlated source signals and only few available snapshots. Recently, the partial relaxation framework has been proposed as an optimization-based technique that accounts for the existence of multiple signals while performing the estimation task through a simple spectral search. Its performance is superior to conventional methods but tends to deteriorate drastically when the source signals are highly correlated due to information loss associated with the relaxation. On the other hand, from a compressed sensing point of view, the recently proposed sparse row-norm reconstruction method formulates the parameter estimation problem as a compact $\ell_{2,1}$-mixed-norm minimization problem. One of its prominent advantages is its robustness under highly correlated sources and a low number of snapshots; an intrinsic bias induced by the ℓ_1-norm approximation, however, affects the estimation performance. In this paper, we propose a method that integrates the $\ell_{2,1}$-mixed-norm minimization formulation into the spectral search of the partial relaxation estimators. Simulation results show that the proposed estimator has superior error performance in difficult scenarios and alleviates the disadvantages of both methods.**

***Index Terms*—DOA estimation, partial relaxation, sparse signal recovery, joint sparsity, mixed-norm minimization**

I. Introduction

Estimating the Direction-of-Arrival (DOA) of the incoming source signals from data collected at a sensor array is an important and long-standing problem in many fields of engineering. Concrete examples include wireless communication, seismic exploration, and automatic monitoring [1], [2], [3].

Numerous approaches have been developed in the past, with different focuses on improving computational efficiency, applicability to general array geometries, increasing resolution capability, or enhancing robustness in difficult scenarios. For example, the Maximum Likelihood Estimator (MLE) [4], [5] has remarkable error performance in both the threshold and the asymptotic region by exploiting the full statistical information, but its application in practice is greatly hindered due to the high computational cost. The root-MUSIC method [6] and the ESPRIT method [7] avoid any spectral search and reduce the computational cost by explicitly utilizing the array structures.

Recently, a new approach, namely the Partial Relaxation (PR) method [8], [9], has been proposed to tackle the problem of DOA estimation from the perspective of relaxation optimization. In contrast, the SPARse ROW-norm reconstruction (SPARROW) method [10] has been proposed as an efficient reformulation of the classical multiple measurement problem of Malioutov [11]. The PR method transforms a multidimensional optimization problem into a simple spectral search by partially relaxing the manifold structure under consideration. Theoretical analysis and simulations show that the PR estimators have superior performance in many setups, but a theoretical gap exists between its Cramér-Rao Bound (CRB) and that of the MLE when the source signals are correlated [12], [13], [14]. In contrast, the SPARROW method recovers the DOAs by reconstructing a row-sparse signal matrix from a densely sampled dictionary manifold. It makes no assumptions on the source signal covariance matrix. However, its error performance tends to saturate in the asymptotic region due to a bias induced by the sparse regularization.

In this paper, we propose a novel DOA estimation method that improves the frequency resolution and alleviates the intrinsic bias of the SPARROW estimator and evidently diminishes the weakness of the PR estimators under correlated sources. Specifically, the proposed method recovers the DOAs by performing a spectral search on the solution of the SPARROW estimator that employs a reduced regularization parameter. We remark that, although only the Uniform Linear Array (ULA) structure is considered throughout the paper, the method can be easily generalized to cope with other array geometries, e.g., shift-invariant or fully-augmentable arrays, by incorporating the correspondingly designed SPARROW estimator [15], [16].

This paper is organized as follows. Section II introduces the signal model for the DOA estimation problem. In Section III, we briefly review the PR framework and the SPARROW method. Based on the above two methods, in Section IV, we propose a novel DOA estimator. Section V contains simulation results, and the concluding remarks are given in Section VI.

Notation: Matrices, vectors, and scalars are denoted by boldface uppercase letters $\boldsymbol{X}$, boldface lowercase letters $\boldsymbol{x}$, and regular letters x, respectively. The symbols $(\cdot)^{\mathsf{T}}$, $(\cdot)^{\mathsf{H}}$, and $(\cdot)^{-1}$ denote transpose, Hermitian transpose, and inverse, respectively. The expectation operator is represented by $\mathbb{E}\{\cdot\}$. $\|\cdot\|_{\mathrm{F}}$ stands for the Frobenius norm and $\|\cdot\|_2$ is the ℓ_2-norm.

II. Signal Model

In this section, we define a signal model for estimating DOAs of incoming signals with data collected at a sensor array. Consider a ULA of M omnidirectional sensors that receives narrowband signals located in the far-field of the array. There are L impinging source signals whose DOAs are denoted by $\boldsymbol{\theta} = [\theta_1, \ldots, \theta_L]^\mathsf{T}$. We assume the number of source signals L to be known. At time instant t, the receive signal at the sensor array is

$$\boldsymbol{y}(t) = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{x}(t) + \boldsymbol{n}(t), \tag{1}$$

where $\boldsymbol{y}(t) \in \mathbb{C}^M$ and $\boldsymbol{x}(t) \in \mathbb{C}^L$ are the received signal vector and the source signal vector, respectively, demodulated to baseband. Matrix $\boldsymbol{A}(\boldsymbol{\theta}) = [\boldsymbol{a}(\theta_1), \ldots, \boldsymbol{a}(\theta_L)] \in \mathbb{C}^{M \times L}$ is the steering matrix with each column $\boldsymbol{a}(\theta_i) = [1, e^{-j\pi \sin(\theta_i)}, \ldots, e^{-j(M-1)\pi \sin(\theta_i)}]^\mathsf{T}$ being the sensor array responses for DOA θ_i for $i = 1, \ldots, L$. Vector $\boldsymbol{n}(t) \in \mathbb{C}^M$ denotes independent and identically distributed circular and spatio-temporal white Gaussian noise with covariance matrix $\boldsymbol{R}_\mathsf{n} = \mathbb{E}\{\boldsymbol{n}(t)\boldsymbol{n}(t)^\mathsf{H}\} = \sigma^2 \boldsymbol{I}_M$, where σ^2 is the noise power at each sensor. We assume that the geometry of the sensor array involves no ambiguity and the steering matrix $\boldsymbol{A}(\boldsymbol{\theta})$ is always of full rank for all possible DOAs $\boldsymbol{\theta}$ [17].

In the case of multiple snapshots collected at time $t = 1, \ldots, N$, the signal model in (1) can be compactly expressed as $\boldsymbol{Y} = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{X} + \boldsymbol{N}$ with $\boldsymbol{Y} = [\boldsymbol{y}(1), \ldots, \boldsymbol{y}(N)]$, $\boldsymbol{X} = [\boldsymbol{x}(1), \ldots, \boldsymbol{x}(N)]$, and $\boldsymbol{N} = [\boldsymbol{n}(1), \ldots, \boldsymbol{n}(N)]$ being the receive signal matrix, source signal matrix, and noise matrix, respectively. We apply a stochastic signal model and assume that the source signals $\boldsymbol{x}(t)$ are samples of a stationary process with zero mean, i.e., $\mathbb{E}\{\boldsymbol{x}(t)\} = \boldsymbol{0}$. Let $\boldsymbol{R}_x = \mathbb{E}\{\boldsymbol{x}(t)\boldsymbol{x}(t)^\mathsf{H}\}$ denote the source covariance matrix. The receive covariance matrix, denoted by $\boldsymbol{R}_y$, is then given by

$$\boldsymbol{R}_y = \mathbb{E}\{\boldsymbol{y}(t)\boldsymbol{y}(t)^\mathsf{H}\} = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{R}_x\boldsymbol{A}(\boldsymbol{\theta})^\mathsf{H} + \sigma^2 \boldsymbol{I}_M. \tag{2}$$

In practice, the receive covariance matrix is not known and, therefore, is often replaced by the sample covariance matrix

$$\widehat{\boldsymbol{R}}_y = \tfrac{1}{N}\boldsymbol{Y}\boldsymbol{Y}^\mathsf{H}.$$

III. Review of PR and SPARROW

We start by briefly reviewing the two methods underlying our proposed method in Section IV.

A. The Partial Relaxtion Framework

The conventional multi-source estimation is formulated as

$$\widehat{\boldsymbol{\theta}} = \arg\min_{\boldsymbol{\theta}} f(\boldsymbol{A}(\boldsymbol{\theta}), \boldsymbol{Y}), \tag{3}$$

where f is a general multi-source cost function, e.g. the deterministic maximum likelihood cost function [5], and $\boldsymbol{A}(\boldsymbol{\theta})$ is the multi-source array manifold. Under the partial relaxation framework, by maintaining only the manifold structure of the first column $\boldsymbol{a}(\theta_1)$ of $\boldsymbol{A}(\boldsymbol{\theta})$ and relaxing the manifold structure of the remaining sources $[\boldsymbol{a}(\theta_2), \ldots, \boldsymbol{a}(\theta_L)]$, problem (3) is transformed into a spectral search as

$$\widehat{\boldsymbol{\theta}} = {}^{L}\arg\min_{\theta} \min_{\boldsymbol{B} \in \mathbb{C}^{M \times (L-1)}} f(\boldsymbol{a}(\theta), \boldsymbol{B}, \boldsymbol{Y}),$$

with ${}^{L}\arg\min$ denoting the set of arguments where the searched spectrum takes the L-deepest minima [8].

On the one hand, by partially relaxing the L-source array manifold, the computational cost of the L-dimensional search is remarkably reduced to that of a simple one-dimensional spectral search, while the estimation accuracy is maintained via accounting for multiple sources. On the other hand, the relaxation induces statistical information loss and, hence, the CRB of the PR framework is generally higher than the conventional CRB [9]. In particular, the PR estimators tend to degrade severely when the sources are highly correlated or when the number of available snapshots is very low.

The Partially Relaxed Unconstrained Covariance Fitting (PR-UCF) estimator is obtained under the PR framework as

$$\widehat{\boldsymbol{\theta}}_{\text{PR-UCF}} = {}^{L}\arg\min_{\theta} \min_{\sigma_x^2 \geqslant 0} \textstyle\sum_{k=L}^{M} \lambda_k^2(\widehat{\boldsymbol{R}}_y - \sigma_x^2 \boldsymbol{a}(\theta)\boldsymbol{a}(\theta)^\mathsf{H})$$

based on the conventional unconstrained covariance fitting method. It has the advantage of being applicable to general array geometries. Other DOA estimators, e.g., the Partially Relaxed Deterministic Maximum Likelihood (PR-DML) estimator based on the conventional deterministic maximum likelihood estimator, also have been derived in [8]. For the sake of conciseness, we refrain from listing them here.

B. The SPARROW Method

In this subsection, we briefly review the SPARROW method that is developed in the field of sparse signal reconstruction.

Sparse signal reconstruction techniques are applied to the DOA estimation problem by recovering a sparse representation of the signal matrix with an overcomplete dictionary matrix sampled from the Field-of-View (FOV). Specifically, consider the classical $\ell_{2,1}$-mixed-norm minimization problem

$$\min_{\boldsymbol{Z} \in \mathbb{C}^{K \times N}} \tfrac{1}{2}||\boldsymbol{A}(\boldsymbol{\nu})\boldsymbol{Z} - \boldsymbol{Y}||_\mathsf{F}^2 + \lambda\sqrt{N}||\boldsymbol{Z}||_{2,1}, \tag{4}$$

where $\boldsymbol{\nu} = \{\nu_1, \ldots, \nu_K\}$ is the densely and sufficiently sampled FOV with $K \gg L$ and $\theta_i \in \boldsymbol{\nu}$, for $i = 1, \ldots, L$, $\boldsymbol{Z} = [\boldsymbol{z}_1, \ldots, \boldsymbol{z}_K]^\mathsf{T}$ is the row-wise sparse signal matrix, and $\boldsymbol{A}(\boldsymbol{\nu}) = [\boldsymbol{a}(\nu_1), \ldots, \boldsymbol{a}(\nu_K)] \in \mathbb{C}^{M \times K}$ is an overcomplete dictionary matrix. The $\ell_{2,1}$-mixed-norm is defined as

$$\|\boldsymbol{Z}\|_{2,1} = \textstyle\sum_{k=1}^{K} \|\boldsymbol{z}_k\|_2$$

and $\lambda > 0$ is the regularization parameter inducing row-sparsity in matrix $\boldsymbol{Z}$. The DOAs are then estimated from the solution $\widehat{\boldsymbol{Z}} = [\widehat{\boldsymbol{z}}_1, \ldots, \widehat{\boldsymbol{z}}_K]^\mathsf{T}$ of (4) by

$$\{\widehat{\theta}_i\}_{i=1}^{L} := \{\nu_k : ||\widehat{\boldsymbol{z}}_k||_2 \neq 0, k = 1, \ldots, K\}.$$

It is proved in [10, Theorem 1] that problem (4) is equivalent to the convex problem

$$\min_{\boldsymbol{S} \in \mathbb{D}_+^K} \mathrm{Tr}\left((\boldsymbol{A}(\boldsymbol{\nu})\boldsymbol{S}\boldsymbol{A}(\boldsymbol{\nu})^\mathsf{H} + \lambda \boldsymbol{I}_M)^{-1}\widehat{\boldsymbol{R}}_y\right) + \mathrm{Tr}(\boldsymbol{S}), \tag{5}$$

where $\mathbb{D}_+^K$ is the set of $K \times K$ nonnegative diagonal matrices. Let $s_1, \ldots, s_K \geqslant 0$ be the diagonal entries of $\boldsymbol{S}$. The solutions of (4) and (5) are related by

$$\widehat{\boldsymbol{Z}} = \widehat{\boldsymbol{S}}\boldsymbol{A}(\boldsymbol{\nu})^\mathsf{H}(\boldsymbol{A}(\boldsymbol{\nu})\widehat{\boldsymbol{S}}\boldsymbol{A}(\boldsymbol{\nu})^\mathsf{H} + \lambda \boldsymbol{I}_M)^{-1}\boldsymbol{Y},$$
$$\widehat{s}_k = \tfrac{1}{\sqrt{N}}||\widehat{\boldsymbol{z}}_k||_2 \quad \text{for } k = 1, 2, \ldots, K.$$

The SPARROW formulation in (5) requires a densely sampled grid in order to reduce any estimation bias caused by spectral leakage effects, which results in a high computational cost. When the array geometry forms a uniform linear structure, problem (5) can be reformulated to allow for a gridless estimation. In the case of a uniform linear array with M sensors, the overcomplete dictionary matrix $\boldsymbol{A}(\boldsymbol{\nu})$ has a Vandemonde structure. The convex combination of $\boldsymbol{a}(\nu_k)\boldsymbol{a}(\nu_k)^{\mathsf{H}}$ forms a positive semidefinite Toeplitz matrix

$$\boldsymbol{T} = \boldsymbol{A}(\boldsymbol{\nu})\boldsymbol{S}\boldsymbol{A}(\boldsymbol{\nu})^{\mathsf{H}} = \sum_{k=1}^{K} s_k \boldsymbol{a}(\nu_k)\boldsymbol{a}(\nu_k)^{\mathsf{H}} \in \mathcal{T}_+^M, \quad (6)$$

where $\mathcal{T}_+^M$ is the set of $M \times M$ positive semidefinte Toeplitz matrices. Without loss of generality, the nonnegative coefficients $s_1, \dots, s_K$ are assumed to be sorted in descending order. By the Caratheodory theorem [18], [19], if the number of distinct DOAs $\nu_1, \dots, \nu_r$ with positive coefficients $s_1, \dots, s_r$ does not exceed the number of sensors, i.e., $r \leqslant M$, then the mapping between $s_1, \dots, s_r$ and the matrix $\boldsymbol{T}$ in (6) is unique and, moreover, $r = \operatorname{rank}(\boldsymbol{T})$. Thus, a straightforward gridless extension of problem (5) is jointly learning a dictionary, which can be equivalently reformulated as the convex problem [20]

$$\widehat{\boldsymbol{T}} = \underset{\boldsymbol{T} \in \mathcal{T}_+^M}{\arg\min} \quad \operatorname{Tr}\left((\boldsymbol{T} + \lambda \boldsymbol{I}_M)^{-1} \widehat{\boldsymbol{R}}_y \right) + \frac{1}{M} \operatorname{Tr}(\boldsymbol{T}). \quad (7)$$

The DOAs $\nu_1, \dots, \nu_K$ are then recovered from the solution $\widehat{\boldsymbol{T}}$ in (7) according to the Vandermonde decomposition in (6), which can be obtained by, e.g., the linear prediction methods [21]. In addition, the convex problems (5) and (7) can be solved by a generic semidefinite program solver such as MOSEK [22].

IV. Motivation and Proposed Method

Both the SPARROW method in (7) and the PR estimators discussed in Section III-A have their disadvantages. An intrinsic bias is introduced in the SPARROW method in (7) by the row-sparsity regularization, whereas the PR estimators tend to degrade drastically when the source signals are highly correlated or when few snapshots are available. In this section, we propose a novel DOA estimator that integrates the SPARROW formulation (7) into the spectral search of the PR framework and alleviates the disadvantages of both methods.

Specifically, the proposed method consists of the two steps:

Step 1: solve the gridless SPARROW problem in (7) with a reduced regularization parameter as specified below;

Step 2: with the additional knowledge of the model order L, recover the DOAs by a spectral search on the solution $\widehat{\boldsymbol{T}}$ of the gridless SPARROW using one of the PR estimators in [8].

In the case of correlated source signals, the source covariance matrix $\boldsymbol{R}_x$ in (2) is not diagonal, and, consequently, the source signals remain correlated in the receive covariance matrix $\boldsymbol{R}_y$ in (2). In contrast, a diagonal structure persists in the sparse matrix $\boldsymbol{S}$ in (5). Furthermore, by the Vandermonde decomposition in (6), the sources are decorrelated in the solution $\widehat{\boldsymbol{T}}$ of the gridless SPARROW in (7). Thus, our proposed method aims at addressing the shortcoming of PR estimators for correlated sources, by first decorrelating the sources using the gridless SPARROW.

On the other hand, since the information on the number of sources is known and utilized by the PR estimators in Step 2, we reduce the sparsity-promoting parameter λ for the gridless SPARROW in Step 1 in order to reduce the bias, which, consequently, leads to solutions of higher rank. In particular, we employ the heuristic tuning rule given in [10] with an additional reduction factor, i.e.,

$$\lambda = C_\lambda \sqrt{\sigma^2 M \log M}, \quad (8)$$

where the reduction factor C_λ is typically chosen between 0.4 and 0.6. In contrast, $C_\lambda = 1$ is used by the SPARROW estimator reported in [10].

In Step 2, instead of computing the Vandermonde decomposition of the Toeplitz solution $\widehat{\boldsymbol{T}}$, the proposed method performs a spectral search on $\widehat{\boldsymbol{T}}$ with a PR estimator. We remark that, if the solution $\widehat{\boldsymbol{T}}$ of SPARROW in (7) has a rank of L, i.e., $\widehat{\boldsymbol{T}}$ only contains L nonzero components in the Vandermonde decomposition in (6), then the spectral search of PR returns the same DOAs as those in the Vandermonde decomposition due to the uniqueness of the Vandermonde decomposition. In this case, the proposed method reduces to the original gridless SPARROW. This is another motivation for the reduction of the sparsity-promoting parameter λ in the first step of our proposed method.

In particular, we consider the following estimator, termed SP-PR-UCF, derived under the framework proposed above based on the PR-UCF estimator in Section III-A:

$$\widehat{\boldsymbol{\theta}}_{\text{SP-PR-UCF}} = {}^{L}\underset{\theta}{\arg\min} \min_{\sigma_x^2 \geqslant 0} \sum_{k=L}^{M} \lambda_k^2 (\widehat{\boldsymbol{T}} - \sigma_x^2 \boldsymbol{a}(\theta)\boldsymbol{a}(\theta)^{\mathsf{H}}).$$

We remark that other DOA estimators can also be derived under our proposed framework in the same manner based on different PR estimators.

V. Simulation Results

In this section, we conduct numerical experiments to evaluate and analyze the performance of the proposed method. Specifically, we compare, based on Monte-Carlo trials, the error performance of the proposed method with that of the PR estimator, the root-MUSIC method, the SPARROW estimator (7) with the original regularization parameter, the SPARROW estimator (7) with the same regularization parameter as the proposed method, and the root-MUSIC method applied to the Toeplitz matrix solution of (7) with the same regularization parameter as the proposed method (termed SP-root-MUSIC). The signal-to-noise ratio (SNR) is defined as SNR $= 1/\sigma^2$. To evaluate the performance of each method, we compute the root-mean-square-error (RMSE) as

$$\text{RMSE} = \sqrt{\tfrac{1}{N_R L} \sum_{i=1}^{N_R} \sum_{l=1}^{L} (\hat{\theta}_l^{(i)} - \theta_l)^2},$$

where N_R is the total number of Monte-Carlo trials, $\widehat{\boldsymbol{\theta}}^{(i)} = [\hat{\theta}_1^{(i)}, \dots, \hat{\theta}_L^{(i)}]^{\mathsf{T}}$ contains the estimated DOAs at the ith trial, and $\boldsymbol{\theta} = [\theta_1, \dots, \theta_L]^{\mathsf{T}}$ is the true DOAs. For the SPARROW

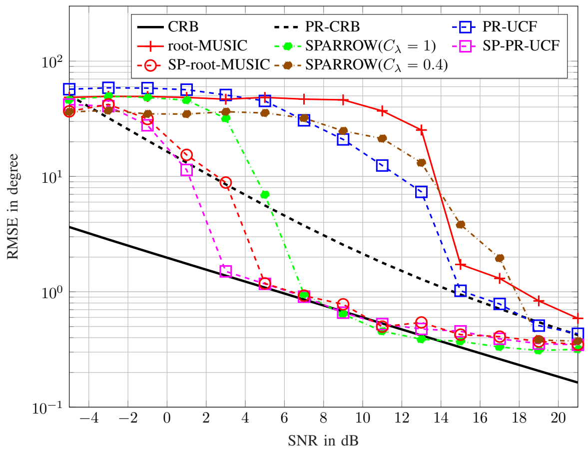

Fig. 1. RMSE vs SNR for correlated sources with $N = 40$.

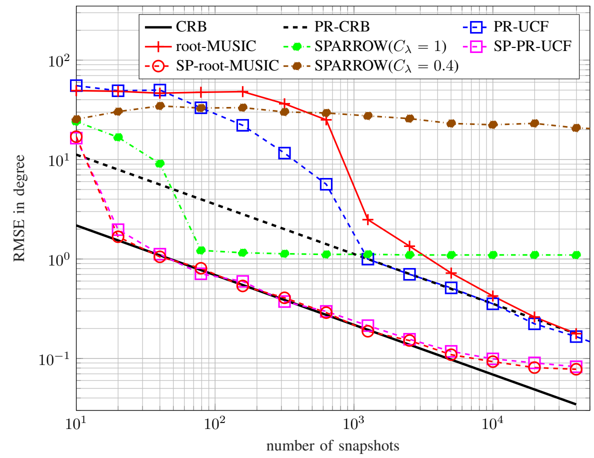

Fig. 2. RMSE vs N for correlated sources with SNR= 5dB.

estimator, only the DOAs of the L leading components in the Vandermonde decomposition in (6) are taken into account in the evaluation of RMSE. The number of Monte-Carlo runs in the following simulations is 200.

A. *Highly-correlated Source Signals*

In the first simulation, we study the error performance of various estimators when the sources are highly correlated. Two sources with the correlation factor $\rho = 0.95$ are placed closely at 45° and 50°. We use a uniform linear array of size $M = 10$. For the proposed method, we choose $C_\lambda = 0.4$ in the heuristic formula (8). We exclude 2% of the runs with the largest estimation errors before calculating RMSE. Simulation results are presented in Fig. 1-3.

From Fig. 1-3, we observe a clear gap between the CRB of PR estimators and the conventional CRB. The PR estimator exhibits better error behavior than the root-MUSIC method in the threshold region and converges to PR-CRB at relatively high SNR values or snapshot sizes. In comparison, in Fig. 1 and Fig. 2, the SPARROW method with the original regularization parameter outperforms the PR estimator, but its RMSE tends to saturate due to the bias induced by the ℓ_1-norm approximation for the sparsity. The proposed estimator is superior to the PR estimator and the SPARROW method, even though its error performance is still affected by an intrinsic bias. Note that, in Fig. 1 as the noise power decreases, the regularization parameter difference between the original SPARROW method and the proposed method gets smaller accordingly. In Fig. 2, where SNR is fixed, it is clearly seen that the RMSEs of the proposed estimator have lower values than those of the SPARROW method with the original regularization parameter after both have saturated. By applying the root-MUSIC method to the Toeplitz matrix solution of (7) with the same regularization parameter as the proposed method, the error rate achieves the CRB at higher SNR values or snapshot sizes in Fig. 1 and Fig. 3. In Fig. 3, the SPARROW estimator cannot separate two sources anymore, whereas the proposed estimator still achieves the best estimation quality amongst the compared methods.

B. *Few Snapshots*

This simulation compares the performance of the aforementioned DOA estimators in the scenario where only a few snapshots are available. Two uncorrelated sources are placed at -30° and 30°. The number of snapshots is $N = 3$, which is smaller than the array size $M = 6$. For the proposed method, we choose $C_\lambda = 0.4$. We exclude 2% of the runs with the largest estimation errors before calculating RMSE. In Fig. 4, the estimation errors of different estimators are depicted as a function of SNR.

It can be observed in Fig. 4 that the proposed method is superior to the PR estimator and the original SPARROW estimator. It has the lowest error rates and converges to the CRB at around SNR = 2 dB.

The performance differences in these two simulations lead to the conclusion that the proposed estimator exhibits superior robustness with respect to highly-correlated source signals and very low snapshots and achieves an improved error performance and frequency resolution. We remark that in both simulations, other PR estimators than PR-UCF are also tested under the proposed framework and they demonstrate comparable or worse estimation performance than the SP-PR-UCF estimator. For the sake of conciseness, we do not present them here. Finally, the procedure of applying the root-MUSIC method to the Toeplitz matrix solution of (7) with the same regularization parameter as the proposed method achieves the CRB at higher SNR value or snapshot size compared to the proposed estimator. Moreover, it is only applicable to uniform linear arrays. The proposed method, in comparison, can be extended to other array geometries, e.g., fully augmentable arrays, by employing the variant of gridless SPARROW derived with a co-array [15, (17)].

VI. Conclusion and Future Work

In this paper, we propose a method that integrates the sparse signal reconstruction techniques under the partial relaxation framework. Simulation results in two difficult setups with highly-correlated sources and very few snapshots show that

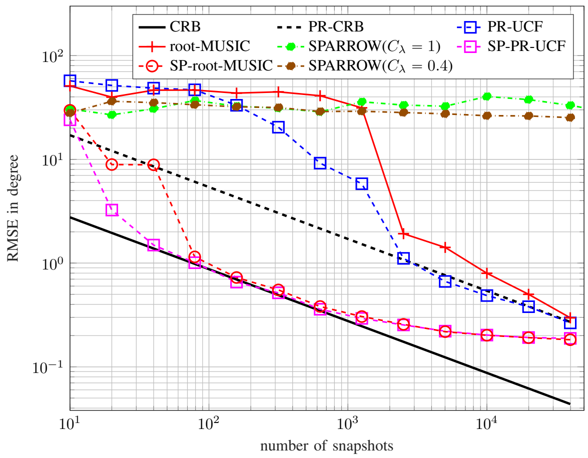

Fig. 3. RMSE vs N for correlated sources with SNR= 3dB.

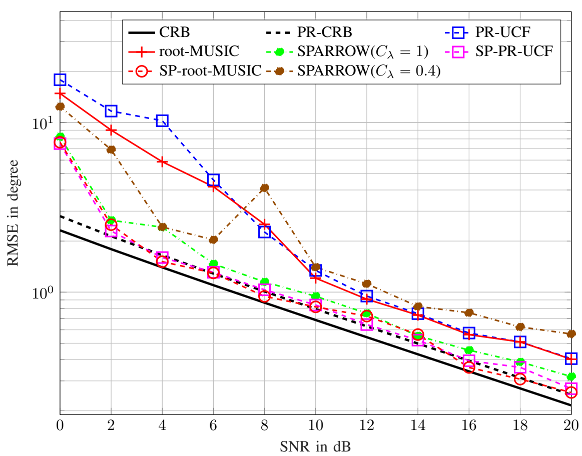

Fig. 4. RMSE vs SNR for uncorrelated sources with $M = 6$ and $N = 3$.

the proposed method alleviates both the intrinsic bias of sparse signal reconstruction techniques and the weakness of the partial relaxation estimators and achieves an improved DOA estimation performance. Moreover, the proposed method possesses an enhanced frequency resolution compared to sparse signal reconstruction techniques. We remark that the design of such an estimator can be easily generalized from the uniform linear array to other array geometries, such as partially shift-invariant arrays and partially augmentable arrays, by incorporating the appropriate SPARROW formulations [23].

The method we propose combines the SPARROW estimator and the PR framework in a separate "two-step" manner. For future work, it is of great interest to search for an estimator that achieves the goal in a "one-step" manner. Furthermore, we choose the regularization parameter based on the original heuristic formula reported in [10]. By experimentally designating a heuristic formula for the proposed estimator, we could potentially further improve the estimation performance.

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