Polynomial Procrustes Problem: Paraunitary Approximation of Matrices of Analytic Functions

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***Abstract*—In the narrowband case, the best least squares approximation of a matrix by a unitary one is given by the Procrustes problem. In this paper, we expand this idea to matrices of analytic functions, and characterise a broadband equivalent to the narrowband case: the polynomial Procrustes problem. Its solution is based on an analytic singular value decomposition, and for the case of spectrally majorised, distinct singular values, we demonstrate the application of a suitable algorithm to three problems — time delay estimation, paraunitary matrix completion, and general paraunitary approximations — in simulations.**

I. Introduction

Matrices of transfer functions, dependent on the complex variable z, occur in a number of applications, such as broadband MIMO systems in telecommunications [1], as polyphase analysis and synthesis matrices when describing filter bank systems in signal processing [2], [3], or reverberation filters and scattering matrices in multichannel audio [4], [5]. A special role amongst those matrices play so-called paraunitary matrices $\boldsymbol{Q}(z) : \mathbb{C} \rightarrow \mathbb{C}^{M\times M}$, such that with the parahermitian operation $\boldsymbol{Q}^{\rm P}(z) = \{\boldsymbol{Q}(1/z^*)\}^{\rm H}$ we have

$$\boldsymbol{Q}(z)\,\boldsymbol{Q}^{\rm P}(z) = \boldsymbol{Q}^{\rm P}(z)\,\boldsymbol{Q}(z) = \mathbf{I}\;. \qquad (1)$$

A paraunitary matrix describes a lossless system. If for a filter bank, e.g. the polyphase analysis matrix $\boldsymbol{H}(z)$ satisfied the paraunitary property, then perfect reconstruction is possible with a system employing a polyphase synthesis matrix $\boldsymbol{G}(z) = \boldsymbol{H}^{-1}(z) = \boldsymbol{H}^{\rm P}(z)$.

To exploit the benefits of paraunitarity, in a number of cases we would like to create a matrix that is paraunitary from one that initially is not. This may be in order to obtain a simpler system inverse as in the case of lossless filter banks, either for signal analysis [3] or coding [6]. In a broadband generalised sidelobe cancelling beamformer, the system design hinges on the identification of the nullspace of a polynomial constraint equation, which requires the completion of a paraunitary matrix [7]. In audio effect processing, the estimation of a lossless scattering matrix from measurements perturbs the system to an extent such that the expected paraunitarity [4] is denied.

In the narrowband case, a unitary matrix closest to an arbitrary matrix in the least-squares sense is given by the Procrustes problem. Therefore, in this paper we want to explore utilising polynomial matrix techniques in order to create a paraunitary matrix that in the least squares sense is closest to some given polynomial matrix, or more generally is closest to a matrix of functions that are analytic in $z \in \mathbb{C}$. We term this the polynomial Procrustes problem.

In order to address the polynomial Procrustes problem, Sec. II reviews the standard Procrustes approach, which in Sec. III is extended to matrices of functions. Following its implementation in Sec. IV, Sec. V will demonstrate the application to time delay estimation, paraunitary matrix completion, and general paraunitary approximations before Sec. VI draws conclusions.

II. Best Approximation in the Narrowband Case

The Procrustes problem seeks a unitary operator $\mathbf{Q}_* \in \mathbb{C}^{M\times M}$ to rotate a matrix $\mathbf{A} \in \mathbb{C}^{N\times M}$ to match another matrix $\mathbf{B} \in \mathbb{C}^{N\times M}$ as best as possible in the least squares sense. This can be formulated as [8]

$$\mathbf{Q}_* = \arg\min_{\mathbf{Q}} \|\mathbf{A}\mathbf{Q} - \mathbf{B}\|_{\rm F}^2\;, \quad \text{s.t.} \quad \mathbf{Q}\mathbf{Q}^{\rm H} = \mathbf{I}\;, \qquad (2)$$

where $\|\cdot\|_{\rm F}$ is the Frobenius norm. With the trace operator ${\rm tr}\{\cdot\}$, for the cost term $\xi = \|\mathbf{A}\mathbf{Q} - \mathbf{B}\|_{\rm F}^2$ we have

$$\begin{aligned}\xi &= {\rm tr}\big\{(\mathbf{Q}^{\rm H}\mathbf{A}^{\rm H} - \mathbf{B}^{\rm H})(\mathbf{A}\mathbf{Q} - \mathbf{B})\big\} & (3)\\ &= {\rm tr}\big\{\mathbf{A}\mathbf{A}^{\rm H}\big\} + {\rm tr}\big\{\mathbf{B}\mathbf{B}^{\rm H}\big\} - 2{\rm Re}\big\{{\rm tr}\big\{\mathbf{Q}^{\rm H}\mathbf{A}^{\rm H}\mathbf{B}\big\}\big\}\;, & (4)\end{aligned}$$

where the rule ${\rm tr}\{\mathbf{ABC}\} = {\rm tr}\{\mathbf{CAB}\}$ has been exploited.

The minimisation of ξ is equivalent to the maximisation of ${\rm Re}\big\{{\rm tr}\big\{\mathbf{Q}\mathbf{A}^{\rm H}\mathbf{B}\big\}\big\}$, where we insert the singular value decomposition $\mathbf{A}^{\rm H}\mathbf{B} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\rm H}$,

$${\rm Re}\big\{{\rm tr}\big\{\mathbf{Q}^{\rm H}\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\rm H}\big\}\big\} = {\rm Re}\big\{{\rm tr}\big\{\mathbf{V}^{\rm H}\mathbf{Q}^{\rm H}\mathbf{U}\boldsymbol{\Sigma}\big\}\big\} \leq {\rm tr}\{\boldsymbol{\Sigma}\}\,. \qquad (5)$$

This step is possible because the singular values are real and non-negative. Equality in (5) is then accomplished by ensuring that for the product of unitary matrices, we have $\mathbf{V}^{\rm H}\mathbf{Q}_*^{\rm H}\mathbf{U} = \mathbf{I}$. This is achieved by setting

$$\mathbf{Q}_* = \mathbf{U}\mathbf{V}^{\rm H}\;. \qquad (6)$$

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It is straightforward to show that (6) also solves the problem

$$\mathbf{Q}_{*} = \arg\min_{\mathbf{Q}} \left\|\mathbf{Q} - \mathbf{A}^{\mathrm{H}}\mathbf{B}\right\|_{\mathrm{F}}^{2} \ , \quad \text{s.t.} \quad \mathbf{Q}\mathbf{Q}^{\mathrm{H}} = \mathbf{I} \ , \tag{7}$$

i.e. the challenge of finding the best unitary approximation of a matrix $\mathbf{A}^{\mathrm{H}}\mathbf{B}$ in the least squares sense.

III. Polynomial Procrustes Problem

A. Formulation

For the case of $N \times M$ matrices $\boldsymbol{A}(z), \boldsymbol{B}(z)$ that are analytic functions in $z \in \mathbb{C}$, the aim is to find a paraunitary matrix $\boldsymbol{Q}(z) : \mathbb{C} \rightarrow \mathbb{C}^{M\times M}$ that extends the classic Procrustes problem to the broadband case. To formulate a least squares fit in the polynomial domain, analogously to [7] we can write

$$\boldsymbol{Q}_{*}(z) = \arg\min_{\boldsymbol{Q}(z)} \oint_{|z|=1} \left\|\boldsymbol{A}(z)\,\boldsymbol{Q}(z) - \boldsymbol{B}(z)\right\|_{\mathrm{F}}^{2} \frac{\mathrm{d}z}{z} \tag{8}$$

$$\text{s.t.} \quad \boldsymbol{Q}(z)\,\boldsymbol{Q}^{\mathrm{P}}(z) = \mathbf{I} \ . \tag{9}$$

Since we are also looking for a paraunitary matrix $\boldsymbol{Q}(z)$ that is analytic in z, we are dealing exclusively with analytic functions and can restrict our investigation below to the unit circle, i.e. to $z = \mathrm{e}^{\mathrm{j}\Omega}$, where Ω is the normalised angular frequency parameter.

Following (3), for the term $\xi(\Omega) = \left\|\boldsymbol{A}(\mathrm{e}^{\mathrm{j}\Omega})\,\boldsymbol{Q}(\mathrm{e}^{\mathrm{j}\Omega}) - \boldsymbol{B}(\mathrm{e}^{\mathrm{j}\Omega})\right\|_{\mathrm{F}}^{2}$ we have

$$\begin{aligned}\xi(\Omega) = \mathrm{tr}\big\{\mathbf{A}(\mathrm{e}^{\mathrm{j}\Omega})\mathbf{A}^{\mathrm{H}}(\mathrm{e}^{\mathrm{j}\Omega})\big\} + \mathrm{tr}\big\{\mathbf{B}(\mathrm{e}^{\mathrm{j}\Omega})\mathbf{B}^{\mathrm{H}}(\mathrm{e}^{\mathrm{j}\Omega})\big\} \\ - 2\mathrm{Re}\Big\{\mathrm{tr}\big\{\boldsymbol{Q}^{\mathrm{H}}(\mathrm{e}^{\mathrm{j}\Omega})\boldsymbol{A}^{\mathrm{H}}(\mathrm{e}^{\mathrm{j}\Omega})\boldsymbol{B}(\mathrm{e}^{\mathrm{j}\Omega})\big\}\Big\} \ .\end{aligned} \tag{10}$$

Therefore, minimising (8) is equivalent to maximising

$$\xi = \frac{1}{2\pi}\int_{0}^{2\pi} \mathrm{Re}\Big\{\mathrm{tr}\big\{\boldsymbol{Q}^{\mathrm{H}}(\mathrm{e}^{\mathrm{j}\Omega})\boldsymbol{A}^{\mathrm{H}}(\mathrm{e}^{\mathrm{j}\Omega})\boldsymbol{B}(\mathrm{e}^{\mathrm{j}\Omega})\big\}\Big\}\,\mathrm{d}\Omega \ . \tag{11}$$

To solve this optimisation problem analogously to the narrowband case with (5), we require the SVD of the term $\boldsymbol{A}^{\mathrm{H}}(\mathrm{e}^{\mathrm{j}\Omega})\boldsymbol{B}(\mathrm{e}^{\mathrm{j}\Omega})$. This is addressed by the analytic version of the SVD, which is capable of providing a frequency-dependent decomposition. To be valid beyond the unit circle, we are seeking a decomposition with analytic factors, which we discuss next.

B. Analytic Singular Value Decomposition

Given a matrix $\boldsymbol{R}(z) : \mathbb{C} \rightarrow \mathbb{C}^{M\times M}$ that is analytic in z, then unless $\boldsymbol{R}(z)$ is connected to a multiplexing operation or afflicted by a condition described below, there exists a singular value decomposition

$$\boldsymbol{R}(z) = \boldsymbol{U}(z)\boldsymbol{\Sigma}(z)\,\boldsymbol{V}^{\mathrm{P}}(z) \tag{12}$$

with analytic factors [9], [10]. The matrices $\boldsymbol{U}(z) : \mathbb{C} \rightarrow \mathbb{C}^{M\times M}$ and $\boldsymbol{V}(z) : \mathbb{C} \rightarrow \mathbb{C}^{M\times M}$ are paraunitary, and $\boldsymbol{\Sigma}(z) : \mathbb{C} \rightarrow \mathbb{C}^{M\times M}$ is diagonal and real on the unit circle, such that $\boldsymbol{\Sigma}(\mathrm{e}^{\mathrm{j}\Omega}) \in \mathbb{R}$.

Within $\boldsymbol{\Sigma}(z) = \mathrm{diag}\{\sigma_1(z), \ldots, \sigma_M(z)\}$, the singular values $\sigma_m(z)$, $m = 1, \ldots, M$ are unique up to an ordering. Note that different to the ordinary SVD, where the singular values typically appear sorted in descending order, analytic singular values may intersect such that the same majorisation is not necessarily meaningful. As a major difference to the ordinary SVD, it can be necessary to let singular values $\sigma_m(\mathrm{e}^{\mathrm{j}\Omega})$ become negative in order for $\sigma_m(z)$ to be analytic [10]–[12]. The zero-crossings that are responsible for such a sign change may, if of odd order, also necessitate an oversampling by a factor of two, such that an analytic SVD can be found for $\boldsymbol{R}(z^2)$ while it does not exist for $\boldsymbol{R}(z)$ [10].

The M left- and right-singular vectors in $\boldsymbol{U}(z)$ and $\boldsymbol{V}(z)$ of (12) are unique up to arbitrary allpass functions, but there is a coupling of this ambiguity across the two matrices [10]: if $\boldsymbol{U}(z)$ and $\boldsymbol{V}(z)$ contain valid left- and right-singular vectors, then so do $\boldsymbol{U}(z)\boldsymbol{\Psi}(z)$ and $\boldsymbol{V}(z)\boldsymbol{\Psi}(z)$, where $\boldsymbol{\Psi}(z) = \mathrm{diag}\{\psi_1(z), \ldots, \psi_M(z)\}$ with $\psi_m(z)$, $m = 1, \ldots, M$, representing arbitrary allpass functions.

C. Best Paraunitary Approximation

We now exploit the analytic SVD in (12) to solve the polynomial Procrustes problem, i.e. the maximisation of (11) subject to (9). For this, with analytic $\boldsymbol{A}^{\mathrm{P}}(z)\boldsymbol{B}(z)$, we obtain the SVD

$$\boldsymbol{A}^{\mathrm{P}}(z)\boldsymbol{B}(z) = \boldsymbol{U}(z)\boldsymbol{\Sigma}(z)\,\boldsymbol{V}^{\mathrm{P}}(z) \ , \tag{13}$$

where $\boldsymbol{U}(z)$ and $\boldsymbol{V}(z)$ are $M \times M$ paraunitary matrices, and $\boldsymbol{\Sigma}(z) = \mathrm{diag}\{\sigma_1(z), \ldots, \sigma_M(z)\}$ is an $M \times M$ diagonal matrix that is real-valued on the unit circle, such that $\boldsymbol{\Sigma}(\mathrm{e}^{\mathrm{j}\Omega}) \in \mathbb{R}$.

For the maximisation of (11), with the insertion of (13), we obtain

$$\begin{aligned}\xi &= \frac{1}{2\pi}\int_{0}^{2\pi} \mathrm{Re}\Big\{\mathrm{tr}\big\{\boldsymbol{Q}^{\mathrm{H}}(\mathrm{e}^{\mathrm{j}\Omega})\,\boldsymbol{U}(\mathrm{e}^{\mathrm{j}\Omega})\boldsymbol{\Sigma}(\mathrm{e}^{\mathrm{j}\Omega})\,\boldsymbol{V}^{\mathrm{H}}(\mathrm{e}^{\mathrm{j}\Omega})\big\}\Big\}\,\mathrm{d}\Omega \\ &= \frac{1}{2\pi}\int_{0}^{2\pi} \mathrm{Re}\Big\{\mathrm{tr}\big\{\boldsymbol{V}^{\mathrm{H}}(\mathrm{e}^{\mathrm{j}\Omega})\boldsymbol{Q}^{\mathrm{H}}(\mathrm{e}^{\mathrm{j}\Omega})\,\boldsymbol{U}(\mathrm{e}^{\mathrm{j}\Omega})\boldsymbol{\Sigma}(\mathrm{e}^{\mathrm{j}\Omega})\big\}\Big\}\,\mathrm{d}\Omega \\ &= \frac{1}{2\pi}\int_{0}^{2\pi} \mathrm{tr}\big\{\mathrm{Re}\big\{\boldsymbol{P}(\mathrm{e}^{\mathrm{j}\Omega})\big\}\,\boldsymbol{\Sigma}(\mathrm{e}^{\mathrm{j}\Omega})\big\}\mathrm{d}\Omega \ ,\end{aligned} \tag{14}$$

where $\boldsymbol{P}(z) = \boldsymbol{V}^{\mathrm{P}}(z)\,\boldsymbol{Q}^{\mathrm{P}}(z)\,\boldsymbol{U}(z)$ is paraunitary. As long as the singular values are non-negative, i.e. $\boldsymbol{\Sigma}(\mathrm{e}^{\mathrm{j}\Omega})$ is positive semidefinite for all values of Ω, (14) is maximised for $\boldsymbol{P}(z) = \mathbf{I}$, such that the optimal paraunitary matrix

$$\boldsymbol{Q}_{*}(z) = \boldsymbol{U}(z)\,\boldsymbol{V}^{\mathrm{P}}(z) \tag{15}$$

is obtained from the analytic left- and right-singular vectors of the matrix product $\boldsymbol{A}^{\mathrm{P}}(z)\boldsymbol{B}(z)$.

In case $\boldsymbol{\Sigma}(\mathrm{e}^{\mathrm{j}\Omega})$ is not positive semidefinite, the optimisation of (14) requires some additional consideration. We focus on the contribution of some singular value $\sigma_m(\mathrm{e}^{\mathrm{j}\Omega})$; assume that for reasons of analyticity, $\sigma_m(\mathrm{e}^{\mathrm{j}\Omega})$ is negative for $\Omega \in [\Omega_1, \Omega_2]$. It is possible to create a non-negative singular value

$$\sigma'_m(\mathrm{e}^{\mathrm{j}\Omega}) = \left\{ \begin{array}{ll} \sigma_m(\mathrm{e}^{\mathrm{j}\Omega}), & \Omega \in [0; 2\pi]/[\Omega_1, \Omega_2] \\ -\sigma_m(\mathrm{e}^{\mathrm{j}\Omega}), & \Omega \in [\Omega_1, \Omega_2] \ , \end{array} \right. \tag{16}$$

which however is only piecewise analytic and non-differentiable in the points Ω_1 and Ω_2. The sign change must also be absorbed into either the corresponding left- or right-singular vector. Let us assume that this sign change is applied to the left-sigular vector, such that

$$\boldsymbol{u}'_m(\mathrm{e}^{\mathrm{j}\Omega}) = \begin{cases} \boldsymbol{u}_m(\mathrm{e}^{\mathrm{j}\Omega}), & \Omega \in [0; 2\pi]/[\Omega_1, \Omega_2] \\ -\boldsymbol{u}_m(\mathrm{e}^{\mathrm{j}\Omega}), & \Omega \in [\Omega_1, \Omega_2] \ . \end{cases} \tag{17}$$

Again, $\boldsymbol{u}'_m(\mathrm{e}^{\mathrm{j}\Omega})$ is only piecewise analytic, but due to $\|\boldsymbol{u}'_m(\mathrm{e}^{\mathrm{j}\Omega})\|_2 = 1 \ \forall\Omega$ will now include discontinuities at the frequency points Ω_1 and Ω_2.

Since functions with non-differentiabilities or discontinuities are difficult to approximate, we here focus on maintaining analyticity for the SVD factors. Generally, for the type of approximation problems, we assume that the differences between $\boldsymbol{A}(z)$ and $\boldsymbol{B}(z)$ may be small, such that the probability of a sign change in the singular values of $\boldsymbol{A}^{\mathrm{P}}(z)\boldsymbol{B}(z)$ is negligible. Alternatively, we can, under the constraint of analyticity, demand that a frequency-independent sign-change $\gamma_m \in \{\pm 1\}$ be applied to the singular values, such that

$$\int_0^{2\pi} \gamma_m \sigma_m(\mathrm{e}^{\mathrm{j}\Omega})\mathrm{d}\Omega \geq 0 \tag{18}$$

is satisfied for $m = 1, \ldots, M$. This frequency-independent sign change γ_m must also be absorbed into either the corresponding left- or right-singular vector. The frequency-independence of these sign changes guarantees that analyticity is maintained, and (15) can be amended as

$$\boldsymbol{Q}_*(z) = \boldsymbol{U}(z)\mathrm{diag}\{\gamma_1, \ldots, \gamma_M\} \ \boldsymbol{V}^{\mathrm{P}}(z) \tag{19}$$

to provide the solution to the polynomial Procrustes problem.

IV. Procrustes Algorithm Implementation

A. Assumptions

We assume wanting to find a paraunitary approximation for a square $M \times M$ polynomial matrix $\boldsymbol{R}(z)$, and assume that the analytic SVD $\boldsymbol{R}(z) = \boldsymbol{U}(z)\boldsymbol{\Sigma}(z)\,\boldsymbol{V}^{\mathrm{P}}(z)$ exists. For $\boldsymbol{\Sigma}(z) = \mathrm{diag}\{\sigma_1(z), \ldots, \sigma_M(z)\}$, we further assume that the singular values are distinct, such that on the unit circle, $\sigma_m(\mathrm{e}^{\mathrm{j}\Omega}) = \sigma_\mu(\mathrm{e}^{\mathrm{j}\Omega}) \ \forall\Omega$ holds only if $m = \mu$, with $m, \mu = 1, \ldots, M$, and spectrally majorised, such that

$$\sigma_1(\mathrm{e}^{\mathrm{j}\Omega}) \geq \sigma_2(\mathrm{e}^{\mathrm{j}\Omega}) \geq \ldots \geq \sigma_M(\mathrm{e}^{\mathrm{j}\Omega}) \ , \qquad \forall\Omega \ . \tag{20}$$

This is a realistic assumption in case $\boldsymbol{R}(z)$ is estimated, as its singular values will not intersect with probability one [13].

B. Analytic SVD in the DFT Domain

For a sufficiently large DFT size K, we determine the SVD in every frequency bin Ω_k of $\boldsymbol{R}(\mathrm{e}^{\mathrm{j}\Omega_k})$, such that

$$\boldsymbol{R}(\mathrm{e}^{\mathrm{j}\Omega_k}) = \mathbf{U}_k \boldsymbol{\Sigma}_k \mathbf{V}_k^{\mathrm{H}} \ , \tag{21}$$

where $\boldsymbol{\Sigma}_k = \mathrm{diag}\{\sigma_{1,k}, \ldots \sigma_{M,k}\}$ contains the M majorised singular values. Due to this spectral majorisation and the uniqueness of the distinct singular values [10], we know that $\boldsymbol{\Sigma}_k = \boldsymbol{\Sigma}(\mathrm{e}^{\mathrm{j}\Omega_k})$. Therefore, both the singular values as well as their corresponding left- and right-singular vectors in (21) are correctly ordered across DFT bins. This avoids having to work out bin-dependent permutations such that left- and right-singular vectors are correctly associated in order to re-establish the spectral coherence that is otherwise denied to a frequency-bin approach [14], [15].

In contrast to the uniqueness of the analytic singular values, their corresponding left- and right-singular vectors possess an ambiguity w.r.t. arbitrary allpass filters, which however are coupled [10]. Thus, with given $\boldsymbol{Q}(z)$ and $\boldsymbol{V}(z)$, $\boldsymbol{U}(z)\boldsymbol{\Phi}(z)$ and $\boldsymbol{V}(z)\boldsymbol{\Phi}(z)$ with $\boldsymbol{\Phi}(z) = \mathrm{diag}\{\varphi_1(z), \ldots, \varphi_M(z)\}$ comprising of arbitrary allpass filters $\varphi_m(z)$, $m = 1, \ldots, M$, also contain valid left- and right-singular vectors. Due to the coupling of the ambiguity across both vectors, in the product $\boldsymbol{U}(z)\,\boldsymbol{V}^{\mathrm{P}}(z)$ the allpass filters cancel out, and the analytic SVD thus presents a unique solution to the polynomial Procrustes problem.

With the phase ambiguity to the left- and right-singular vectors removed, it is therefore possible to determine the solution to (19) in each frequency bin, such that with (21), we have $\mathbf{Q}_k = \mathbf{U}_k \mathbf{V}_k^{\mathrm{H}}$, $k = 0, \ldots, (K-1)$. Provided that the DFT length K suffices, we can obtain the coefficients of $\boldsymbol{Q}(z)$ via an inverse DFT of $\mathbf{Q}_k$.

C. Procrustes and Order Trimming

Even if the matrix $\boldsymbol{R}(z)$ is of finite order, the analytic SVD in (12) can have factors of infinite length [10], [16], [17]. Thus, despite the cancellation of the allpass-ambiguity of the singular vectors, the product $\boldsymbol{U}(z)\,\boldsymbol{V}^{\mathrm{P}}(z)$ can potentially have infinite support. Since the product shares the analyticity of its factors, the coefficients of $\boldsymbol{Q}_*(z)$ are absolutely convergent, and the best approximation by a polynomial in the least squares sense is achieved by truncation. Hence trimming any small outer coefficients of $\boldsymbol{Q}_*(z)$ that fall below a given threshold can help to obtain a finite order solution $\hat{\boldsymbol{Q}}_*(z)$ that arbitrarily closely approximates the ground truth.

V. Application Examples and Simulations

To demonstrate the polynomial Procrustes problem, we first consider a simple 1-d example related to time delay estimation, followed by a 2-d case of paraunitary matrix completion and higher-dimensional simulations over an ensemble of matrices with known ground truth.

A. Time-Delay Estimation

Estimating a potentially fractional delay between two signals $x[n]$ and $y[n]$ has been a challenge addressed over decades, see e.g. [18]–[21], that is confounded if the signals possess a strong lowpass auto-correlation. Typically a time delay, particularly in the presence of noise, can be determined from the cross-correlation $r_{yx}[\tau] = \mathcal{E}\{y[n]x^*[n-\tau]\}$, which is equivalent to matched filtering [22]. The delay can then be determined by a maximum search over $r_{yx}[\tau]$. As an example, Fig. 1 shows both the auto-correlation $r_{xx}[\tau]$ of $x[n]$, and the cross-correlation with $y[n]$ delayed by 7.3 samples with respect to $x[n]$ [23]. Due to the lowpass nature of $x[n]$, the maximum search for $r_{yx}[\tau]$ is difficult due to the ill-defined peak of the function, as seen in the inset of Fig. 1.

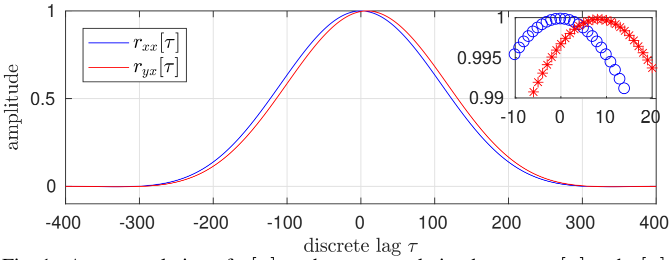

Fig. 1. Auto-correlation of $x[n]$, and cross-correlation between $y[n]$ and $x[n]$.

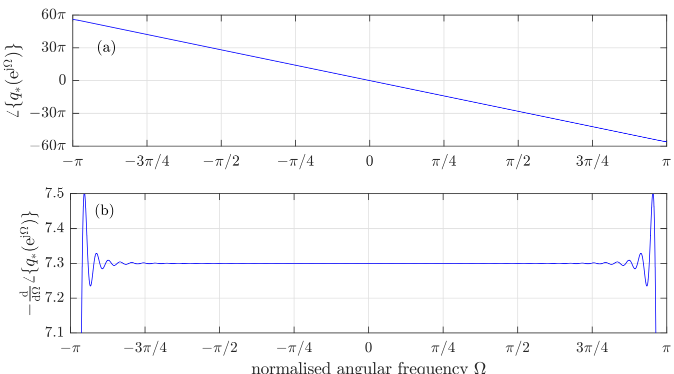

Fig. 2. (a) angle of $q_*(\mathrm{e}^{\mathrm{j}\Omega})$, and (b) extraction of delay by differentiation.

For finite-length sequences, we set $a[n] = x[n]$ and $b[n] = y[n]$. With the respective z-transforms $a(z)$ and $b(z)$, we want to determine the time delay by minimising $\oint_{|z|=1} |a(z)q(z) - b(z)|_2^2 \frac{\mathrm{d}z}{z}$. Thus determining the fractional delay filter $q(z)$ is equivalent to an $M = 1$-d Procrustes problem, where we need to obtain the analytic SVD of

$$r(z) = a^{\mathrm{P}}(z)b(z) = u(z)\sigma(z)v^{\mathrm{P}}(z) \ , \tag{22}$$

with $r(z)$ being a cross-correlation estimate. In (22), $\sigma(z)$ is real on the unit circle, such that $\sigma(\mathrm{e}^{\mathrm{j}\Omega}) \in \mathbb{R}$, and $u(z)$ and $v(z)$ are allpass functions. Without loss of generality, we set $v(z) = 1$ and assume that the entire allpass functionality is provided by $u(z)$. Thus, we have $q_*(z) = \gamma u(z)$. The parameter γ applies a sign change if necessitated by (18).

For the above example, Fig. 2(a) shows the phase response of the identified fractional delay filter $q_*(z)$, from which by differentiation, a group delay-type response is extracted in Fig. 2(b). The delay of 7.3 samples can be extracted fairly precisely from this group delay by integrating over the lowpass band where the signal is active. Inspecting the group delay is known from time delay estimation [18], but places it in the context of the polynomial Procrustes problem.

B. Paraunitary Matrix Completion

As an example for paraunitary matrix completion, we assume that for a matrix $\boldsymbol{R}(z) : \mathbb{C} \to \mathbb{C}^{2\times 2}$, the first column is given by the lowpass filter of a Daubechies-2 wavelet [24]; the unknown second column is set to zero. Using the approach in Sec. IV, we want to obtain a near-paraunitary matrix $\hat{\boldsymbol{Q}}_*(z)$, whereby the reconstruction error

$$e_{\mathrm{r}} = \oint_{|z|=1} \| \hat{\boldsymbol{Q}}_*(z) \hat{\boldsymbol{Q}}_*^{\mathrm{P}}(z) - \mathbf{I} \|_{\mathrm{F}}^2 \frac{\mathrm{d}z}{z} \ , \tag{23}$$

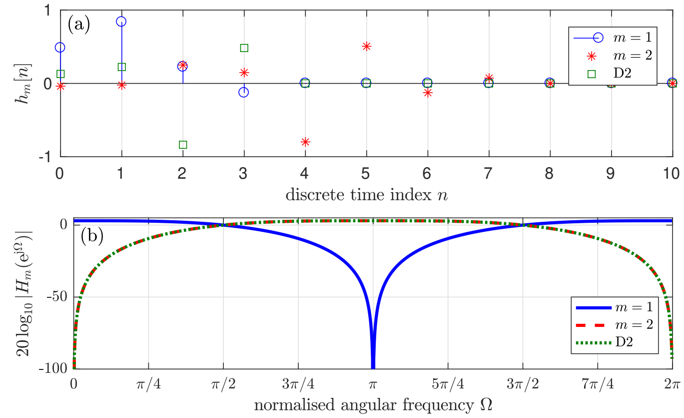

Fig. 3. Paraunitary matrix completion problem with (a) impulse responses and (b) magnitude responses of the two extracted filters; for comparison, D2 marks the highpass filter of the Daubechies-2 filter bank.

measures the precision with which the constraint in the Procrustes problem in (9) is satisfied.

Solving the polynomial Procrustes problem, we measure a reconstruction error $10\log_{10} e_{\mathrm{r}} = -308$ dB, which is close to machine accuracy. Interpreting $\hat{\boldsymbol{Q}}_*(z)$ as a polyphase matrix, we can extract two filters $h_m[n]$, $m = 1, 2$, from its columns; their impulse and magnitude responses are shown in Fig. 3. While the filter $h_1[n]$ matches the lowpass filter of the Daubechies-2 wavelet, $h_2[n]$ does not satisfy the Daubechies-2 solution in Fig. 3(a), which is a quadrature mirror filter to the lowpass system. In Fig. 3(b) we see that the magnitude response of $h_2[n]$ matches the Daubechies-2 solution. Because the second column of $\boldsymbol{R}(z)$ was set to zero, $h_2[n]$ needs to satisfy orthogonality, but is permitted to differ by an arbitrary allpass from other valid solutions including Daubechies-2. In this case, all such solutions have the same distance from $\boldsymbol{R}(z)$.

C. Paraunitary Matrix Recovery

In this section, we investigate a wider and more complex range of paraunitary approximations. In order to assess the solution in each case, we need to know the ground truth solution to the polynomial Procrustes problem. For this, we construct an ensemble of randomised ground truth paraunitary matrices $\boldsymbol{Q}_*(z) = \boldsymbol{Q}_1(z)\boldsymbol{Q}_2(z)$, with $\boldsymbol{Q}_i(z)$, $i = 1, 2$, assembled from random elementary paraunitary operations [3],

$$\boldsymbol{Q}_i(z) = \prod_{\ell=1}^{N} \left((\mathbf{I} - \mathbf{e}_{i,\ell}\mathbf{e}_{i,\ell}^{\mathrm{H}}) + \mathbf{e}_{i,\ell}\mathbf{e}_{i,\ell}^{\mathrm{H}} z^{-1} \right) \ , \tag{24}$$

where $\mathbf{e}_{i,\ell} \in \mathbb{C}^M$, $\ell = 1, \ldots, N$, $i = 1, 2$, are unit norm vectors such that $\|\mathbf{e}_{i,\ell}\|_2 = 1$ with complex Gaussian distributed entries. The matrix $\boldsymbol{Q}_*(z)$ is perturbed by randomised matrices $\boldsymbol{\Sigma}(z)$ with symmetric and spectrally majorised entries, which can be controlled via a source model [22], [25]. Thus, the matrix $\boldsymbol{R}(z) = \boldsymbol{Q}_1(z)\boldsymbol{\Sigma}(z)\boldsymbol{Q}_2^{\mathrm{P}}(z)$ possesses the matrix $\boldsymbol{Q}_*(z)$ as its closest paraunitary approximation.

For an ensemble simulation, the spatial dimension is varied over the parameters $M \in \{2, 4, 8, 16, 32\}$. The orders of $\boldsymbol{Q}_i(z)$, $i = 1, 2$, are set to $\{2, 4, 8, 16, 32, 64\}$, which determines the order $\mathcal{O}\{\boldsymbol{Q}_*(z)\}$ of the ground truth paraunitary

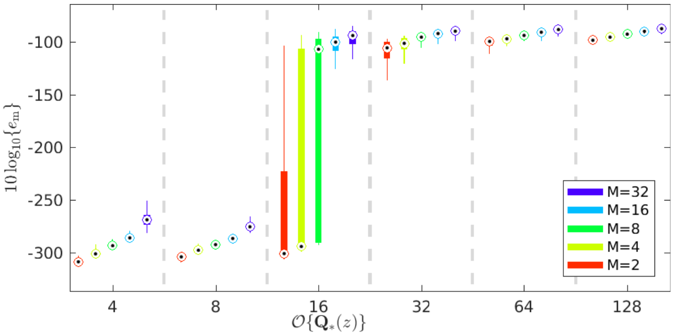

Fig. 4. Box plot of ensemble results for mismatch between the obtained solution and the ground truth, dependent on the order $\mathcal{O}\{\boldsymbol{Q}_*(z)\}$ of the ground truth and the spatial dimension M.

matrix. For each value of M and the orders of $\boldsymbol{Q}_i(z)$, $i = 1, 2$, we generate 100 random instances of $\{\boldsymbol{R}(z), \boldsymbol{Q}_*(z)\}$.

To characterise the mismatch e_{m} between the ideal ground truth paraunitary matrix $\boldsymbol{Q}_*(z)$ and the matrix $\hat{\boldsymbol{Q}}_*(z)$ recovered by the polynomial Procrustes problem, we measure

$$e_{\mathrm{m}} = \oint_{|z|=1} \| \boldsymbol{Q}_*(z) - \hat{\boldsymbol{Q}}_*(z) \|_{\mathrm{F}}^2 \frac{\mathrm{d}z}{z} \; . \qquad (25)$$

As with (23), the metric in (25) can be evaluated in the time domain using Parseval's theorem. Since according to Sec. IV-C, small trailing values in $\hat{\boldsymbol{Q}}_*(z)$ falling below a threshold of 10^{-10} are truncated [26], a shift-alignment is necessary. Note however, that after a similar truncation of trailing values below 10^{-10} was applied to $\boldsymbol{Q}_*(z)$, both the truncated ground truth and the Procrustes solution exhibited the same orders across the entire ensemble.

Over the untrimmed order $\mathcal{O}\{\boldsymbol{Q}_*(z)\}$ of the ground truth, Fig. 4 shows the mismatch e_{m} for different values of M. This metric is close to machine precision for small values of $\mathcal{O}\{\boldsymbol{Q}_*(z)\}$. For larger values of $\mathcal{O}\{\boldsymbol{Q}_*(z)\} \geq 16$, the construction of $\hat{\boldsymbol{Q}}_*(z)$ leads to the truncation of small trailing values, which causes a loss in precision in the results around the truncation value of 10^{-10}. Nonetheless, the size of the mismatch error e_{m} highlights that the correct paraunitary ground truth matrix is recovered.

VI. Conclusions

We have investigated the challenge of finding the best least-squares fit of a paraunitary matrix to a given matrix of analytic functions. For this purpose, we have extended the narrowband Procrustes problem, which is based on the factorisation afforded by an analytic SVD. The restriction here has been to square matrices with spectrally majorised, distinct singular values. We have shown that a unique solution to the polynomial Procrustes problem exists, for which we have presented an algorithmic implementation.

In simulations, we have demonstrated the approach in the domains of time delay estimation, filter bank completion, and paraunitary matrix recovery. Given a sufficient DFT size, the achievable precision can be close to machine accuracy, or to the level of truncation if trimming is applied to control the polynomial order of the matrices.

For further investigation, there remains the case of multiple identical, in particular zero singular values, the case of general rectangular matrices where singular vectors may be arbitrary within some subspace, and the application to problems where benchmark techniques exist for comparison.

References

[1] A. Scaglione, P. Stoica, S. Barbarossa, G.B. Giannakis, and H. Sampath, "Optimal designs for space-time linear precoders and decoders," *IEEE Trans. SP*, **50**(5):1051–1064, May 2002.

[2] P.P. Vaidyanathan, "Multirate digital filters, filter banks, polyphase networks, and applications: a tutorial," *Proc. IEEE*, **78**(1):56–93, Jan. 1990.

[3] ——, *Multirate Systems and Filter Banks*. Englewood Cliffs: Prentice Hall, 1993.

[4] S.J. Schlecht and E.A.P. Habets, "Scattering in feedback delay networks," *IEEE/ACM Trans. ASLP*, **28**:1915–1924, 2020.

[5] S.J. Schlecht, "Allpass feedback delay networks," *IEEE Trans. SP*, **69**:1028–1038, 2021.

[6] S. Redif, J. McWhirter, and S. Weiss, "Design of FIR paraunitary filter banks for subband coding using a polynomial eigenvalue decomposition," *IEEE Trans. SP*, **59**(11):5253–5264, Nov. 2011.

[7] S. Weiss, S. Bendoukha, A. Alzin, F. Coutts, I. Proudler, and J. Chambers, "MVDR broadband beamforming using polynomial matrix techniques," in *EUSIPCO*, Nice, France, pp. 839–843, Sep. 2015.

[8] G.H. Golub and C.F. Van Loan, *Matrix computations*, 3rd ed. Baltimore, Maryland: John Hopkins University Press, 1996.

[9] G. Barbarino and V. Noferini, "On the Rellich eigendecomposition of para-Hermitian matrices and the sign characteristics of ∗-palindromic matrix polynomials," *arXiv:2211.15539*, 2022.

[10] S. Weiss, I.K. Proudler, G. Barbarino, J. Pestana, and J.G. McWhirter, "On properties and structure of the analytic singular value decomposition," *IEEE Trans. SP*, to be submitted 2023.

[11] B. De Moor and S. Boyd, "Analytic properties of singular values and vectors," KU Leuven, Tech. Rep., 1989.

[12] A. Bunse-Gerstner, R. Byers, V. Mehrmann, and N. K. Nicols, "Numerical computation of an analytic singular value decomposition of a matrix valued function," *Numer. Math*, **60**:1–40, 1991.

[13] F.A. Khattak, S. Weiss, I.K. Proudler, and J.G. McWhirter, "Space-time covariance matrix estimation: Loss of algebraic multiplicities of eigenvalues," in *Asilomar Conf. SSC*, Pacific Grove, CA, Oct. 2022.

[14] S. Weiss, I.K. Proudler, and F.K. Coutts, "Eigenvalue decomposition of a parahermitian matrix: Extraction of analytic eigenvalues," *IEEE Trans. SP*, **69**:722–737, 2021.

[15] S. Weiss, I. Proudler, F. Coutts, and F. Khattak, "Eigenvalue decomposition of a parahermitian matrix: Extraction of analytic eigenvectors," *IEEE Trans. SP*, **71**:1642–1656, Apr. 2023.

[16] S. Weiss, J. Pestana, and I.K. Proudler, "On the existence and uniqueness of the eigenvalue decomposition of a parahermitian matrix," *IEEE Trans. SP*, **66**(10):2659–2672, May 2018.

[17] S. Weiss, J. Pestana, I.K. Proudler, and F.K. Coutts, "Corrections to 'on the existence and uniqueness of the eigenvalue decomposition of a parahermitian matrix'," *IEEE Trans. SP*, **66**(23):6325–6327, Dec. 2018.

[18] S. L. Marple, "Estimating group delay and phase delay via discrete-time "analytic" cross-correlation," *IEEE Trans. SP*, **47**(9):2604–2607, Sep. 1999.

[19] L. Zhang and X. Wu, "On cross correlation based-discrete time delay estimation," in *ICASSP*, **4**:981–984, Mar. 2005.

[20] L. Svilainis, "Review on time delay estimate subsample interpolation in frequency domain," *IEEE Trans. Ultrason. Ferr.*, **66**(11):1691–1698, Nov. 2019.

[21] H. Rosseel and T. v. Waterschoot, "Improved acoustic source localization by time delay estimation with subsample accuracy," in *Immersive and 3D Audio: from Architecture to Automotive*, pp. 1–8, Sep. 2021.

[22] A. Papoulis, *Probability, random variables, and stochastic processes*, 3rd ed. New York: McGraw-Hill, 1991.

[23] T.I. Laakso, V. Välimäki, M. Karjalainen, and U.K. Laine, "Splitting the unit delay," *IEEE SP Mag.*, **13**(1):30–60, Jan. 1996.

[24] I. Daubechies, *Ten Lectures on Wavelets*. Philadelphia: SIAM, 1992.

[25] S. Redif, S. Weiss, and J. McWhirter, "Sequential matrix diagonalization algorithms for polynomial EVD of parahermitian matrices," *IEEE Trans. SP*, **63**(1):81–89, Jan. 2015.

[26] J. Corr, K. Thompson, S. Weiss, I. Proudler, and J. McWhirter, "Row-shift corrected truncation of paraunitary matrices for PEVD algorithms," in *EUSIPCO*, Nice, France, pp. 849–853, Aug. 2015.