# Robust Adaptive Beamforming with Multiple Signal Mismatch Constraints: A Sequential Convex Approximation Method

Xianlian Lin Yongwei Huang Wenzheng Yang
*School of Information Engineering*
*Guangdong University of Technology*
Guangzhou, China
ywhuang@gdut.edu.cn
{2112003015,2112003058}@mail2.gdut.edu.cn

Jingwei Xu
*School of Electronic Engineering*
*Xidian University*
Xi'an, Shaanxi, China
jwxu@xidian.edu.cn

***Abstract*—A robust adaptive beamforming (RAB) problem of maximizing the worst-case (the minimal) signal-to-interference-plus-noise ratio (SINR) over the union of small uncertainty sets of the desired signal steering vector is formulated and recast into a quadratic minimization problem with nonconvex constraints. In existing works, the semidefinite programming relaxation technique is applied to approximately solve the quadratic problem, incurring heavy computational burden or low array output SINR. Herein, a sequential convex second-order cone programming (SOCP) approximation algorithm is proposed. In particular, an SOCP problem is constructed and solved in each step, and it is shown that the sequence of the optimal values of the SOCPs is nonincreasing and bounded, and the optimal solutions of the SOCPs are feasible for the quadratic problem and converge to a locally optimal solution. Example simulations are performed to demonstrate the improved performance of the proposed algorithm in terms of the beamformer output SINR, as well as the average CPU time and average number of iterations of the algorithm.**

***Index Terms*—Robust adaptive beamforming, worst-case SINR maximization, multiple signal mismatch constraints, sequential SOCP approximation, SDP approximation.**

## I. Introduction

In recent years, robust adaptive beamforming (RAB) techniques have been devised to significantly increase the array output performance and alleviate the array sensitivity caused by the estimation error and uncertainty in the parameters, in terms of, e.g., maximizing the signal-to-interference-plus-noise ratio (SINR). The RAB designs output an optimal/suboptimal complex-valued weight vector (known as a RAB vector), which must efficiently address any imperfect information about the source, propagation, and sensor array [1]–[3]. In particular, the optimal RAB vector established in seminal paper [4] performs well when there is small to medium mismatch between the true desired signal steering vector and the presumed steering vector (see also [3, Sec. 2.8]).

This work was supported partially by the National Natural Science Foundation of China (11871168) and Guangdong Basic and Applied Basic Research Foundation (2022A1515011782).

Among all the modern RAB techniques in the literature, the worst-case optimization based RAB appears to be popular and interesting [4]–[9]. For example, the worst-case SINR (the minimal SINR) over a ball constraint of the error vector was maximized in [4], and the error vector was the difference between the actual desired signal steering vector and the presumed steering vector. Subsequently, the maximin problem was equivalently converted into a second-order cone programming (SOCP) problem, which was solved very efficiently. In addition to the signal steering vector errors, interference nonstationarity, which affects the array data matrix and sample covariance matrix, was considered in [5]. Thus the problem of maximizing the minimal SINR over the constraints of two classes of errors was formulated and solved by reexpressing it as an SOCP problem.

Assuming that the desired signal steering vector and covariance matrix are uncertain and the corresponding uncertainty set is a general convex compact set, the optimal RAB design via the maximization problem of the worst-case SINR over the uncertainty model was studied in [6]. Specifically, it was shown therein that when the uncertainty model can be represented by linear matrix inequalities (LMIs), the worst-case SINR maximization problem can be solved using semidefinite programming (SDP). In contrast, when the uncertainty set of the desired signal steering vector is nonconvex, the maximization problem of the minimal SINR over the nonconvex uncertainty set can be no longer recast into a convex optimization problem [7], but can be rewritten as a quadratic matrix inequality problem, which was solved approximately via a restricted LMI relaxation therein. Besides, there are many other useful RAB techniques (see, e.g., [3]) that we cannot review due to the page limit.

When a large uncertainty set is required to model a significant mismatch between the actual and presumed signal steering vectors, or other types of errors, an optimal RAB vector obtained using the method reported in [4] often causes the array performance deterioration [8], [9]. In the previous both papers, the union of several small uncertainty sets (modelling a small mismatch between the signal steering vectors) was employed to replace the large uncertainty set, with the aim of improving the array performance in terms of the output SINR. Therein, leveraging on the technique detailed in [4], the maximization problem of the minimal SINR over the union of small uncertainty sets was equivalently transformed into a nonconvex quadratic minimization problem with multiple constraints representing the small signal mismatch. A relaxation-restriction-relaxation method was designed in [8] to address the quadratic problem while two iterative SDP approximation algorithms were proposed in [9] to solve the same problem.

In this paper, we present a sequential convex SOCP approximation algorithm for the previous nonconvex quadratic minimization problem studied in [8], [9], where SDP-based approximation algorithms were developed. We claim that the approximation algorithm exhibits better properties compared with the existing SDP-based algorithms. Specifically, (i) the computational complexity of the SOCP in each iterative step is less than that of the SDP in each step of the iterative SDP algorithms; (ii) the feasible region of each SOCP is included in that of the original quadratic minimization problem, which means that any optimal solutions for the SOCPs are feasible for the quadratic problem; in contrast, a rank-one solution for the SDP obtained from a higher-rank solution in the SDP-based algorithms may or may not be feasible for the original quadratic problem; (iii) the sequence consisting of the optimal solutions for the SOCPs converges to a locally optimal solution. In the simulation, it is demonstrated that the beamformers generated by the sequential convex SOCP approximation algorithm and one of the iterative SDP approximation algorithms exhibit equal performances in terms of the output SINR. However the former algorithm appears to be faster in the sense that average number of iterations for solving an instance of the nonconvex quadratic problem is less, along with a shorter average CPU time.

Therefore, the main contribution of this study is twofold: (i) An SOCP-based iterative algorithm is designed to solve the RAB problem with multiple signal mismatch constraints, which is faster than the existing algorithms; (ii) we demonstrate that the algorithm outputs a locally optimal solution for the RAB problem (in fact, the global optimality can be achieved numerically).

## II. Signal Model, Problem Formulation, and Existing Approaches

### A. Signal Model

The array output at instance $t$ is expressed as

$$x(t) = \boldsymbol{w}^H \boldsymbol{y}(t) \tag{1}$$

where $\boldsymbol{w} \in \mathbb{C}^N$ is a complex-valued weight vector (RAB vector), the superscript $(\cdot)^H$ denotes the conjugate transpose, $\boldsymbol{y}(t) \in \mathbb{C}^N$ is the snapshot vector of array observations, and $N$ is the number of antenna elements of the array. In the point signal source case, the observation vector is given by

$$\boldsymbol{y}(t) = s(t)\boldsymbol{a} + \boldsymbol{i}(t) + \boldsymbol{n}(t) \tag{2}$$

where $s(t)\boldsymbol{a}$, $\boldsymbol{i}(t)$, and $\boldsymbol{n}(t)$ are the statistically independent components of the signal of interest (SOI), interference, and noise, respectively. In (2), $s(t)$ is the SOI waveform and $\boldsymbol{a}$ is the target steering vector.

Therefore, the output SINR of the beamformer is given by

$$\text{SINR} = \frac{\sigma_s^2 \boldsymbol{w}^H \boldsymbol{a}\boldsymbol{a}^H \boldsymbol{w}}{\boldsymbol{w}^H \boldsymbol{R}_{i+n} \boldsymbol{w}} \tag{3}$$

where $\sigma_s^2$ is the SOI power and $\boldsymbol{R}_{i+n} \triangleq \mathsf{E}[(\boldsymbol{i}(t)+\boldsymbol{n}(t))(\boldsymbol{i}(t)+\boldsymbol{n}(t))^H]$ is the interference-plus-noise covariance matrix. In practical scenarios, the covariance $\boldsymbol{R}_{i+n}$ is not available. Thus, the sample covariance for $\boldsymbol{R} = \mathsf{E}[\boldsymbol{y}(t)\boldsymbol{y}^H(t)]$:

$$\hat{\boldsymbol{R}} = \frac{1}{T}\sum_{t=1}^{T} \boldsymbol{y}(t)\boldsymbol{y}^H(t) \tag{4}$$

is employed as a compromise. In (4), $T$ represents the number of training snapshots.

### B. Problem Formulation

Often, the true steering vector $\boldsymbol{a}$ is not exactly known in many applications. In other words, there is always a mismatch between the true $\boldsymbol{a}$ and the presumed steering vector $\hat{\boldsymbol{a}}$. However, the beamforming vector obtained by maximizing the SINR (3) with $\boldsymbol{a}$ replaced by $\hat{\boldsymbol{a}}$ can lead to dramatic performance degradation of the array. Therefore, to improve the beamformer performance (e.g., in terms of the output S-INR), a RAB technique must be considered, and the following interesting and popular problem of maximizing the worst-case SINR is adopted herein:

$$\underset{\boldsymbol{w}\neq\boldsymbol{0}}{\mathsf{maximize}} \quad \underset{\boldsymbol{a}\in\mathcal{A}}{\mathsf{minimize}} \quad \frac{\boldsymbol{a}^H \boldsymbol{w}\boldsymbol{w}^H \boldsymbol{a}}{\boldsymbol{w}^H \hat{\boldsymbol{R}} \boldsymbol{w}}, \tag{5}$$

where the uncertainty set $\mathcal{A}$ is the set of all possible actual steering vectors. Clearly, it is equivalent to the following problem:

$$\underset{\boldsymbol{w}\neq\boldsymbol{0}}{\mathsf{minimize}} \ \boldsymbol{w}^H \hat{\boldsymbol{R}} \boldsymbol{w} \ \ \mathsf{subject\ to} \ |\boldsymbol{w}^H \boldsymbol{a}|^2 \geq 1, \ \forall \boldsymbol{a} \in \mathcal{A}, \tag{6}$$

in the sense that if the RAB vector $\boldsymbol{w}^\star$ is optimal for (6), then it is also optimal for (5).

In [4], the uncertainty set

$$\mathcal{A} = \{\boldsymbol{a} \mid \|\boldsymbol{a} - \hat{\boldsymbol{a}}\| \leq \epsilon\} \tag{7}$$

is considered, where $\hat{\boldsymbol{a}}$ is the presumed steering vector with $\|\hat{\boldsymbol{a}}\|^2 = N$, and $\|\cdot\|$ stands for the $l_2$-norm. Then, problem (6) can be reformulated into the SOCP:

$$\underset{\boldsymbol{w}}{\mathsf{minimize}} \ \boldsymbol{w}^H \hat{\boldsymbol{R}} \boldsymbol{w} \ \ \mathsf{subject\ to} \ \Re\{\boldsymbol{w}^H \hat{\boldsymbol{a}}\} \geq \sqrt{\epsilon}\|\boldsymbol{w}\| + 1, \tag{8}$$

where $\Re(\cdot)$ means the real part of a complex-valued argument.

When a large uncertainty set $\mathcal{A}$ is modelled in a practical scenario, where a large direction-of-arrival mismatch along with other types of errors occurs, the RAB problem (6) becomes more conservative (namely, the radius $\epsilon$ is larger and the feasible set of (6) is smaller) such that a solution for (6), i.e., an optimal RAB vector, exhibits worse performance (cf. [8], [9]). In this case, we instead assume that there are multiple possible presumed steering vectors $\hat{\boldsymbol{a}}_m$ and the new uncertainty set is defined by

$$\mathcal{A}' = \cup_{m=1}^{M} \mathcal{A}_m, \tag{9}$$

where

$$\mathcal{A}_m = \{\boldsymbol{a} \mid \|\boldsymbol{a} - \hat{\boldsymbol{a}}_m\| \leq \epsilon_m\},\ m = 1, \ldots, M, \tag{10}$$

with each $\epsilon_m$ smaller than $\epsilon$ (see e.g. Fig. 1 in [8] or [9]), and $M$ is the number of the multiple presumed steering vectors $\hat{\boldsymbol{a}}_m$. It can be observed that the new uncertainty set $\mathcal{A}'$ is nonconvex, which represents the main difficulty in solving the corresponding RAB problem (i.e. problem (6)). To avoid trivial cases, we assume that $\epsilon_m > 0$ for all $m$ throughout the paper.

It should be noted that

$$\begin{aligned} &\{\boldsymbol{w} \mid |\boldsymbol{w}^H \boldsymbol{a}| \geq 1,\ \forall \boldsymbol{a} \in \mathcal{A}'\} \\ &= \cap_{m=1}^{M} \{\boldsymbol{w} \mid |\boldsymbol{w}^H \boldsymbol{a}| \geq 1,\ \forall \boldsymbol{a} \in \mathcal{A}_m\}. \end{aligned} \tag{11}$$

Therefore, substituting $\mathcal{A}'$ into $\mathcal{A}$ in (6), we obtain

$$\begin{aligned} \underset{\boldsymbol{w}}{\text{minimize}} \quad & \boldsymbol{w}^H \hat{\boldsymbol{R}} \boldsymbol{w} \\ \text{subject to} \quad & |\boldsymbol{w}^H \boldsymbol{a}| \geq 1,\ \forall \boldsymbol{a} \in \mathcal{A}_1 \\ & \vdots \\ & |\boldsymbol{w}^H \boldsymbol{a}| \geq 1,\ \forall \boldsymbol{a} \in \mathcal{A}_M. \end{aligned} \tag{12}$$

Using the same technique reported in [4], the previous problem is further tantamount to the following problem:

$$\begin{aligned} \underset{\boldsymbol{w}}{\text{minimize}} \quad & \boldsymbol{w}^H \hat{\boldsymbol{R}} \boldsymbol{w} \\ \text{subject to} \quad & |\boldsymbol{w}^H \hat{\boldsymbol{a}}_m| - 1 \geq \epsilon_m \|\boldsymbol{w}\|,\ m = 1, \ldots, M. \end{aligned} \tag{13}$$

Evidently, the problem cannot be transformed equivalently into the SOCP:

$$\begin{aligned} \underset{\boldsymbol{w}}{\text{minimize}} \quad & \boldsymbol{w}^H \hat{\boldsymbol{R}} \boldsymbol{w} \\ \text{subject to} \quad & \Re(\boldsymbol{w}^H \hat{\boldsymbol{a}}_m) - 1 \geq \epsilon_m \|\boldsymbol{w}\|,\ m = 1, \ldots, M, \end{aligned} \tag{14}$$

which is indeed a convex restriction for (13).

### C. Existing Approaches

In [9], problem (13) is solved by relaxing it into

$$\begin{aligned} \underset{\boldsymbol{w}}{\text{minimize}} \quad & \boldsymbol{w}^H \hat{\boldsymbol{R}} \boldsymbol{w} \\ \text{subject to} \quad & (|\boldsymbol{w}^H \hat{\boldsymbol{a}}_m| - 1)^2 \geq \epsilon_m^2 \|\boldsymbol{w}\|^2,\ m = 1, \ldots, M, \end{aligned} \tag{15}$$

which can be recast into the following problem

$$\begin{aligned} \underset{\boldsymbol{w}}{\text{minimize}} \quad & \boldsymbol{w}^H \hat{\boldsymbol{R}} \boldsymbol{w} \\ \text{subject to} \quad & \boldsymbol{w}^H \hat{\boldsymbol{a}}_m \hat{\boldsymbol{a}}_m^H \boldsymbol{w} = \beta_m \\ & \epsilon_m^2 \boldsymbol{w}^H \boldsymbol{w} \leq \beta_m - 2\sqrt{\beta_m} + 1,\ m = 1, \ldots, M, \end{aligned} \tag{16}$$

and then $\sqrt{\beta_m}$ is linearized and the problem is relaxed into the SDP:

$$\begin{aligned} \underset{\boldsymbol{W}, \{\beta_m\}}{\text{minimize}} \quad & \text{tr}\,(\hat{\boldsymbol{R}} \boldsymbol{W}) \\ \text{subject to} \quad & \text{tr}\,(\hat{\boldsymbol{a}}_m \hat{\boldsymbol{a}}_m^H \boldsymbol{W}) = \beta_m \\ & \epsilon_m^2 \text{tr}\, \boldsymbol{W} \leq \beta_m - \frac{\beta_m + \beta_m^{l-1}}{\sqrt{\beta_m^{l-1}}} + 1,\ m = 1, \ldots, M, \\ & \boldsymbol{W} \succeq \boldsymbol{0}, \end{aligned} \tag{17}$$

where $\beta_m^{l-1}$, $m = 1, \ldots, M$, are obtained by solving (17) at iteration $l-1$, $l = 1, 2, \ldots$ (here, we assume that at iteration $l = 0$ the initial point $\{\beta_m^0\}$ is prefixed by picking up randomly), and tr $(\cdot)$ represents the trace. In order to solve (16) (which is equivalent to (13)), the SDP problem (17) is solved with the solution $(\boldsymbol{W}_l, \{\beta_m^l\})$ at the $l$th step with $l = 1$, and set $l = l + 1$; the previous step is repeated, and an iterative algorithm is formed. When the algorithm terminates, the output $\boldsymbol{W}^\star$ is decomposed into $\sum_{r=1}^{R} \lambda_r \boldsymbol{w}_r \boldsymbol{w}_r^H$, where $R$ is the rank of $\boldsymbol{W}^\star$, $\lambda_1 \geq \cdots \geq \lambda_R > 0$ are the eigenvalues and $\boldsymbol{w}_1, \cdots, \boldsymbol{w}_R$ are the corresponding eigenvectors, respectively. Finally, $\boldsymbol{w}^\star = \sqrt{\lambda_1} \boldsymbol{w}_1$ is outputted as a solution for the RAB problem (13). It should be noted that this solution may not be feasible for (13), and thus is not necessarily feasible for (12). We would also like to remark that the worst-case computational complexity for the SDP problem (17) is $O(N^4 M^{2.5})$ (as stated in [9]).

In [8], the beamforming problem (13) is relaxed into the following problem:

$$\begin{aligned} \underset{\boldsymbol{w}}{\text{minimize}} \quad & \boldsymbol{w}^H \hat{\boldsymbol{R}} \boldsymbol{w} \\ \text{subject to} \quad & (|\boldsymbol{w}^H \hat{\boldsymbol{a}}_m| - \epsilon_m \|\boldsymbol{w}\|)^2 \geq 1,\ m = 1, \ldots, M. \end{aligned} \tag{18}$$

Then, the relaxed problem is further restricted to the approximation problem:

$$\begin{aligned} \underset{\boldsymbol{w}}{\text{minimize}} \quad & \boldsymbol{w}^H \hat{\boldsymbol{R}} \boldsymbol{w} \\ \text{subject to} \quad & \boldsymbol{w}^H \boldsymbol{P}_m \boldsymbol{w} \geq 1,\ m = 1, \ldots, M, \end{aligned} \tag{19}$$

where matrices

$$\boldsymbol{P}_m = (\epsilon_m^2 - 2\epsilon_m \|\hat{\boldsymbol{a}}_m\|) \boldsymbol{I} + \hat{\boldsymbol{a}}_m \hat{\boldsymbol{a}}_m^H,\ m = 1, \ldots, M. \tag{20}$$

Then, the SDP relaxation problem for (19) is solved:

$$\begin{aligned} \underset{\boldsymbol{W}}{\text{minimize}} \quad & \text{tr}\,(\hat{\boldsymbol{R}} \boldsymbol{W}) \\ \text{subject to} \quad & \text{tr}\,(\boldsymbol{P}_m \boldsymbol{W}) \geq 1,\ m = 1, \ldots, M \\ & \boldsymbol{W} \succeq \boldsymbol{0}. \end{aligned} \tag{21}$$

It can be observed that the procedure to obtain a rank-one solution $\boldsymbol{w}^\star \boldsymbol{w}^{\star H}$ for (21) is not mentioned therein for a general $M$. It should be noted that the solution $\boldsymbol{w}^\star$ is feasible for (19) or (18), but not necessarily feasible for (13) (i.e., for the original RAB problem (12)) because problem (21) is relaxation-restriction-relaxation for problem (13).

In this paper, we solve the RAB problem (13) (rather than the relaxed problem, either (15) or (18)) using a sequential convex approximation algorithm, where an SOCP (instead of a computationally heavier SDP) is solved in each iteration, and the solution for the SOCP in every iteration is feasible for (13), which eventually leads to a locally optimal solution.

## III. A Sequential convex Approximation Algorithm for RAB problem (13)

It can be observed that

$$|\boldsymbol{w}^H \hat{\boldsymbol{a}}_m| = \frac{|\boldsymbol{w}^H \hat{\boldsymbol{a}}_m \hat{\boldsymbol{a}}_m^H \boldsymbol{w}^{l-1}|}{|\hat{\boldsymbol{a}}_m^H \boldsymbol{w}^{l-1}|} \geq \frac{\Re(\boldsymbol{w}^H \hat{\boldsymbol{a}}_m \hat{\boldsymbol{a}}_m^H \boldsymbol{w}^{l-l})}{|\hat{\boldsymbol{a}}_m^H \boldsymbol{w}^{l-l}|}, \tag{22}$$

where $\boldsymbol{w}^{l-1}$ is an optimal solution in the $(l-1)$-th iteration of an iterative algorithm for (13) (the initial point $\boldsymbol{w}^0$ is randomly provided that $\hat{\boldsymbol{a}}_m^H \boldsymbol{w}^0 \neq 0$, $m = 1, \ldots, M$). Thereby, the following SOCP problem is a restriction problem for (13):

$$\begin{aligned} \min_{\boldsymbol{w}} \quad & \boldsymbol{w}^H \hat{\boldsymbol{R}} \boldsymbol{w} \\ \text{s.t.} \quad & \frac{\Re(\boldsymbol{w}^H \hat{\boldsymbol{a}}_m \hat{\boldsymbol{a}}_m^H \boldsymbol{w}^{l-1})}{|\hat{\boldsymbol{a}}_m^H \boldsymbol{w}^{l-1}|} - 1 \geq \epsilon_m \|\boldsymbol{w}\|, \; m = 1, \ldots, M. \end{aligned} \tag{23}$$

Namely, any feasible solution for (23) is feasible for (13), which implies that an optimal solution for (23) is always feasible for (13). Then, (23) is solved, outputing an optimal solution $\boldsymbol{w}^l$ and we set $l = l + 1$. The steps are repeated and an iterative algorithm is established for (13). Accordingly, the proposed sequential convex approximation algorithm is summarized as follows.

**Algorithm 1** A Sequential Convex Approximation Algorithm for (13)

**Input:** $\hat{\boldsymbol{R}}$, $\{\hat{\boldsymbol{a}}_m\}$, $\{\epsilon_m\}$, $\xi$;
**Output:** A solution $\boldsymbol{w}$ for problem (13);
1: let $\boldsymbol{w}^0$ be an initial feasible point; set $l = 1$ and $v_0 = \boldsymbol{w}^{0H} \hat{\boldsymbol{R}} \boldsymbol{w}^0$;
2: **do**
3: solve SOCP (23) with the given point $\boldsymbol{w}^{l-1}$, obtaining an optimal solution $\boldsymbol{w}^l$ and the optimal value $v_l$;
4: $l := l + 1$;
5: **until** $|v_{l-2} - v_{l-1}| \leq \xi$
6: output $\{\boldsymbol{w}^{l-1}\}$.

Suppose that $v_l$ and $\boldsymbol{w}^l$ ($l \geq 1$) are, respectively, the optimal value and an optimal solution for (23) in the $l$th iteration.

**Lemma III.1** *It holds that $v_l \geq v_{l+1}$ for $l = 1, 2, \ldots$.*

*Proof:* It can be seen that $\boldsymbol{w}^l$ ($l \geq 1$) is feasible for (23) in the $(l+1)$-th iteration, where $\boldsymbol{w}^l$ is an optimal solution for (23) in the $l$-th iteration. In fact,

$$|\boldsymbol{w}^{lH} \hat{\boldsymbol{a}}_m| - 1 \geq \frac{\Re(\boldsymbol{w}^{lH} \hat{\boldsymbol{a}}_m \hat{\boldsymbol{a}}_m^H \boldsymbol{w}^{l-1})}{|\hat{\boldsymbol{a}}_m^H \boldsymbol{w}^{l-1}|} - 1 \geq \epsilon_m \|\boldsymbol{w}^l\|, \tag{24}$$

where the first inequality can be attributed to (22) while the second inequality is due to the fact that the optimal solution $\boldsymbol{w}^l$ is also feasible for (23) in the $l$-th iteration (in other words, the constraints in (23) must be satisfied at $\boldsymbol{w}^l$). It can be observed that

$$\frac{\Re(\boldsymbol{w}^{lH} \hat{\boldsymbol{a}}_m \hat{\boldsymbol{a}}_m^H \boldsymbol{w}^l)}{|\hat{\boldsymbol{a}}_m^H \boldsymbol{w}^l|} - 1 = \frac{\Re(|\hat{\boldsymbol{a}}_m^H \boldsymbol{w}^l|^2)}{|\hat{\boldsymbol{a}}_m^H \boldsymbol{w}^l|} - 1 = |\boldsymbol{w}^{lH} \hat{\boldsymbol{a}}_m| - 1 \tag{25}$$

Combining (24) with (25), we obtain

$$\frac{\Re(\boldsymbol{w}^{lH} \hat{\boldsymbol{a}}_m \hat{\boldsymbol{a}}_m^H \boldsymbol{w}^l)}{|\hat{\boldsymbol{a}}_m^H \boldsymbol{w}^l|} - 1 \geq \epsilon_m \|\boldsymbol{w}^l\|. \tag{26}$$

This implies that $\boldsymbol{w}^l$ is feasible for (23) in the $(l+1)$-th iteration. Therefore,

$$v_l = \boldsymbol{w}^{lH} \hat{\boldsymbol{R}} \boldsymbol{w}^l \geq \boldsymbol{w}^{(l+1)H} \hat{\boldsymbol{R}} \boldsymbol{w}^{l+1} = v_{l+1}, \tag{27}$$

for $l = 1, 2, \ldots$, as $\boldsymbol{w}^{l+1}$ is optimal for (23) in the $(l+1)$-th iteration. ■

Based on Lemma III.1, we claim that the solution output by Algorithm 1 is a locally optimal solution because the optimal values $\{v_l\}$ correspond to a nonincreasing and bounded sequence. It follows from [10, page 423] that the worst-case computational complexity for (23) is given by $O(M^{1.5} N^2)$.

## IV. Simulation Results

Let us consider a uniform linear array of 10 omni-directional antenna elements (i.e., $N = 10$) with an inter-element spacing of half wavelength. The power of additive noise in every antenna is assumed to be 0 dB. There are two interferers from directions $-5°$ and $15°$, both with an interference-to-noise ratio (INR) of 30 dB. We suppose that the desired signal is always present in the training data cell. The training sample size $T$ is preset to 100. The actual signal impinges upon the array from the direction of $5°$ while the angle of the presumed steering vector is $8°$.

We also consider the desired signal steering vector mismatch caused by wavefront distortion in an inhomogeneous medium [11, Simulation Example 2], besides the signal look direction mismatch. Precisely, we assume that the signal steering vector is distorted by wave propagation effects in the way that independent-increment phase distortions are accumulated by the components of the steering vector, and assume that the phase increments are independent Gaussian variables, each with a zero mean and standard deviation of 0.03, and they are randomly generated and remain unaltered in each simulation run.

In the big uncertainty set (7), we set the radius to $\epsilon = \sqrt{0.5N}$. It is assumed that in small uncertainty sets (10), there are five presumed steering vectors ($M = 5$) with angles $\{3°, 4°, 5°, 6°, 7°\}$, and each radius $\epsilon_m = \sqrt{0.15N}$ is fixed. All results are averaged over 200 simulation runs.

The proposed sequential convex algorithm is compared with the second iterative SDP approximation method detailed in [9], the single-step SDP relaxation approach reported in [8], and the SOCP equivalent reformulation for the RAB problem with the big uncertainty set (7) reported in [4]. In the figures, the previous four methods are termed in turn as "Proposed beamformer", "FLXZZ beamformer 1", "FLXZZ beamformer 2", and "VGL beamformer". Fig. 1 displays the beamformer output SINR versus the signal-to-noise ratio (SNR). It can be observed that the proposed beamformer has equal performance with the FLXZZ beamformer 1, which is better than that of the FLXZZ beamformer 2 (with a difference of 4 dB), and all the three beamformers outperform the VGL beamformer under the previous setups of $\epsilon$ and $\epsilon_m$. The equal performance between the proposed beamformer and the FLXZZ beamformer 1 implies that both the sequential convex SOCP method and iterative SDP algorithm output a globally optimal solution for (13) in the numerical example.

On the other hand, we compare the performance between the proposed beamformer and FLXZZ beamformer 1 in terms of the CPU time and number of iterations required to solve problem (13). Fig. 2 shows the average CPU time versus SNR and the average iteration numbers versus SNR. It can be seen that the CPU time of the proposed method appears shorter, which is reasonable because an SOCP (as opposed to an SDP) is solved in each iteration of the proposed method. In addition, the proposed one requires fewer iterations, which means that it converges faster. In Fig. 2, we also included the average CPU time for the FLXZZ beamformer 2. It can be observed that the beamformer has the least CPU time. This is reasonable because it is obtained by solving the SDP problem (21) in a single time, unlike the other two beamformers where the iterative algorithms are used. However, there is a trade-off because the array output SINR for the FLXZZ beamformer 2 is worse than that for the other two beamformers (see Fig. 1), and from an optimization perspective, an optimal solution for (19) may not be robust feasible for the original RAB problem (12).

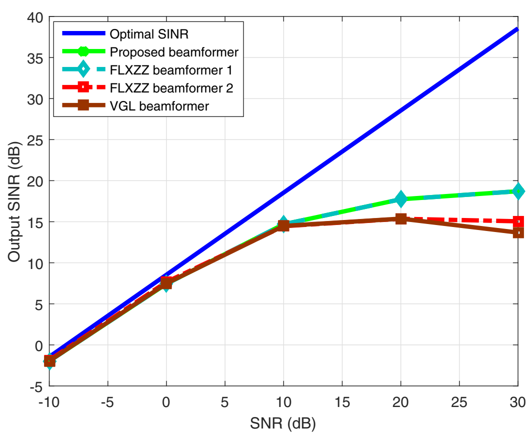


Fig. 1. Average beamformer output SINR versus SNR.

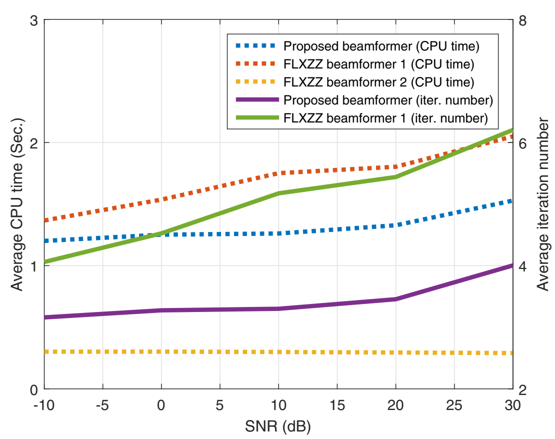


Fig. 2. Average CPU time versus SNR, and average number of iterations versus SNR.

## V. Conclusion

In this study, we considered the RAB problem of maximizing the worst-case (the minimal) SINR over the union of multiple small uncertainty sets of the desired signal steering vector. The problem was formulated to a quadratic minimization problem with nonconvex constraints. As opposed to solving the problem using iterative SDP approximation algorithms, we proposed a sequential convex SOCP approximation algorithm. In particular, we showed that the sequence of optimal values of the SOCPs is nonincreasing and bounded. Furthermore, it was shown that optimal solutions for the SOCPs are feasible for the nonconvex quadratic minimization problem and the sequence consisting of them converges to a locally optimal solution. Because only an SOCP must be solved in each iterative step of the proposed algorithm, the (theoretical) computational complexity is lower than that of an SDP solved in every iterative step of the existing SDP-based approximation algorithm. By harnessing the proposed approximation algorithm, we demonstrated the improved performance of the array using simulations, in terms of the array output SINR, average CPU time, and average number of iterations required to solve an instance of the quadratic minimization problem.

## References


[1] J. Li and P. Stoica, *Robust Adaptive Beamforming*, John Wiley & Sons, Hoboken, NJ, 2006.

[2] A. B. Gershman, N. D. Sidiropoulos, S. Shahbazpanahi, M. Bengtsson, and B. Ottersten, "Convex optimization-based beamforming: from receive to transmit and network designs," *IEEE Signal Processing Magazine*, vol. 27, no. 3, pp. 62-75, May 2010.

[3] S. A. Vorobyov, "Principles of minimum variance robust adaptive beamforming design," *Signal Processing*, vol. 93, pp. 3264-3277, 2013.

[4] S. A. Vorobyov, A. B. Gershman, and Z.-Q. Luo, "Robust adaptive beamforming using worst-case performance optimization: A solution to the signal mismatch problem," *IEEE Transactions on Signal Processing*, vol. 51, no. 2, pp. 313-324, February 2003.

[5] S. A. Vorobyov, A. B. Gershman, Z.-Q. Luo, and N. Ma, "Adaptive beamforming with joint robustness against mismatched signal steering vector and interference nonstationarity," *IEEE Signal Processing Letters*, vol. 11, no. 2, pp. 108-111, February 2004.

[6] S.-J. Kim, A. Magnani, A. Mutapcic, S. P. Boyd, and Z.-Q. Luo, "Robust beamforming via worst-case SINR maximization," *IEEE Transactions on Signal Processing*, vol. 56, no. 4, pp. 1359-1547, April 2008.

[7] Y. Huang, H. Fu, S. A. Vorobyov and Z. -Q. Luo, "Worst-case SINR maximization based robust adaptive beamforming problem with a nonconvex uncertainty set," *IEEE Transactions on Signal Processing*, vol. 71, pp. 218-232, March 2023.

[8] Y. Feng, G. Liao, J. Xu, S. Zhu, and C. Zeng, "Robust beamforming using multiple constraints relaxation," *Proceedings of 2018 IEEE 10th Sensor Array and Multichannel Signal Processing Workshop (SAM)*, pp. 514-518, 2018.

[9] Y. Feng, G. Liao, J. Xu, S. Zhu, and C. Zeng, "Robust adaptive beamforming against large steering vector mismatch using multiple uncertainty sets," *Signal Processing*, vol. 153, pp. 320-330, 2018.

[10] A. Ben-Tal and A. Nemirovski, *Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications*. MPS-SIAM Series on Optimization, SIAM, Philadelphia, 2001.

[11] A. Khabbazibasmenj, S. A. Vorobyov, and A. Hassanien, "Robust adaptive beamforming based on steering vector estimation with as little as possible prior information," *IEEE Transactions on Signal Processing*, vol. 60, no. 6, pp. 2974-2987, June 2012.