# Broadband frequency-invariant broadside beamforming with a differential loudspeaker array

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***Abstract*—The directional loudspeaker array generating sound beam to the target listener is highly demanded in the application. The null-constraint-based differential beamforming has recently been applied to the loudspeaker line array to produce a broadside frequency-invariant radiation pattern. However, its effective frequency range is limited since it only pursues sound pressure matching in a few directions. In this paper, we develop the modal matching approach of the null-constrained method to control the beam pattern better. Specifically, we derive the modal domain target beam pattern of a broadside differential loudspeaker array from the information about the nulls. Then, we use the Jacobi-Anger expansion and modal matching to optimise the loudspeaker gains in the modal domain. In addition, a distortionless constraint in the broadside direction is included to achieve constant performance over the frequency band of interest. Simulation results show that the proposed method achieved more than double the effective frequency range with an invariant main lobe (broadside beam) up to 4 kHz, and better sound attenuation of the side lobes, compared to the existing methods.**

***Index Terms*—loudspeaker line array, differential beamforming, frequency-invariant beamforming, broadside radiation, Jacobi-Anger expansion**

## I. Introduction

Directional loudspeaker arrays have been used in various scenarios, such as public address systems [1, 2], reducing room reverberation [3, 4], and creating personalized sound zones [5-8]. These arrays can be either additive or differential [9, 10]. Additive arrays achieve high directivity by controlling the interference of the sound fields generated by multiple loudspeakers. However, due to the diffraction limit, additive arrays with small aperture sizes cannot generate high directivity at low frequencies.

On the other hand, differential arrays can radiate a narrow beam pattern and benefit from small array sizes [11]. Due to their compact size, frequency-invariant beam pattern, and high spatial directivity, differential arrays have been widely studied in microphone array applications over the past few years [9, 12-17]. Many differential beamforming methods based on microphone array have been developed, such as multistage manner [18, 19], null-constrained method [20-22], and series approximation method [23-25].

Differential beamformers have also been utilized in loudspeaker arrays to produce highly directional patterns [10, 11, 26-28]. Choi studied the creation of a second-order differential broadside beam using a three-element line array for sound zones in a car cabin [27]. For a higher-order broadside differential beam pattern, Wang *et al.* devised a null-constrained method [10]. This approach is versatile and easy to use for designing differential beamformers. However, the frequency-invariant beam pattern has a limited effective frequency band, which may be undesirable for some applications.

Our recent research studied the series expansion method for designing steerable and frequency-invariant beam patterns using a circular loudspeaker array [28]. This method employed the Jacobi-Anger expansion to approximate the target beam pattern, which led to better-preserved frequency-invariant beam patterns across the frequency range of interest. Nonetheless, this method requires prior knowledge of the target beam pattern, which may not be feasible in real-world applications.

This paper develops the series expansion to the null-constrained method for the broadside differential loudspeaker line array. Unlike traditional series expansion methods that use a cardioid or super-cardioid beam pattern as the target, the proposed method derives the modal domain target beam pattern of a broadside differential loudspeaker array from the information about the nulls, offering greater flexibility. The Jacobi-Anger expansion is used to design the differential beamformer, allowing for beam pattern matching in the modal domain rather than just a few directions. Additionally, a distortionless constraint is applied in the broadside direction to maintain a constant main lobe over the frequency band of interest. This method can also be used for designing a linear differential microphone array with a broadside beam pattern for high-quality acoustic signal acquisition.

## II. Related Work

### A. Loudspeaker Line Array

Assume a regularly spaced line array of $L$ loudspeakers centred at the origin of the polar coordinate system, as shown

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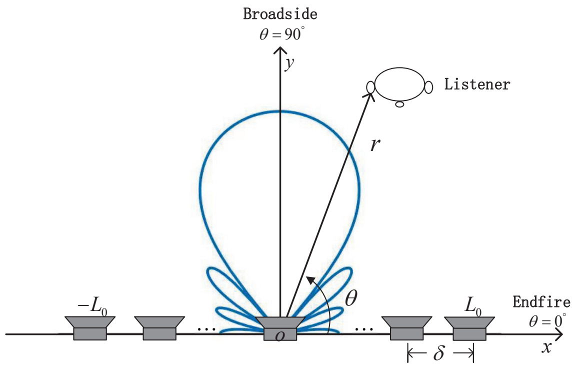



Fig. 1. Broadside beamforming with a differential loudspeaker line array.

in Figure 1. The far-field sound pressure at a listener position $(r, \theta)$ generated by the loudspeaker array is [10, 11]

$$p(k, r, \theta) \approx \frac{e^{ikr}}{4\pi r} \sum_{l=-L_0}^{L_0} w_l^*(k) e^{-ikx_l \cos\theta}, \tag{1}$$

where $k$ is the wave number, the superscript $*$ represents the complex-conjugate operator, $w_l(k)$ denotes the weight of the $l$-th loudspeaker at $(x_l, 0)$, where $x_l = l\delta$, $\delta$ is the spacing between the loudspeakers, the loudspeaker index $l \in [-L_0, L_0]$ and $L_0 = (L-1)/2 \geq 1$. For a differential loudspeaker array, $L$ is an odd number with $L > 1$, and each loudspeaker acts as a monopole source for the far-field response.

The directivity pattern of the far-field radiation is

$$\begin{aligned} B(k, \theta) &= p(k, r, \theta)/(e^{ikr}/4\pi r) \\ &= \sum_{l=-L_0}^{L_0} w_l^*(k) e^{-ikx_l \cos\theta}, \end{aligned} \tag{2}$$

with vector form

$$B(k, \theta) = \mathbf{w}^{\mathrm{H}}(k)\mathbf{g}(k, \theta), \tag{3}$$

where the superscript $(\cdot)^{\mathrm{H}}$ is the conjugate-transpose operator,

$$\mathbf{w}(k) = [w_{-L_0}(k), \ldots, w_{L_0}(k)]^{\mathrm{T}}, \tag{4}$$

$$\mathbf{g}(k, \theta) = [e^{-ikx_{-L_0}\cos\theta}, \ldots, e^{-ikx_{L_0}\cos\theta}]^{\mathrm{T}}, \tag{5}$$

the superscript $(\cdot)^{\mathrm{T}}$ is the transpose operator, and $\mathbf{w}(k)$ is the weighting vector to be designed for a radiation pattern $B(k, \theta)$.

### B. Target Broadside Radiation Pattern

The broadside loudspeaker line array optimises the weighting vector $\mathbf{w}(k)$ to maximise the generated sound energy at the broadside direction $\theta' = \pi/2$. Assuming the spacing between neighbouring loudspeakers to be much smaller than the wavelength, the $2N$th-order broadside differential beam pattern is [10]

$$B^{(2N)}(k, \theta) \approx \prod_{n=1}^{N} \left(1 - \frac{\cos^2\theta}{\beta_n^2}\right), \tag{6}$$

where the superscript $(\cdot)^{(2N)}$ denotes the $2N$th-order, $N$ is a positive integer, the tuning parameters $\beta_n, n = 1, 2, \ldots, N$ determine the $2N$ null directions $\theta_{null}^{(2n)} = \pm\arccos(\beta_n), n = 1, 2, \ldots, N$. Equation (6) shows the $2N$th-order beam pattern is mainly determined by the nulls' directions and is independent of the wave number $k$.

### C. Minimum-Norm Method with Additional Constraints

The state-of-the-art method, the Minimum-Norm method with Additional constraints (MNA) [10], is null-constrained to design a broadside differential radiation pattern with the optimization of the loudspeaker weighting $\mathbf{w}(k)$. A $2N$th-order broadside differential pattern has $2N$ distinct nulls and requires a loudspeaker line array of $L > 2N + 1$ speakers. Due to the broadside pattern being symmetric, the minimum norm (MN) method [10] uses only $N$ distinct nulls $\{\theta_{N,n} \in [0^\circ, 90^\circ), n = 1, 2, \ldots, N\}$ as the $2N + 1$ fundamental beam pattern constraints.

The MNA method additionally applies $N'$ more constraints at $\theta_{N',n'}$, where $0^\circ \leqslant \theta_{N',1} < \cdots < \theta_{N',N'} < 90^\circ$ and $\theta_{N',n'}$ is different from $\theta_{N,n}$, and $b_{n'}$ is the desired gain at $\theta_{N',n'}$. So the controlled $(1 + N + N')$ directions and loudspeakers' directional responses at those directions are

$$\tilde{\boldsymbol{\theta}} = [90^\circ, \theta_{N,1}, \ldots, \theta_{N,N}, \theta_{N',1}, \ldots, \theta_{N',N'}]^{\mathrm{T}}, \tag{7}$$

$$\mathbf{G}(k, \tilde{\boldsymbol{\theta}}) = [\mathbf{g}(k, 90^\circ), \mathbf{g}(k, \theta_{N,1}), \ldots, \mathbf{g}(k, \theta_{N',N'})]^{\mathrm{H}}. \tag{8}$$

The optimization problem of the MNA method is

$$\min_{\mathbf{w}(k)} \mathbf{w}^{\mathrm{H}}(k)\mathbf{w}(k) \qquad s.t. \tilde{\mathbf{G}}(k, \tilde{\boldsymbol{\theta}})\mathbf{w}(k) = \tilde{\mathbf{i}}, \tag{9}$$

where

$$\tilde{\mathbf{G}}(k, \tilde{\boldsymbol{\theta}}) = [\mathbf{G}(k, \tilde{\boldsymbol{\theta}}); \mathbf{C}], \tag{10}$$

$$\tilde{\mathbf{i}} = [1, 0, \ldots, 0, b_1, \ldots, b_{N'}, 0, \ldots, 0]^{\mathrm{T}}, \tag{11}$$

$$\mathbf{C} = [\mathbf{I} \quad \mathbf{0} \quad -\breve{\mathbf{I}}], \tag{12}$$

where $\mathbf{I}$ is an $L_0 \times L_0$ identity matrix, and $\mathbf{0}$ is an $L_0$ element all-zero column vector matrix, $\breve{\mathbf{I}}$ is the flip of the matrix $\mathbf{I}$. The minimum-norm solution of (9) is

$$\mathbf{w}_{\mathrm{MNA}}(k) = \tilde{\mathbf{G}}^{\mathrm{H}}(\tilde{\mathbf{G}}\tilde{\mathbf{G}}^{\mathrm{H}})^{-1}\tilde{\mathbf{i}}. \tag{13}$$

## III. Proposed Differential Beamformer Design

We propose an approach based on the series expansion method to design a differential loudspeaker line array with a broadside frequency-invariant beam pattern in the modal domain, including (1) calculating the target radiation pattern according to the null information from an ideal broadside differential pattern, and (2) designing the differential beamformer using the Jacobi-Anger expansion.

### A. Target Broadside Radiation Pattern in the Modal Domain

Equation (6) can be formulated in a sum form

$$B^{(2N)}(k,\theta) = \sum_{n=0}^{N} \alpha_{N,n} \cos^{2n}\theta, \tag{14}$$

where

$$\alpha_{N,n} = \begin{cases} 1, & n = 0, \\ (-1)^n e_n(\frac{1}{\beta_1^2},\ldots,\frac{1}{\beta_N^2}), & n = 1,\ldots,N. \end{cases} \tag{15}$$

where the $n$th elementary symmetric function [29]

$$e_n(\frac{1}{\beta_1^2},\ldots,\frac{1}{\beta_N^2}) = \sum_{1\le i_1<i_2\ldots<i_n\le N} \frac{1}{\beta_{i_1}^2\ldots\beta_{i_n}^2}. \tag{16}$$

Following (1.320-5) in [30],

$$\begin{aligned}\cos^{2n}\theta &= \frac{1}{2^{2n}}\Big\{\sum_{m=0}^{n-1}\binom{2n}{m}\Big[e^{j2(n-m)\theta} + e^{-j2(n-m)\theta}\Big] + \binom{2n}{n}\Big\} \\ &= \sum_{m=-n}^{n} \eta_m(n) e^{j2m\theta},\end{aligned} \tag{17}$$

$$\eta_m(n) = \frac{1}{2^{2n}}\binom{2n}{n-|m|}, \quad m = 0,\pm1,\ldots,\pm n, \tag{18}$$

where $\binom{\cdot}{\cdot}$ is combinations, and $|\cdot|$ represents absolute value. Inserting (17) into (14), the $2N$th-order broadside differential beampattern in a symmetrical form is

$$B^{(2N)}(k,\theta) = \sum_{n=-2N}^{2N} \gamma_n e^{jn\theta}, \tag{19}$$

where

$$\gamma_n = \begin{cases} 0, & n = \pm1,\pm3,\ldots,\pm(2N-1), \\ \sum\limits_{p=|n|/2}^{N} \alpha_{N,p}\eta_{|n|/2}(p), & n = 0,\pm2,\pm4,\ldots,\pm2N. \end{cases} \tag{20}$$

### B. Modal Matching with Jacobi-Anger Expansion and Distortionless Constraint

In the series expansion method, the Jacobi-Anger expansion translates the resulting beam pattern into the modal domain to match the target beam pattern in (19). The Jacobi-Anger expansion is

$$e^{-ikx\cos\theta} = \sum_{n=-\infty}^{+\infty} i^{-n} J_n(kx) e^{in\theta}, \tag{21}$$

where $J_n(kx)$ is the $n$th-order Bessel function of the first kind. Inserting (21) into the exponential of (2) yields

$$B(k,\theta) = \sum_{l=-L_0}^{L_0} w_l^*(k) \sum_{n=-\infty}^{+\infty} i^{-n} J_n(kx_l) e^{in\theta}. \tag{22}$$

To obtain a $2N$th-order broadside differential beampattern, the infinite series is truncated to the order $2N$, that

$$B(k,\theta) = \sum_{n=-2N}^{2N} e^{in\theta} \sum_{l=-L_0}^{L_0} i^{-n} J_n(kx_l) w_l^*(k). \tag{23}$$

If the radiation pattern in (23) is consistent with the target beam pattern in (19), there is

$$\sum_{l=-L_0}^{L_0} i^{-n} J_n(kx_l) w_l^*(k) = \gamma_n \quad n = 0,\pm1,\ldots,\pm2N. \tag{24}$$

For the design of a broadside differential beamformer, the distortionless constraint in the broadside direction is required

$$\mathbf{w}^{\mathrm{H}}(k)\mathbf{g}(k,\pi/2) = 1. \tag{25}$$

Combining (24) and (25),

$$\mathbf{\Phi}\mathbf{w} = \Upsilon_{2N+1}, \tag{26}$$

where

$$\mathbf{\Phi} = \begin{bmatrix} e^{ikx_{-L_0}\cos(\pi/2)} & \ldots & e^{ikx_{L_0}\cos(\pi/2)} \\ J_{-2N}(kx_{-L_0})*(i^{-2N}) & \ldots & J_{-2N}(kx_{L_0})*(i^{-2N}) \\ \vdots & \ddots & \vdots \\ J_{2N}(kx_{-L_0})*(i^{2N}) & \ldots & J_{2N}(kx_{L_0})*(i^{2N}) \end{bmatrix}, \tag{27}$$

$$\Upsilon_{2N+1} = [1 \quad \gamma_{-2N}\ldots\gamma_0\ldots\gamma_{2N}]^{\mathrm{T}}. \tag{28}$$

Using the symmetric property of the Bessel function,

$$\tilde{\mathbf{\Phi}}\mathbf{w} = \tilde{\Upsilon}_{2N+1}, \tag{29}$$

where

$$\tilde{\mathbf{\Phi}} = \begin{bmatrix} e^{ikx_{-L_0}\cos(\pi/2)} & \ldots & e^{ikx_{L_0}\cos(\pi/2)} \\ J_0(kx_{-L_0})*(i^0) & \ldots & J_0(kx_{L_0})*(i^0) \\ \vdots & \ddots & \vdots \\ J_{2N}(kx_{-L_0})*(i^{2N}) & \ldots & J_{2N}(kx_{L_0})*(i^{2N}) \end{bmatrix} \tag{30}$$

is a $(2N+2)\times L$ full-rank matrix,

$$\tilde{\Upsilon}_{2N+1} = [1 \quad \gamma_0\ldots\gamma_{2N}]^{\mathrm{T}}, \tag{31}$$

where $[\gamma_0\ldots\gamma_{2N}]$ is determined by the nulls of the desired beam pattern. With the number of loudspeakers $L > 2N+2$, the minimum-norm solution of (29) is

$$\mathbf{w} = \tilde{\mathbf{\Phi}}^{\mathrm{H}}(\tilde{\mathbf{\Phi}}\tilde{\mathbf{\Phi}}^{\mathrm{H}})^{-1}\tilde{\Upsilon}_{2N+1}. \tag{32}$$

## IV. SIMULATIONS

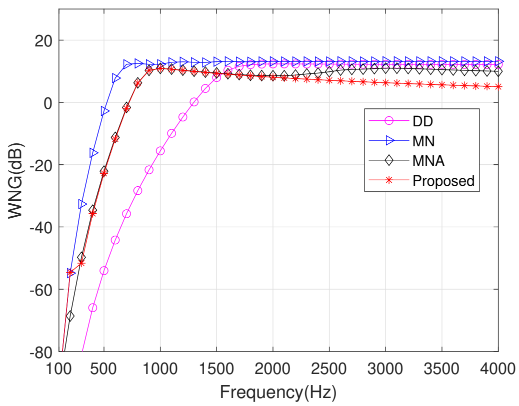
The simulation assumes a line array of 21 loudspeakers with a spacing of 0.05 m. Each loudspeaker is modelled as an ideal point source under the free field condition. The frequency range of interest is from 100 to 4000 Hz, covering the frequency range of speech. The simulation considers the design of a sixth-order broadside differential beam, where the six positive nulls are set as 10°, 30°, 50°, 130°, 150° and 170°. MNA additionally has a constraint on the angle of 75°, corresponding to the main lobe's half-power beamwidth [10].

The evaluation metrics include the white noise gain (WNG) and the directivity index (DI). WNG represents radiation efficiency as well as robustness, that

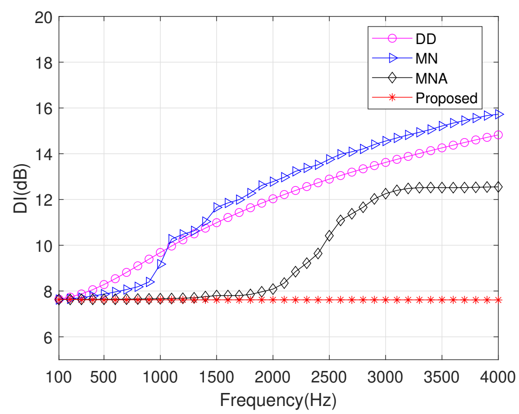
$$\mathrm{WNG} = 10\log_{10}(\frac{|B(k,\pi/2)|^2}{\mathbf{w}^{\mathrm{H}}(k)\mathbf{w}(k)}). \tag{33}$$

DI is the ratio between the power radiated in the broadside direction and the spatial average of the radiated intensity over the half-plane where the loudspeaker array is located, that

$$\mathrm{DI} = 10\log_{10}\left(\frac{\pi|B(k,\pi/2)|^2}{\int_0^\pi |B(k,\theta)|^2 d\theta}\right). \tag{34}$$



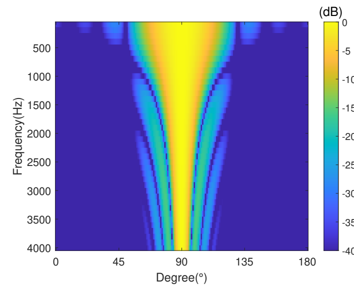

(a)



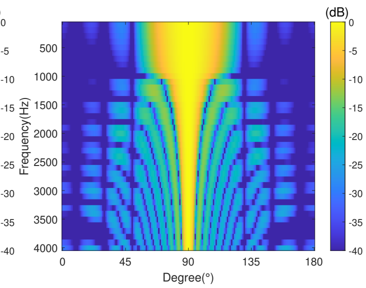

(b)

Fig. 2. Performance of generating a sixth-order broadside differential pattern with nulls at 10°, 30°, 50°, 130°, 150° and 170°, regarding (a) white noise gain (WNG) and (b) the directivity index (DI).

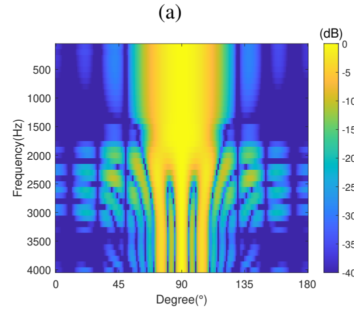
Figure 2 shows the WNG and DI of the proposed method, compared with the existing methods, DD (combining the differential and delay-and-sum patterns [11]), MN and MNA. In Figure 2(a), MN has the maximum WNG in the 100 Hz - 4000 Hz frequency range. DD has the lowest WNG among the four methods below 1500 Hz, indicating the worst anti-perturbation ability at low frequencies. The proposed method has the same WNG as MNA below 2000 Hz. Although the WNG of the proposed method is the lowest above 2000 Hz, the value is still greater than 0 dB, which is considered a proper level of robustness for practical usage. Figure 2(b) shows the DI of MN and DD increases with frequency. DI of MNA maintains almost the same below 2000 Hz, but increases with frequency when above 2000 Hz where the frequency-invariant beampattern cannot hold. The DI of the proposed method can maintain a constant over the frequency range of interest (100 Hz - 4000 Hz), due to the frequency-invariant radiation pattern achieved by the proposed method.



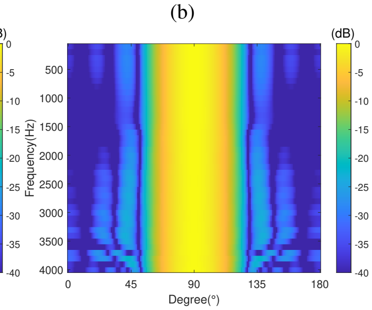

(a)




(b)




(c)




(d)

Fig. 3. Radiation patterns for the design of a sixth-order broadside differential patterns with nulls at 10°, 30°, 50°,130°, 150° and 170°: (a) DD, (b) MN, (c) MNA, and (d) Proposed method.

Figure 3 shows the radiation patterns obtained with the four methods over 100 Hz - 4000 Hz, respectively. The main lobes of the DD and MN methods become narrower as the frequency increases. Below 2000 Hz, the main lobe of MNA kept almost the same. However, grating lobes appear above 2000 Hz. In contrast, the proposed method maintained the frequency-invariant pattern in the evaluated frequency range of up to 4 kHz.

## V. CONCLUSIONS

This paper proposes a new method to design a broadside frequency-invariant beam pattern with a differential loudspeaker line array. The proposed method establishes the relationship between the series expansion method and the null-constrained method and combines their advantages in designing a broadside differential beamformer, that allows flexibility in pattern design as the null-constraint method and broadband control as the series expansion method. Specifically, we first calculate the analytic form of the target radiation pattern according to the null information of an ideal broadside differential beam pattern, and then design the beamformer using the Jacobi-Anger expansion method with distortionless constraint. Simulations show that the proposed method outperforms the existing methods and achieves frequency-invariant broadside beamforming over the frequency range of 100 Hz - 4000 Hz.

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