PERFECT RECONSTRUCTION OF CLASSES OF 3D NON-BANDLIMITED SIGNALS FROM TOMOGRAPHIC PROJECTIONS AT UNKNOWN ANGLES

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ABSTRACT

In this paper, we consider the problem of reconstructing 3D objects from sampled 2D tomographic projections at unknown angles. We consider 3D polyhedrons and we provide a constructive solution to recover the 3D structure of the polyhedron and the projection directions. The proposed method is based on the observation that the 2D projections are actually signals with finite rate of innovation, and this allows us to retrieve the locations of the projected vertices using Prony's method. We are then able to unveil the 3D geometric information of the projection directions, as well as the structure of the polyhedron with an algebraic approach. The reconstruction method can also be applied to 3D point source models and is resilient to noise.

***Index Terms*—** 3D tomography, polyhedron reconstruction, projection angle estimation, finite rate of innovation (FRI), sampling theory

1. INTRODUCTION

The problem of reconstructing a 3D structure from its 2D projections at unknown angles has gained a lot of prominence over the last two decades as it appears in a host of applications that range from biomedical imaging [1, 2, 3] and geophysical imaging [4] to industrial radiography [5]. In particular, single-particle cryo-electron microscopy has been intensively studied in the past decades for its ability to reconstruct high-resolution 3D structures of macro molecules from noisy 2D tomographic projections at unknown view angles without crystallization [1, 6, 7].

Much of the current research targeting this problem can be divided into two main categories. In the first category, the projection directions are first estimated using approaches such as common-line methods [8] or moment-based methods [9, 10, 11]. The 3D structure is then recovered by inverting the operation of projection [12]. The estimation is further refined through projection matching [13] or expectation maximization based methods [14]. These approaches can be computationally demanding, and high order statistics may be required, which leads to low resilience to noise. In the second category, the reconstruction of the 3D structure relies on the estimation of rotational-invariant features from the projections [15, 16, 17]. Normally a good estimation of these features requires a huge number of projections, and there is no constructive proof for the minimum number of projections required.

In this paper, we address the fundamental sampling question of when the perfect reconstruction of the 3D structure can be achieved from a set of samples of its 2D projections. To do this, we incorporate sparsity in the signal model considered. Specifically, we consider the 3D tomography problem for specific classes of signals, bilevel convex polyhedron models and point source models, for which we provide a constructive solution to the perfect reconstruction of the 3D structure, as well as the estimation of projection angles, up to an orthogonal transformation.

The key insight of our proposed framework is that the tomographic projection of the convex polyhedron model is piecewise linear with convex polygonal boundaries, which can be regarded as a signal with finite rate of innovation (FRI) [18, 19]. This allows us to retrieve the vertices of the polygonal boundaries using Prony's method [18]. We then adopt an approach similar in spirit to [20] to perform reconstruction of both the 3D structure and the projection directions.

This paper is organized as follows. In Section 2, we formulate the problem. In Section 3, we present our estimation algorithm, which is composed of 1) retrieval of 2D parameters and 2) simultaneous estimation of the projection directions and the 3D structure from 2D parameters. We then validate our method with experiments done with synthetic data under both noiseless and noisy settings in Section 4 and conclude in Section 5.

2. PROBLEM FORMULATION

Let us consider the following 3D bilevel convex polyhedron model:

$$g(\boldsymbol{r}) = \begin{cases} 1, & \text{for } \boldsymbol{r} \text{ inside } \Gamma, \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

where Γ is the boundary surface of the polyhedron specified by K vertices $\{\boldsymbol{S}_k\}_{k=1}^K$ in $\mathbb{R}^3$ space. We assume that all vertices lie within a sphere region of known radius R centered at the origin, and that $\sum_{k=1}^{K} \boldsymbol{S}_k = \mathbf{0}$. We consider a set of J tomographic projections of the 3D volume $g(\boldsymbol{r})$ onto 2D planes whose normal vectors are denoted as $\boldsymbol{d}_j \in \mathbb{R}^3,\ j = 1, ..., J$. The 2D projection is inside a rectangular plane Π_j of dimension (R_1, R_2) with $R_1, R_2 \geq 2R$. The unit norm vectors $\boldsymbol{l}_{j_1}, \boldsymbol{l}_{j_2} \in \mathbb{R}^3$ such that $\boldsymbol{l}_{j_1} \perp \boldsymbol{l}_{j_2}$, $\boldsymbol{l}_{j_1} \times \boldsymbol{l}_{j_2} = \boldsymbol{d}_j$, are used to denote the two directions of the plane Π_j as shown in Fig. 1, where $\times$ denotes the vector product. We further assume that the center of the jth observation window deviates from the projection of the origin $\boldsymbol{O}_j$ by $\boldsymbol{\eta}_j = [\eta_{j_1}, \eta_{j_2}]^T$.

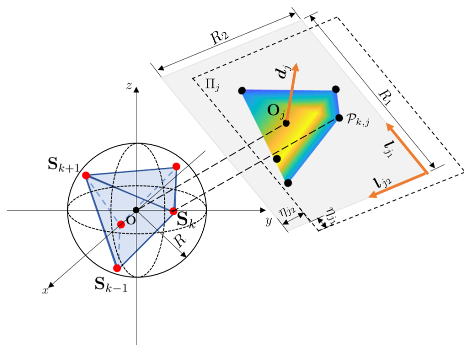


Fig. 1: The bilevel convex polyhedron model $g(\mathbf{r})$. The projection is taken onto the 2D plane Π_j whose normal vector is $\boldsymbol{d}_j$.

Using the above notations, the 2D projection $\mathcal{P}_j(x, y)$ within the observation window can be expressed as:

$$\mathcal{P}_j(x, y) = \int_{\mathbb{B}^3} g(\mathbf{r})\delta(x - \mathbf{r}^T \boldsymbol{l}_{j_1} - \eta_{j_1})\delta(y - \mathbf{r}^T \boldsymbol{l}_{j_2} - \eta_{j_2})d\mathbf{r}, \tag{2}$$

where $\mathbb{B}^3$ is the ball region of radius R on $\mathbb{R}^3$. We assume the signal $\mathcal{P}_j(x, y)$ is filtered with a 2D sampling kernel $\varphi(x, y)$ and uniformly sampled. Therefore, we observe

$$\mathcal{P}_j[m, n] = \langle \mathcal{P}_j(x, y), \varphi(x - mT_1, y - nT_2) \rangle, \tag{3}$$

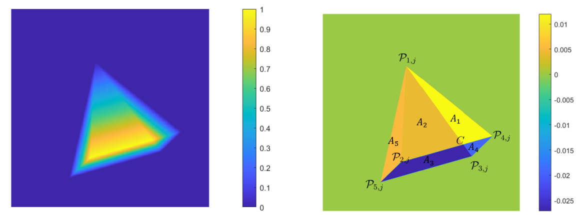
where $m, n \in \mathbb{N}$, T_1, T_2 are the uniform sampling periods, and $\langle \cdot, \cdot \rangle$ denotes the inner product. We assume $T_1 = T_2$ without loss of generality.

The problem we want to solve is the simultaneous estimation of the shape of the convex polyhedron $g(\mathbf{r})$, the direction vectors $\{\boldsymbol{d}_j\}_{j=1}^{J}$ and the unknown shifts $\{\boldsymbol{\eta}_j\}_{j=1}^{J}$. We notice that the shape of $g(\mathbf{r})$ is completely specified by the locations of the vertices $\{\boldsymbol{S}_k\}_{k=1}^{K}$, therefore, estimating $g(\mathbf{r})$ is equivalent to estimating the locations of its vertices.

Since convexity is preserved through projection, $\mathcal{P}_j(x, y)$ is actually piecewise linear with convex polygonal boundaries as shown in Fig. 2, where parts of the vertices correspond to the projection of $\{\mathbf{S}_k\}_{k=1}^{K}$. As evident from (2), the projected location of $\boldsymbol{S}_k$ onto the plane Π_j can be written as:

$$\mathcal{P}_{k,j} = [\boldsymbol{S}_k^T \boldsymbol{l}_{j_1} - \eta_{j_1}, \boldsymbol{S}_k^T \boldsymbol{l}_{j_2} - \eta_{j_2}]^T. \tag{4}$$

As we shall explain later, the exact estimation of $\mathcal{P}_{k,j}$ is essential to the reconstruction method.



(a) Projection data $\mathcal{P}_j(x, y)$.

(b) The differentiated projection data $\partial_x \mathcal{P}_j(x, y)$.

Fig. 2: An example of the projection data and its partial derivative. Note that in (b), the point C comes from the intersection of two edges.

3. RECONSTRUCTING METHOD

Under the assumptions we have made in the previous section, we can state the following result:

Proposition 1. *Let $g(\mathbf{r}) \in \mathbb{R}^3$ be a convex polyhedron with $K \geq 5$ vertices. If the projections are taken on $J \geq 6$ unknown 2D planes with non-coplanar normal vectors, the in-plane shifts $\{\boldsymbol{\eta}_j\}_{j=1}^{J}$, the model $g(\mathbf{r})$ and the normal vectors of the unknown planes $\{\boldsymbol{d}_j\}_{j=1}^{J}$ can be estimated, and for the latter two the estimation is up to an orthogonal transformation.*

Proof. As we mentioned above, the retrieval of locations of vertices and direction vectors rely on the exact estimation of projected locations of vertices. Therefore, we first show how to estimate the parameters $\mathcal{P}_{k,j}$. We then show how to simultaneously retrieve the 3D direction vectors $\boldsymbol{d}_j$, as well as the locations of the vertices $\boldsymbol{S}_k$ using the estimated parameters.

The projection data $\mathcal{P}_j(x, y)$ is composed of H disjoint piecewise linear convex polygons as shown in Fig.2 (a), therefore, its partial derivatives $\partial_x \mathcal{P}_j(x, y)$ and $\partial_y \mathcal{P}_j(x, y)$ contain H piecewise constant L_h-sided convex polygons A_h with amplitudes a_h, as shown in Fig.2(b). The vertices of each patch A_h are denoted in the complex plane in counter clockwise sequence as $z_{h,l} = x_{h,l} + iy_{h,l}$, where $l = 1, ..., L_h$. We can then leverage Davis's theorem [21] and write the following equation for each polygonal closure A_h:

$$\int\int_{A_h} \partial_x \mathcal{P}_j(x, y) f''(z) dxdy = a_h \sum_{l=1}^{L_h} \rho_{h,l} f(z_{h,l}), \tag{5}$$

where $f(z)$ can be any regular function and $\rho_{h,l} = \frac{i}{2}(\frac{\bar{z}_{h,l-1} - \bar{z}_{h,l}}{z_{h,l-1} - z_{h,l}} - \frac{\bar{z}_{h,l} - \bar{z}_{h,l+1}}{z_{h,l} - z_{h,l+1}})$. Since the polygon closures do not overlap, we can further compute the above integral on the whole observation window as follows:

$$\int\int_{\Pi_j} \partial_x \mathcal{P}_j(x, y) f''(z) dxdy = \sum_{h=1}^{H} a_h \sum_{l=1}^{L_h} \rho_{h,l} f(z_{h,l}). \tag{6}$$

It can be proved that the coefficients of the polygon vertices that come from the intersection of two edges of the polyhedron will vanish, while only the ones corresponding to the projection of the vertices of the polyhedron will remain [1]. If we further choose $f(z) = z^n$, Eq. (6) can be written as a power sum series of the projected locations of the polyhedron:

$$\begin{aligned}\sum_{k=1}^{K} \lambda_{k,j} z_{k,j}^n &= \int\int_{\Pi_j} \partial_x \mathcal{P}_j(x,y)(z^n)'' dxdy \\ &= n(n-1)\int\int_{\Pi_j} \partial_x \mathcal{P}_j(x,y)(x+iy)^{n-2} dxdy \\ &= n(n-1)\tau_{n-2} = \hat{\tau}_n, \end{aligned} \tag{7}$$

where $z_{k,j}$ corresponds to $\mathcal{P}_{k,j}$ in the complex plane and τ_n is the simple complex moment. It is clear that $\hat{\tau}_n$ are the linear combinations of exponentials of $z_{k,j}^n$, thus, we can apply Prony's method [18, 22] to retrieve the K locations $z_{k,j}$ given $2K$ weighted complex moments, or equivalently, $2K-2$ simple complex moments. Now the question has become how to obtain the complex moments τ_n from samples of the projection data $\mathcal{P}_j[m,n]$.

We make the connection by assuming the sampling kernel $\varphi(x,y)$ is a 2D polynomial reproducing kernel that can reproduce polynomials $x^\alpha y^\beta$, $0 \le \alpha, \beta \le 2K-3$, which is to say there exist some proper coefficients $c_{m,n}^{\alpha,\beta}$ such that:

$$\sum_{m,n} c_{m,n}^{\alpha,\beta} \varphi(x-m, y-n) = x^\alpha y^\beta, 0 \le \alpha, \beta \le 2K-3, \tag{8}$$

where $T_1 = T_2 = 1$ is assumed here. We take the finite differences of the samples $Z_j[m,n] = \mathcal{P}_j[m+1,n] - \mathcal{P}_j[m,n]$:

$$\begin{aligned} Z_j[m,n] &= \langle \mathcal{P}_j(x,y), \varphi(x-m-1, y-n) - \varphi(x-m,y-n)\rangle \\ &\overset{(a)}{=} \langle \mathcal{P}_j(x,y), -\partial_x(\varphi(x-m,y-n) * \beta_x^0(x-m))\rangle \\ &= \langle \partial_x \mathcal{P}_j(x,y), \underbrace{\varphi(x-m,y-n) * \beta_x^0(x-m)}_{\psi(x-m,y-n)}\rangle, \end{aligned}$$

where (a) can be derived using integration by parts, and β_x^0 denotes the box function along x direction. It can be shown that, $\psi(x,y)$ is able to reproduce polynomial up to degree $x^{2K-2}y^{2K-3}$ with some coefficients $b_{m,n}^{\alpha,\beta}$ [19], therefore:

$$\begin{aligned} &\sum_{m,n} b_{m,n}^{\alpha,\beta} Z_j[m,n] \\ &= \sum_{m,n} b_{m,n}^{\alpha,\beta} \langle \partial_x \mathcal{P}_j(x,y), \psi(x-m,y-n)\rangle \\ &= \int_{\Pi_j} \partial_x \mathcal{P}_j(x,y) \sum_{m,n} b_{m,n}^{\alpha,\beta} \psi(x-m,y-n) dxdy \\ &= \int_{\Pi_j} \partial_x \mathcal{P}_j(x,y) x^\alpha y^\beta dxdy = \mu_{\alpha,\beta}, \end{aligned}$$

[1] We omit this derivation due to lack of space.

where $\mu_{\alpha,\beta}$ are the geometric moments of $\partial_x \mathcal{P}_j(x,y)$, and they relate to the simple complex moments by:

$$\tau_n = \sum_{\beta=0}^{n} \binom{n}{\beta} i^\beta \mu_{\alpha,\beta},\ \alpha+\beta = n. \tag{9}$$

We have shown above how to use samples $\mathcal{P}_j[m,n]$ to obtain the weighted complex moments $\hat{\tau}_n$, from which the parameters $\mathcal{P}_{k,j}$ are retrieved using Prony's method. Next we show that by using an approach similar to the one introduced in our previous work [20], we can simultaneously retrieve the unknown shifts $\{\boldsymbol{\eta}_j\}_{j=1}^J$, direction vectors $\{\boldsymbol{d}_j\}_{j=1}^J$, as well as the locations of the vertices $\{\boldsymbol{S}_k\}_{k=1}^K$.

We can solve for the unknown shifts $\boldsymbol{\eta}_j$ by summing the retrieved parameters for all vertices on one projection:

$$\sum_{k=1}^{K} \mathcal{P}_{k,j}^v = (\sum_{k=1}^{K} \mathbf{S}_k^T)\boldsymbol{l}_{j_v} - K\eta_{j_v} \overset{(b)}{=} -K\eta_{j_v},$$

where $v = 1,2$, and (b) follows from the assumption that $\sum_{k=1}^K \mathbf{S}_k = \mathbf{0}$.

If the parameters $\mathcal{P}_{k,j}$ are paired correctly across different projections, that is, if we can decide whether two parameters $\mathcal{P}_{k,j}$ and $\mathcal{P}_{k',i}$ on two different projections correspond to the same vertex, we can then build the following matrix with elements $\mathbf{\Omega_1}(k,j) = \mathcal{P}_{k,j}^1 - \mathcal{P}_{k-1,j}^1 = (\mathbf{S}_k - \mathbf{S}_{k-1})^T \boldsymbol{l}_{j_1}$:

$$\mathbf{\Omega_1} = \underbrace{\begin{bmatrix} \mathbf{S}_2 - \mathbf{S}_1 \\ \mathbf{S}_3 - \mathbf{S}_2 \\ \vdots \\ \mathbf{S}_K - \mathbf{S}_{K-1} \end{bmatrix}^T}_{(K-1)\times 3} \underbrace{\begin{bmatrix} \boldsymbol{l}_{1_1} & \boldsymbol{l}_{2_1} & \dots & \boldsymbol{l}_{J_1} \end{bmatrix}}_{3\times J} := \mathbf{S_1}\mathbf{C_1}. \tag{10}$$

Similarly, we can build another difference matrix $\mathbf{\Omega}_2 = \mathbf{S}_2\mathbf{C}_2$ with entries $\mathbf{\Omega}_2(k,j) = (\mathbf{S}_k - \mathbf{S}_{k-1})^T \boldsymbol{l}_{j_2}$. Under the assumption that there are $K \ge 5$ vertices and $J \ge 4$ non-coplanar direction vectors, the matrices $\mathbf{\Omega}_1$ and $\mathbf{\Omega}_2$ are rank deficient with rank($\mathbf{\Omega}_1$) = rank($\mathbf{\Omega}_2$) = 3. The rank deficiency property is enforced to allow the correct pairing of the parameters which can be done in a pairwise manner. Specifically, we can extract two columns $\mathbf{\Gamma}_1, \mathbf{\Gamma}_2$ from $\mathbf{\Omega}_1, \mathbf{\Omega}_2$ respectively with the same column indices, and form a combined pairwise difference matrix $\mathbf{\Gamma} = [\mathbf{\Gamma}_1, \mathbf{\Gamma}_2]$. By permutating the sequence of vertices on one projection, the matrix with rank($\mathbf{\Gamma}$)= 3 will yield the correct pairing. In case of noise, we look for the matrix with the maximum ratio between the third and the forth singular values.

As in our previous paper [20] we obtain the first estimate of the direction vectors $\tilde{\mathbf{C}}_1$, $\tilde{\mathbf{C}}_2$ by performing singular value decomposition on the difference matrices. Obviously $\mathbf{Q}\tilde{\mathbf{C}}_v = \mathbf{C}_v, v = 1,2$, where $\mathbf{Q}$ is a 3×3 matrix. We notice that $\mathbf{C}_1, \mathbf{C}_2$ have unit norm columns, thus, we solve

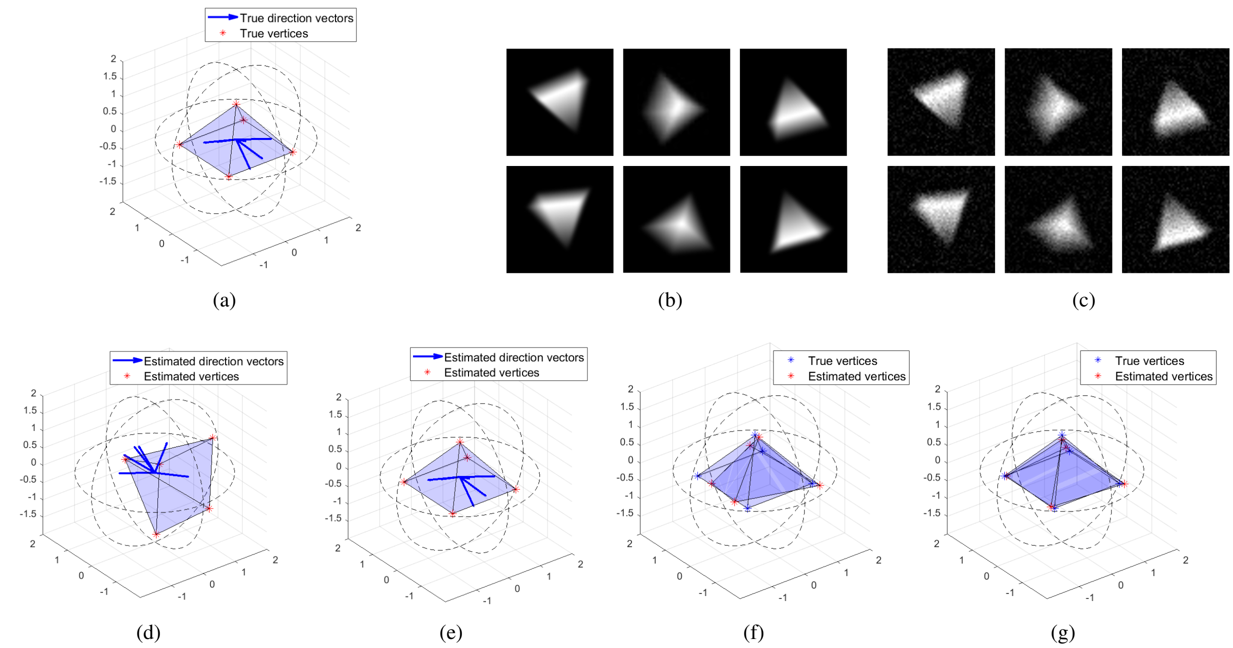


Fig. 3: The ground truth of the polyhedron and direction vectors is shown in (a). In noiseless cases, uniform samples are taken as shown in (b), and the estimation is exact up to an orthogonal transformation as shown in (d). If we assume that three direction vectors are known, this ambiguity can be eliminated and perfect reconstruction can be achieved as in (e). In case the samples are corrupted with noise of SNR = 15dB as in (c), we infer parameters with a deep neural network and the reconstruction for $J = 6$ projections is shown in (f) after further aligning with Procrustes analysis. The reconstruction can be further improved with more projections as shown in (g) with $J = 20$ projections.

for $\mathbf{Q}$ by enforcing $\tilde{\boldsymbol{l}}_{j_v}^T \boldsymbol{M} \tilde{\boldsymbol{l}}_{j_v} = 1$, where $\boldsymbol{M} = \mathbf{Q}^T\mathbf{Q}$ is a 3×3 symmetric matrix, and can be determined given there are $J \geq 6$ projection directions. Consequently, $\mathbf{Q}$ is given by $\mathbf{Q} = \mathbf{B}(\boldsymbol{M})^{\frac{1}{2}}$, where $\mathbf{B}$ is an arbitrary orthogonal matrix.

Once the direction vectors and the unknown shifts are retrieved, the estimation of the vertices is addressed by solving the linear system of equations in Eq. (4).

Finally, given the vertices $\{\mathbf{S}_k\}_{k=1}^K$ the reconstruction of $g(\mathbf{r})$ is unique due to the convexity assumption. □

Remark. *The above estimation algorithm can be applied to the estimation of 3D point sources. The locations and amplitudes of the projected 2D Diracs can be perfectly retrieved using techniques in the area of finite rate of innovation [23, 24]. Given the assumptions as in* ***Proposition 1****, the directions and locations of point sources can be estimated up to an orthogonal transformation using the same approach.*

4. NUMERICAL RESULTS

We consider the model $g(\mathbf{r})$ composed of $K = 5$ vertices in the sphere region of radius $R = 2$, and the projections are taken at $J = 6$ non-coplanar directions. The samples are taken uniformly with 2D B-spline of order 12 along both x and y directions at sampling period $T_1 = T_2 = \frac{1}{2^6}$ as shown in Fig. 3 (b). The result in Fig. 3 (d) shows that the reconstruction of the polyhedron and projection directions using the proposed method are exact up to an orthogonal transformation when compared to the ground truth in Fig. 3 (a). If we assume that three directions are known, the ambiguity can be eliminated and the estimation is exact as shown in Fig. 3 (e).

In practical scenarios, the 2D samples of projections may be corrupted by noise, which leads to errors in the estimation of parameters $\mathcal{P}_{k,j}$, and consequently in the estimation of the 3D structure and projection directions. In this case, we infer the parameters $\mathcal{P}_{k,j}$ directly from the noisy samples $\mathcal{P}_j[m, n]$ using a carefully designed deep neural network. Given the estimated parameters $\mathcal{P}_{k,j}$ we form the difference matrix in Eq. (10) and retrieve the locations and directions. For comparison, we then use Procrustes analysis [25] to estimate a linear transformation between the estimated and true locations of vertices. Fig. 3 (c) shows the corrupted samples with additive white Gaussian noise of SNR $= 15$dB. The reconstruction result is shown in Fig. 3 (f) where the average squared error of the locations of vertices is $\epsilon = 8.67 \times 10^{-2}$. In addition, by increasing the number J of projections to $J = 20$ projections, we reduce the error to only $\epsilon = 1.32 \times 10^{-2}$, see Fig. 3(g). For different noise levels, we repeat 500 experiments with randomly generated convex polyhedra with 5 vertices. In Table. 1 we report the average squared error for the estimated vertices reconstructed from 6 projections.

SNR (dB)	15	20	25
Average squared error ϵ	9.21×10^{-2}	3.83×10^{-2}	5.24×10^{-3}

Table 1: Average squared error of estimated vertices reconstructed from 6 unknown projections. The results are averaged over 500 experiments with randomly generated polyhedra with 5 vertices.

5. CONCLUSION

In this work, we presented a method to simultaneously recover the 3D structure of the convex polyhedron model from uniform samples of 2D projections at unknown angles. Simulations performed on synthetic data validate the proposed method. In noisy settings, we robustify the approach by increasing the number of projections and by using a deep network to estimate the locations of the vertices on each projection.

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