Yule-Walker-based approaches for estimation of noise-corrupted periodic autoregressive model - finite- and infinite-variance cases

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***Abstract*—Periodic autoregressive (PAR) time series with finite variance is considered one of the most known second-order cyclostationary models. However, in the real applications, the signals with the periodic structure very often are disturbed by additional noise (called additive noise, AN) that may be related to measurement device disturbances or other external sources. Thus, the known estimation techniques dedicated to pure PAR models may be inefficient for such cases. In this paper, the problem of parameters estimation for PAR models with AN is considered. Several known Yule-Walker-based methods dedicated to noise-corrupted PAR time series are discussed and compared using Monte Carlo simulations. The efficiency of all methods is verified for selected distributions of AN. Most of all, we consider the model with infinite-variance (alpha-stable) additive noise, which was not analyzed in the literature before. For such a model, a novel estimation method based on fractional lower-order covariance and high-order Yule-Walker equations is proposed.**

***Index Terms*—periodic autoregressive model, additive noise, α-stable distribution, estimation, Monte Carlo simulations**

I. Introduction

Periodic autoregressive (PAR) model is one of the most known cyclostationary processes. It can be considered as a generalization of classical autoregressive model, having periodic coefficients. In the literature, one can find numerous applications of this time series for the analysis of data describing a phenomenon with periodic characteristics, see e.g., [1], [2].

Although the classical (finite-variance distributed, especially Gaussian) PAR model by itself is a useful tool, it might be not suitable for some non-standard cases. For example, when the observed behavior is not only periodic but also impulsive (e.g., many outliers occur), the PAR model based on the Gaussian distribution could be an inappropriate choice. Instead, one can consider its modification, where the classical distribution is replaced with a heavy-tailed distribution which would be able to capture the aforementioned impulsiveness, see e.g. [3]–[5].

Another possible modification of the standard PAR model refers to the presence of additive noise. This is the case considered in this paper. In reality, the observed behavior may not correspond to the pure periodic autoregression, but rather might be additionally disturbed by some external sources, e.g., related to measurement error or specific processes occurring in described phenomenon. In that case we consider the model being a noise-corrupted PAR time series and thus we are dealing with the so-called hidden periodicity, [6], [7]. Depending on applications, one can analyze different distributions (or even models) of the disturbances. In this paper, we consider the additive noise from continuous distribution and separately analyze the finite- and infinite-variance cases. In both cases, the α-stable distribution is proposed [8], [9]. We note this distribution in finite-variance scenario (i.e. when $\alpha = 2$) reduces to the Gaussian one. The α-stable distribution is selected as the general one that may capture specific characteristics of real data, such as heavy-tailed and non-symmetric behavior.

In the applications of the mentioned noise-corrupted models, one of the crucial steps is the estimation of parameters, [10]. Hence, it is of huge importance to develop algorithms dedicated to the considered case. There are several classes of methods suitable for different models, in a more or less general way. In this paper, the main attention is paid to the Yule-Walker-based techniques, i.e. the methods utilizing the sample autocovariance function (ACVF) or corresponding alternative dependence measures. For the classical finite-variance pure PAR model (i.e., without any additive noise), one can mention the classical Yule-Walker algorithm [11], [12]. For the PAR model with the so-called additive outliers (i.e. additive noise with discrete distribution with a finite number of possible values), the classical Yule-Walker method has its robust versions [13]. For the PAR model with general finite-variance additive noise (hence, also for additive outliers), there are considered four Yule-Walker equations-based methods

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dedicated to such cases [14], [15]. In the most extreme case, when the additive noise has infinite-variance distribution, the autocovariance-based algorithms cannot be applied, since in this case the theoretical ACVF is not defined. Thus, a new approach needs to be proposed. Similar as for the PAR model with heavy-tailed innovations [3], [5], one may analyze the modified Yule-Walker methods that are based on alternative dependence measures. In this paper, we propose to modify the classical methodology by including the fractional lower-order covariance (FLOC) and introduce the modified Yule-Walker algorithm for PAR model with additive noise with a heavy-tailed infinite-variance distribution. The efficiency of the mentioned above algorithms for finite- and infinite-variance distributed additive noise is verified by Monte Carlo simulations.

The paper is structured as follows. In Section II, the PAR model with additive noise is recalled. In Section III, the estimation methods for considered cases of additive noise are briefly discussed. In Section IV, the comparison of estimators and its results are presented. The paper is concluded in Section V. Because of a shorter form of this work, not all technical details are described; however, all necessary references are provided. In Appendix A, we present the new estimation algorithm for PAR model with infinite-variance distributed additive noise.

II. Periodic autoregressive model with additive noise

The sequence $\{X_t\}$ is called a PAR(p) process with period T (with $t \in \mathbb{Z}$ and $p, T \in \mathbb{N}$) when it satisfies the following equation [1]:

$$X_t - \phi_1(t)X_{t-1} - \cdots - \phi_p(t)X_{t-p} = \xi_t, \tag{1}$$

where $\{\xi_t\}$ is called innovation sequence and in finite-variance case is assumed to be white noise with mean 0 and variance σ_ξ^2. The coefficients $\{\phi_i(t),\ i = 1, \cdots, p\}$ are periodic in t with the same period T. We assume the considered PAR model is causal. The sequence given in model (1) is second-order cyclostationary with period T, which means its mean $\mu(t) = \mathbb{E}(X_t)$ and its autocovariance function $\gamma(t, h) = cov(X_t, X_{t+h})$ are T-periodic with respect to t. For simplicity, in this paper, we assume the innovations are Gaussian distributed. However, as was mentioned, one may also consider a more general case of model (1), and replace the Gaussian distribution with other ones, i.e. α-stable distribution [3]. In this case, the cyclostationarity is expressed by means of alternative dependence measures, see e.g. [4].

The periodic autoregressive model with additive noise $\{Y_t\}$ (called also noise-disturbed PAR model), which is the main model under consideration here, is constructed as follows [14], [15]:

$$Y_t = X_t + Z_t, \tag{2}$$

where $\{X_t\}$ is a PAR(p) model defined above and $\{Z_t\}$ is an additive noise sequence of independent identically distributed (i.i.d.) random variables, independent of $\{X_t\}$ series. In this paper, we distinguish here between two scenarios of additive noise distribution, finite- and infinite-variance cases. As a general distribution of the noise $\{Z_t\}$ we consider symmetric α-stable one [8], [9]. We remind, the random variable has symmetric α-stable distribution $\mathcal{S}(\alpha, \sigma)$ if its characteristic function is given by

$$\Phi(s) = \exp\left(-\sigma^\alpha |s|^\alpha\right), \qquad s \in \mathbb{R}, \tag{3}$$

where $0 < \alpha \leq 2$ is the stability index and $\sigma > 0$ is the scale parameter. For $\alpha = 2$, the α-stable distribution reduces to the Gaussian one ($\mathcal{N}$), and thus it can be considered as a generalization of this classical distribution. The symmetric α-stable distribution has no closed-form probability density function and cumulative distribution function. The only exception is the Gaussian distribution (that is, for $\alpha = 2$) and the Cauchy distribution (that is, for $\alpha = 1$). The stability index is responsible for the heaviness of this distribution's tail, i.e. for the smaller α, the probability of large values is much higher. For $\alpha < 2$, the variance of α-stable distribution is infinite.

One can show that the sequence $\{Y_t\}$ given in (2) satisfies the following equation:

$$\begin{aligned} Y_t - \phi_1(t)Y_{t-1} - \cdots - \phi_p(t)Y_{t-p} &= \\ = \xi_t + Z_t - \phi_1(t)Z_{t-1} - \cdots - \phi_p(t)Z_{t-p}&, \end{aligned} \tag{4}$$

which indicates it is no longer a PAR model (1), however as it was mentioned in [15], it is still second-order cyclostationary in case the additive noise $\{Z_t\}$ is finite-variance distributed.

Let us note that (4) is useful for designing estimation algorithms — it is a base for discussed estimators, also in the most extreme case, namely for the PAR model with α-stable additive noise with $\alpha < 2$.

III. Estimation methods for periodic autoregressive model with additive noise

As mentioned, the estimation methods analyzed in this paper can be grouped into classes depending on the distribution of the additive noise. In the following subsections, the methods of all classes are listed and briefly discussed. Technical details of the known algorithms are not presented in this paper. However, the appropriate references are given. We remind, in the presented approaches we assume the order p and period T of the model (4) are known. We refer the readers to [10] for the new approach useful for the identification of such parameters for noise-corrupted PAR model.

A. Pure PAR model

- *Yule-Walker (YW) method (also known as low-order Yule-Walker method)* [11], [12]
  The YW algorithm is one of the most common estimation methods for finite-variance PAR (or AR) models, widely used because of its simplicity and high efficiency. In this method, we construct systems of equations with autocovariance-based terms, replace them with empirical counterparts using classical autocovariance estimator (sample autocovariance function) and solve the system

of equations to obtain estimates of the coefficients. This method is also a base for several other approaches presented further in this paper.

B. PAR model with finite-variance distributed additive noise

- *High-order Yule-Walker (hYW) method* [14], [15]
The derivation of this method is very similar to the classical low-order Yule-Walker algorithm recalled above. However, in this case, to construct autocovariance terms present in designed systems of equations, we use more "lagged" values of the now noise-corrupted process. Such an approach was earlier widely considered also for autoregressive models with additive noise, see e.g. [16]. This algorithm allows to overcome the bias in the estimation which is present in the standard YW method for models with additive noise. However, as presented in [15], in some cases this method may return some extremely large variance of the estimators.
- *Errors-in-variables (EIV) method* [14]
This approach is a generalization of the method presented in [17] (from AR model with additive noise to PAR with additive noise). The general idea is to combine low- and high-order Yule-Walker methods to achieve their advantages (respectively, low variance and bias) and negate their drawbacks (respectively, large bias and variance). First, using high-order Yule-Walker equations, we find an estimate of additive noise variance (for each season $v = 1, \cdots, T$ separately) which is then put into low-order Yule-Walker equations (derived for noise-corrupted process) for estimation. In this paper, for this method (as well as for modified errors-in-variables and constrained least squares optimization methods described below) we set $s = 2$, where s is the number of high-order Yule-Walker equations used in the method.
- *Modified errors-in-variables (mEIV) method* [14]
This method is only a slight modification of the EIV approach. Let us remind that previously the estimate of additive noise variance was found separately for each season. However, as we consider the constant variance of additive noise for each season, we can modify this method to include this assumption. Here, we first find a single estimate of additive noise variance and then use it in the estimation of coefficients for each season.
- *Constrained least squares optimization (CLSO) method* [14]
The last method from this class is a generalization of one of the techniques presented in [18] for AR models with additive noise. Similarly to EIV and mEIV approaches, it also can be considered as a combination of low- and high-order Yule-Walker methods. First, using low-order and first high-order equations, the estimate of additive noise variance is found in an iterative procedure. Then, the estimated coefficients are obtained as least-squares solutions of the systems of low- and high-order Yule-Walker equations.

C. PAR model with infinite-variance distributed additive noise

- *Fractional lower-order covariance-based high-order Yule-Walker (hFLOC) method – novel method*
This method can be considered as a novel approach designed for the PAR model with α-stable additive noise. Its derivation is presented in Appendix A. For simulations, we set $B = 0.65$. In practice, initially we calculate $\hat{\alpha}$ from the data and assume $B < \hat{\alpha} - 1$ [5]. Here, we modify the hYW method by replacing autocovariance with fractional lower-order covariance (called FLOC), an alternative dependence measure defined for infinite-variance distributions. The same idea is considered in [5] for the YW method. We remind, the fractional lower-order moments-based approaches are widely applied in signal processing techniques, i.e. for the estimation of the parameters of α-stable distribution, see e.g. [19], [20].
- *Robust Yule-Walker (rYW) method* [13]
In case when the additive noise has impulsive behavior, one may also consider the approaches being straight modifications of the classical Yule-Walker method by replacing the sample ACVF with its robust versions. The only difference between the YW method and the rYW approach is the utilized estimator of autocovariance function terms. The robust version of the YW method was widely discussed in the case of the PAR model with additive outliers, see e.g. [21].

IV. Simulation Study

In this section, we present the Monte Carlo simulation study for the comparison of analyzed methods. For simplicity, they will be assessed for the noise-corrupted PAR(1) model with $T = 2$, $\phi_1(1) = 0.7$, $\phi_1(2) = -1.1$, and $\{\xi_t\} \sim \mathcal{N}(0, \sigma_\xi^2 = 1)$, for different types of additive noise. We consider the three following cases:

- Model 1: $\{Z_t\} = 0$ (pure PAR model)
- Model 2: $\{Z_t\} \sim \mathcal{N}(0, \sigma_Z^2 = 1)$
- Model 3: $\{Z_t\} \sim \mathcal{S}(\alpha = 1.7, \sigma = 1)$

In Fig. 1, the sample trajectories of the considered models are presented. The underlying pure PAR model sample is the same on each plot. In the case of Gaussian additive noise (Model 2), the difference between clean and corrupted samples does not seem to be significant; however, as will be shown, the additive noise presence strongly influences the estimation of the parameters. The case of α-stable additive noise (Model 3), with relatively low $\alpha = 1.7$, is clearly the most challenging one – because of the present outliers, the underlying PAR trajectory becomes even more hidden.

For each model, the simulation procedure is as follows. We simulate $M = 1000$ trajectories of length $N = 3000$ from a given model. For each sample, we estimate values of $\phi_1(1)$ and $\phi_1(2)$ using all analyzed methods. In the end, for each method and each coefficient, we create a boxplot of obtained values. Moreover, for each method, we calculate the average mean absolute error (MAE) in order to compare the results.

The boxplots of the estimators obtained for Model 1 are presented in Fig. 2. One can see that all analyzed methods

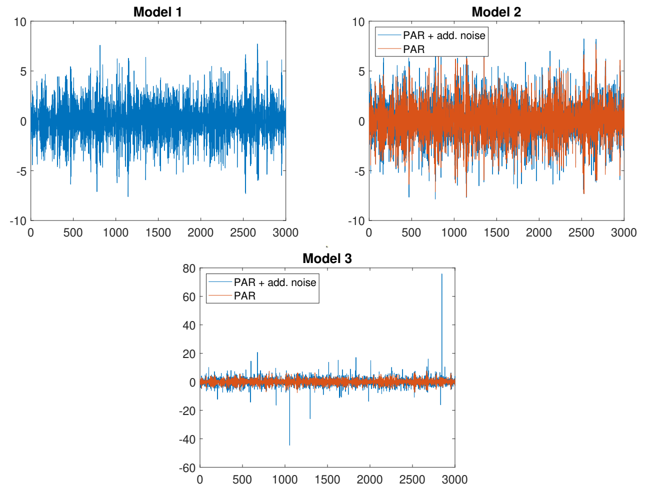

Fig. 1. Sample trajectories of models considered in the simulation study. The red trajectories correspond to the pure PAR model while the blue ones - to the noise-corrupted PAR model.

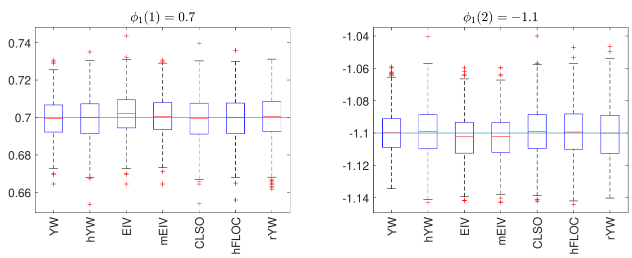

Fig. 2. Boxplots of estimated values for Model 1 (pure PAR).

(including those devoted to cases with additive noise) produce estimates around the true value of a given coefficient. As can be seen in Tab. I, the classical YW method has the lowest MAE, which could be expected for a pure PAR model. However, its advantage over the other approaches is relatively small. These results indicate that the analyzed non-standard methods preserve an acceptable efficiency also for undisturbed PAR models.

In Fig. 3, one can see the illustrated estimation results for Model 2. Here, due to a significant presence of additive noise, the observed behavior is different than in the previous case. One can easily distinguish between methods that take into account the general additive noise presence (hYW, EIV, mEIV, CLSO, hFLOC) and other algorithms (i.e. YW, rYW). The former are able to maintain their efficiency and still produce reliable results, whereas the latter do not, due to a significant

TABLE I
AVERAGE MAE VALUES OBTAINED FOR EACH CONSIDERED MODEL (WITH THE BEST RESULT HIGHLIGHTED).

model	YW	hYW	EIV	mEIV	CLSO	hFLOC	rYW
1	**0.0095**	0.0110	0.0102	0.0098	0.0111	0.0112	0.0115
2	0.1729	0.0214	0.0211	**0.0192**	0.0209	0.0218	0.1719
3	0.5420	0.0727	0.1697	0.2153	0.0656	**0.0653**	0.3076

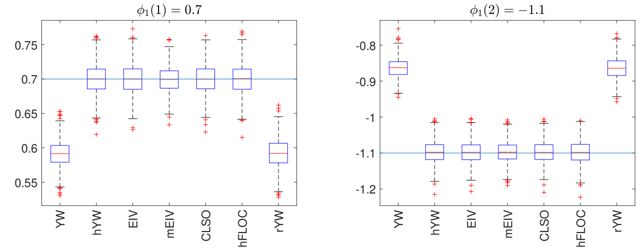

Fig. 3. Boxplots of estimated values for Model 2 (Gaussian additive noise with $\sigma_Z^2 = 1$).

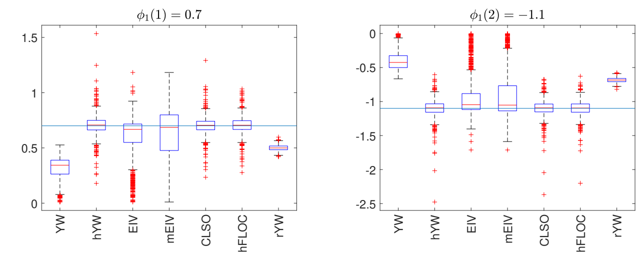

Fig. 4. Boxplots of estimated values for Model 3 (α-stable additive noise with $\alpha = 1.7$, $\sigma = 1$).

bias. This is also confirmed by MAE results presented in Tab. I and their difference between both groups. Among all analyzed methods, the lowest error was obtained for the mEIV method.

For Model 3, the estimation results are illustrated in Fig. 4. As expected, this case was the most challenging one. One can compare the observed behavior to the case of Model 2 — only methods designed for cases with general additive noise seem to be unbiased. However, here all methods produce more outliers (in contrast to the previous models), which is caused by the heavy-tailed distribution of the additive noise. As presented in Tab. I, the best result was obtained for the hFLOC method which is designed particularly for such case. However, let us note that CLSO and hYW methods performed only slightly worse, even though they are not defined for an infinite-variance case. Let us also comment on the performance of the rYW approach. Although it is again biased because of the general corruption of a PAR signal, its variance is very low.

V. CONCLUSIONS

We have discussed the noise-corrupted PAR model. We have recalled estimation techniques for the analyzed model in cases when the additive noise has finite-variance distribution. The main motivation for the current research comes from condition monitoring, where the models of real vibration might be used to develop an inverse filter to remove components related to mesh frequencies in gearbox vibrations. The presence of additive noise is very intuitive in such a case, and the additive noise might be related to electromagnetic disturbances in measurement systems, specific technological processes carried out by machines or any rapid, transient phenomena. The novelty of this paper is the introduction of a new estimation technique for the PAR model with heavy-tailed (infinite-variance) distributed additive noise. The new algorithm is confronted with other Yule-Walker approaches dedicated to finite-variance distributed disturbances. The presented simulation study indicates that the new approach outperforms the known algorithms in most extreme cases.

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Appendix A
Estimation algorithm for PAR model with infinite-variance distributed additive noise

Let us consider the periodic version of (4), setting $t = nT + v$ for $v = 1, \cdots, T$ and $n \in \mathbb{Z}$:

$$\begin{aligned} Y_{nT+v} - \phi_1(v)Y_{nT+v-1} - \cdots - \phi_p(v)Y_{nT+v-p} &= \\ = \xi_{nT+v} + Z_{nT+v} - \phi_1(v)Z_{nT+v-1} - \cdots \\ \cdots - \phi_p(v)Z_{nT+v-p}. \end{aligned} \tag{5}$$

We assume that $\{\xi_t\}$ is white noise and $\{Z_t\} \sim \mathcal{S}(\alpha, \sigma)$. We define the periodic version of fractional lower-order covariance (FLOC) for a process $\{Y_t\}$ as:

$$\psi^Y(w, k, A, B) = \mathbb{E}[Y_{nT+w}^{<A>} Y_{nT+w-k}^{<B>}], \tag{6}$$

where $x^{<a>} = |x|^a \operatorname{sgn}(x)$ is a signed power.

First, for each $v = 1, \cdots, T$, we multiply (5) by $Y_{nT+v-(p+1)}^{<B>}, \cdots, Y_{nT+v-2p}^{<B>}$ and take the expected value of both sides. Using the fact that $x = x^{<1>}$, we obtain the following system of p equations:

$$\begin{cases} \mathbb{E}[Y_{nT+v}^{<1>} Y_{nT+v-(p+1)}^{<B>}] - \phi_1(v)\mathbb{E}[Y_{nT+v-1}^{<1>} Y_{nT+v-(p+1)}^{<B>}] \\ - \cdots - \phi_p(v)\mathbb{E}[Y_{nT+v-p}^{<1>} Y_{nT+v-(p+1)}^{<B>}] = 0, \\ \vdots \\ \mathbb{E}[Y_{nT+v}^{<1>} Y_{nT+v-2p}^{<B>}] - \phi_1(v)\mathbb{E}[Y_{nT+v-1}^{<1>} Y_{nT+v-2p}^{<B>}] \\ - \cdots - \phi_p(v)\mathbb{E}[Y_{nT+v-p}^{<1>} Y_{nT+v-2p}^{<B>}] = 0. \end{cases}$$

Let us note that all right-hand sides can be set to zero, as both ξ_t and Z_t are zero-mean and independent of Y_s for $s < t$. Hence, by rewriting expressions above in a FLOC form, we obtain:

$$\begin{cases} \psi^Y(v, p+1, 1, B) = \phi_1(v)\psi^Y(v-1, p, 1, B) \\ + \cdots + \phi_p(v)\psi^Y(v-p, 1, 1, B), \\ \vdots \\ \psi^Y(v, 2p, 1, B) = \phi_1(v)\psi^Y(v-1, 2p-1, 1, B) \\ + \cdots + \phi_p(v)\psi^Y(v-p, p, 1, B), \end{cases}$$

and, in the matrix form:

$$\boldsymbol{\psi}_v^Y = \boldsymbol{\Psi}_v^Y \boldsymbol{\phi}_v, \tag{7}$$

where:

$$\boldsymbol{\psi}_v^Y = [\psi^Y(v, p+1, 1, B), \cdots, \psi^Y(v, 2p, 1, B)]', \tag{8}$$

$$(\boldsymbol{\Psi}_v^Y)_{ij} = \psi^Y(v-j, p+i-j, 1, B), \; i, j = 1, \cdots, p, \tag{9}$$

$$\boldsymbol{\phi}_v = [\phi_1(v), \cdots, \phi_p(v)]'. \tag{10}$$

The estimator of the FLOC for sample $y_1, \cdots, y_{NT}$ is:

$$\hat{\psi}(w, k, A, B) = \frac{1}{N} \sum_{n=l}^{r} y_{nT+w}^{<A>} y_{nT+w-k}^{<B>}, \tag{11}$$

$$l = \max\left(\left\lceil \frac{1-w}{T} \right\rceil, \left\lceil \frac{1-(w-k)}{T} \right\rceil \right), \tag{12}$$

$$r = \min\left(\left\lfloor \frac{NT-w}{T} \right\rfloor, \left\lfloor \frac{NT-(w-k)}{T} \right\rfloor \right). \tag{13}$$

The hFLOC method is well-defined if $B < \alpha - 1$.