# Bearings-Only Tracking With Speed and Range Constraints

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***Abstract*—Target motion analysis for bearings-only tracking is a challenging task and while estimating the range and velocity of the target, estimation error may diverge. To improve the estimation accuracy and minimize the track divergence, in this paper, we assume that by hearing the sound, a sonar operator is capable of saying the upper and lower limit of range and velocity of the target and we incorporate such information along with the traditional state estimation methods. To do so, after obtaining the posterior estimate a constrained optimization problem which minimizes weighted error square has been solved. The developed method is applied to two bearings-only tracking scenarios. It has been observed that the proposed technique delivers more accurate results in terms of root mean square error and track loss percentage than that obtained from only nonlinear filters.**

***Index Terms*—Bearings-only tracking, target motion analysis, constrained optimization, Lagrange multiplier.**

## I. Introduction

The bearings-only tracking (BOT) problem finds major application in underwater target tracking [1], due to the fact that passive bearing measurements aid in hiding the ownship's position from the enemies [1]. The objective of BOT problems is to obtain a moving target's kinematics with noisy bearing measurements so it is also referred to as target motion analysis (TMA) [2]. In TMA, the observer has to maneuver to make the system observable while tracking a non-maneuvering target [3]–[5]. Thus, TMA is a challenging task and many times it leads to poor estimation accuracy [6] and high track divergence [7].

Several nonlinear filtering techniques are used to solve the TMA. The first among which is the extended Kalman filter (EKF) [8] and its variants [9], [10]. Poor estimation accuracy of the EKF leads to the idea of deterministic sample point filtering which includes cubature Kalman filter (CKF) [11], [12], unscented Kalman filter (UKF) [13], Gauss Hermite filter (GHF) [7], [14] *etc.* A few filters such as the shifted Rayleigh filter (SRF) [15], [16], the batch recursive filter [17], [18], and weighted instrumental variable (WIV) [19] estimator are specifically developed for solving a BOT problem. However, the number of diverging tracks still remains, especially for highly nonlinear scenarios [6], [7].

The sound emitted from the enemy ship is generally received by hydrophones, mounted on the hull of the own ship or towed at the back of the ship using a cable. The received sound is also heard manually by a sonar operator. The operator is trained and experienced enough to guess the limit of the range and velocity of the enemy ship or submarine. Sometimes the operators on the ownship may have prior knowledge about the type of the target vessel, which allows them to leverage valuable information about the target's range and velocity limits. To increase the accuracy of the estimation and decrease the percentage of track loss, the information received from the operator is incorporated with the estimation methods. In [17], [20], [21], it has been assumed that the limit of the target's radial velocity is known and with such limit a constrained optimization problem has been solved.

In this paper, we have incorporated the limit of the target's range in addition to the limit of target's radial velocity as the constraints at each time instant. This has been done by formulating a constrained optimization problem and solving it using the Lagrange multiplier. The cost function is considered as the square of the errors weighted with the error covariance matrix. The optimization process is implemented along with the existing state estimation techniques when the velocity or/and range of the estimators go beyond the said limit.

The developed method is implemented on two BOT scenarios, namely one moderately nonlinear and another highly nonlinear. Simulation results show that the proposed technique of estimation with both the range and velocity constraints is more accurate compared to the unconstrained estimators in terms of root mean square error (RMSE) and track loss percentage. Interestingly, the proposed constrained optimization can be incorporated with any nonlinear state estimation method and is thus capable of improving the accuracy when the operator's experience is included.

## II. Problem Formulation

A two dimensional tracking problem is considered, where the target follows a nearly constant velocity path. The target dynamics and measurement model can be represented as [6]

$$\mathcal{X}_k = F\mathcal{X}_{k-1} - U_{k-1,k} + v_{k-1}, \tag{1}$$

$$\mathcal{Y}_k = \tan^{-1}\left(\frac{x_k}{y_k}\right) + \omega_k, \tag{2}$$

where $\mathcal{X}_k = \mathcal{X}_k^t - \mathcal{X}_k^o = \begin{bmatrix} x_k & y_k & \dot{x}_k & \dot{y}_k \end{bmatrix}^T$ is the relative state vector. $\mathcal{X}_k^t$ and $\mathcal{X}_k^o$ represent the target and the observer state, respectively and $\mathcal{Y}_k$ is the measurement. $U_{k-1,k}$ is the vector of input defined as

$$U_{k-1,k} = \begin{bmatrix} x_k^o - x_{k-1}^o - \Delta\dot{x}_{k-1}^o \\ y_k^0 - y_{k-1}^o - \Delta\dot{y}_{k-1}^o \\ \dot{x}_k^o - \dot{x}_{k-1}^o \\ \dot{y}_k^o - \dot{y}_{k-1}^o \end{bmatrix},$$

where $\Delta$ is the sampling time. $F$ is the system matrix defined as

$$F = \begin{bmatrix} I_{2\times2} & \Delta I_{2\times2} \\ 0_{2\times2} & I_{2\times2} \end{bmatrix}.$$

$v_{k-1}$ and $\omega_k$ are process and measurement noises, respectively which are assumed to be white, following a Gaussian distribution with zero mean and covariance $Q_{k-1}$ and $R_k$, respectively. The expression for $Q_k$ can be derived as,

$$Q_k = \begin{bmatrix} \frac{\Delta^3}{3} I_{2\times2} & \frac{\Delta^2}{2} I_{2\times2} \\ \frac{\Delta^2}{2} I_{2\times2} & \Delta I_{2\times2} \end{bmatrix} q,$$

where $q$ is the process noise intensity.

## III. Tracking Methodology

### A. State constraint

As stated above a trained sonar operator has enough experience to say about the upper and lower limit of range and velocity of a target. So we can write the target's range, $r_k = \sqrt{x_k^2 + y_k^2} \in [r_{k,min}, r_{k,max}]$ and the target's radial velocity $v = \sqrt{\dot{x}_k^2 + \dot{y}_k^2} \in [v_{min}, v_{max}]$. It can be noted that we omit the subscript $k$ from velocity because we assume the target is moving with a near constant velocity.

We define the cost function as

$$J(\hat{\mathcal{X}}'_{k|k}) = \arg\min_{\hat{\mathcal{X}}'_{k|k}} (\hat{\mathcal{X}}'_{k|k} - \hat{\mathcal{X}}_{k|k})^T P_{k|k}^{-1} (\hat{\mathcal{X}}'_{k|k} - \hat{\mathcal{X}}_{k|k}), \quad (3)$$

where $\hat{\mathcal{X}}'_{k|k}$ is the constrained estimate; $\hat{\mathcal{X}}_{k|k}$ and $P_{k|k}$ are the posterior state estimate and posterior error covariance of state obtained from any traditional nonlinear filter. We have to solve Eq. (3) subjected to the constraint,

$$\begin{bmatrix} r_{k,min}^2 \\ v_{min}^2 \end{bmatrix} \leq D_{rv} \hat{\mathcal{X}}'_{k|k} \leq \begin{bmatrix} r_{k,max}^2 \\ v_{max}^2 \end{bmatrix}, \quad (4)$$

where $D_{rv} = \begin{bmatrix} \hat{x}_{k|k} & \hat{y}_{k|k} & 0 & 0 \\ 0 & 0 & \hat{\dot{x}}_{k|k} & \hat{\dot{y}}_{k|k} \end{bmatrix}$.

Four cases are taken under consideration which are: (i) the posterior range estimate, $\hat{r}_{k|k}$ goes beyond the upper bound of the range, $r_{k,max}$, (ii) the $\hat{r}_{k|k}$ goes below the lower bound of the range, $r_{k,min}$, (iii) the posterior velocity estimate, $\hat{v}_{k|k}$ goes beyond the upper bound of velocity, $v_{max}$ and (iv) $\hat{v}_{k|k}$ goes below the lower bound of velocity, $v_{min}$. If the posterior estimate is within the range and velocity limit, the optimization problem mentioned above is not required to be solved.

*1) Range only constrained estimation:* For range only constrained estimation, if the estimated range is out of the bound, either case (i) or (ii) may occur at any time step. Let, the range only constrained estimate to be represented by $\hat{\mathcal{X}}'_{r,k|k}$. When $\hat{r}_{k|k} \geq r_{k,max}$, we try to assign the $x$ axis and $y$ axis position estimate in such a way that the estimated range remains the same as the upper bound of it *i.e.* $r_{k,max}$. In such consideration the inequality constrain can be replaced with the equality constrain and our optimization problem becomes

$$J(\hat{\mathcal{X}}'_{r,k|k}) = \arg\min_{\hat{\mathcal{X}}'_{r,k|k}} (\hat{\mathcal{X}}'_{r,k|k} - \hat{\mathcal{X}}_{k|k})^T P_{k|k}^{-1} (\hat{\mathcal{X}}'_{r,k|k} - \hat{\mathcal{X}}_{k|k}), \quad (5)$$

subjected to,

$$D_r \hat{\mathcal{X}}'_{r,k|k} = r_{k,max}^2, \quad \text{when} \quad \hat{r}_{k|k} \geq r_{k,max}, \quad (6)$$

$$D_r \hat{\mathcal{X}}'_{r,k|k} = r_{k,min}^2, \quad \text{when} \quad \hat{r}_{k|k} \leq r_{k,min}, \quad (7)$$

where $D_r = \begin{bmatrix} \hat{x}_{k|k} & \hat{y}_{k|k} & 0 & 0 \end{bmatrix}$.

*2) Velocity only constrained estimation:* The estimated velocity at any instant may remain within the bound or it may be out of the limit as guessed by the operator. If it is out of the limit, Case (iii) or (iv) may occur at each time step. In such cases, the optimization problem becomes

$$J(\hat{\mathcal{X}}'_{v,k|k}) = \arg\min_{\hat{\mathcal{X}}'_{v,k|k}} (\hat{\mathcal{X}}'_{v,k|k} - \hat{\mathcal{X}}_{k|k})^T P_{k|k}^{-1} (\hat{\mathcal{X}}'_{v,k|k} - \hat{\mathcal{X}}_{k|k}), \quad (8)$$

subjected to

$$D_v \hat{\mathcal{X}}'_{v,k|k} = v_{max}^2, \quad \text{when} \quad \hat{v}_{k|k} \geq v_{max}, \quad (9)$$

$$D_v \hat{\mathcal{X}}'_{v,k|k} = v_{min}^2, \quad \text{when} \quad \hat{v}_{k|k} \leq v_{min}, \quad (10)$$

where $D_v = \begin{bmatrix} 0 & 0 & \hat{\dot{x}}_{k|k} & \hat{\dot{y}}_{k|k} \end{bmatrix}$.

*3) Range and velocity constrained estimate:* In range and velocity constrained estimation, range and velocity obtained from a nonlinear filter may remain inside the limit guessed by the operator or $\hat{r}_{k|k}$ or $\hat{v}_{k|k}$ or both may go beyond their respective boundaries. Thus, while performing both range and velocity constrained optimization, if the filter's states go beyond the limit, any one from Case (i) and (ii) may occur along with another from either Case (iii) and (iv) *i.e.,* at most two out of the four cases may occur at each time step. So, at first, the range only constrained estimation is performed and then the velocity only constrained estimation is performed. On merging both the constraints the range and velocity constrained estimate, $\hat{\mathcal{X}}'_{k|k}$ can be evaluated as

$$\hat{\mathcal{X}}'_{k|k} = \begin{bmatrix} I_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & 0_{2\times2} \end{bmatrix} \hat{\mathcal{X}}'_{r,k|k} + \begin{bmatrix} 0_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & I_{2\times2} \end{bmatrix} \hat{\mathcal{X}}'_{v,k|k}. \quad (11)$$

### B. Solution

The constrained optimization problems described above are solved in this subsection. Without loss of generality, we can express the above optimization problems as

$$J(X) = \arg\min_{X} (X - \hat{\mathcal{X}}_{k|k})^T P_{k|k}^{-1} (X - \hat{\mathcal{X}}_{k|k}), \quad (12)$$

subjected to

$$DX = d. \tag{13}$$

To solve it using Lagrange multiplier we augment the cost function

$$L(X,\lambda) = (X-\hat{\mathcal{X}}_{k|k})^T P_{k|k}^{-1}(X-\hat{\mathcal{X}}_{kk})+2\lambda^T(DX-d), \tag{14}$$

where $\lambda$ is the Lagrange multiplier. To minimize $L$, the necessary first order conditions are as follows [22]

$$\frac{\partial L}{\partial X} = 0 \implies P_{k|k}^{-1}(X-\hat{\mathcal{X}}_{k|k}) + D^T\lambda = 0, \tag{15}$$

and

$$\frac{\partial L}{\partial \lambda} = 0 \implies DX - d = 0. \tag{16}$$

On solving Eqs. (15) - (16), we get

$$\lambda = (DP_{k|k}D^T)^{-1}(D\hat{\mathcal{X}}_{k|k} - d). \tag{17}$$

Substituting Eq. (17) in Eq. (15), we get

$$X = \hat{\mathcal{X}}_{k|k} - P_{k|k}D^T(DP_{k|k}D^T)^{-1}(D\hat{\mathcal{X}}_{k|k} - d). \tag{18}$$

The second order condition that is sufficient for minimization of $L$ is [23, p. 803]

$$\frac{\partial^2 L}{\partial X^2} = P_{k|k}^{-1} \geq 0, \tag{19}$$

where $P_{k|k}^{-1}$ is the inverse of a positive definite matrix and the determinant of Hessian matrix of $L$ from Eq. (14) can be expressed as

$$\frac{\partial^2 L}{\partial X^2}\frac{\partial^2 L}{\partial \lambda^2} - \frac{\partial^2 L}{\partial X \partial \lambda}\frac{\partial^2 L}{\partial \lambda \partial X}, \tag{20}$$

evaluating which we get $DD^T$, which is again positive as on applying range constraint, $DD^T = D_r D_r^T = \hat{x}_{k|k}^2 + \hat{y}_{k|k}^2$ and on applying velocity constraint, $DD^T = D_v D_v^T = \hat{\dot{x}}_{k|k}^2 + \hat{\dot{y}}_{k|k}^2$. Thus, Eq. (18) delivers the local minima.

The pseudocode of the developed method is shown in Algorithm 1, where $\hat{\mathcal{X}}_{k|k-1}$, $P_{k|k-1}$ could be obtained from any nonlinear state estimator. For TMA of underwater BOT problem, deterministic sampling point filters [11], [13], [24], or particle filter [25] or shifted Rayleigh filter [26] or other available filters [8], [9], [12] could be used.

### C. Constrained Cramer-Rao bound

The covariance of $\hat{\mathcal{X}}'_{0|0}$ [17],

$$\mathbb{E}[(\hat{\mathcal{X}}'_{0|0} - \mathcal{X}_0)(\hat{\mathcal{X}}'_{0|0} - \mathcal{X}_0)^T] \geq P_0^{CCR}, \tag{21}$$

where $\mathbb{E}[\cdot]$ represents the expectation. $P_0^{CCR}$ is initial error covariance of constrained Cramer-Rao bound. The unconstrained Fisher information matrix (FIM), $\mathcal{I}_0$ is defined as [19],

$$\mathcal{I}_0 = \mathbb{E}[(\nabla_{\mathcal{X}_0} \ln p(\mathcal{Y}_0, \mathcal{X}_0))(\nabla_{\mathcal{X}_0} \ln p(\mathcal{Y}_0, \mathcal{X}_0))^T], \tag{22}$$

where the expectation is with respect to $\mathcal{Y}_0$ and $\mathcal{X}_0$, and

$$p(\mathcal{Y}_0, \mathcal{X}_0) = p(\mathcal{X}_0)p(\mathcal{Y}_0|\mathcal{X}_0), \tag{23}$$

**Algorithm 1** Range and velocity constrained estimation

Initialize filter, $\hat{\mathcal{X}}_{0|0}$, $P_{0|0}$.
**for** $k = 1 : k_{max}$ **do**
  Perform time update to produce $\hat{\mathcal{X}}_{k|k-1}$, $P_{k|k-1}$.
  Perform measurement update to produce $\hat{\mathcal{X}}_{k|k}$, $P_{k|k}$.
  Evaluate range, $\hat{r}_{k|k} = \sqrt{\hat{x}_{k|k}^2 + \hat{y}_{k|k}^2}$.
  **if** $\hat{r}_{k|k} \leq r_{k,min}$ **then**
    $d = r_{k,min}^2$, $D = D_r$, find $\hat{\mathcal{X}}'_{r,k|k} = X$ using (18).
  **else if** $\hat{r}_{k|k} \geq r_{k,max}$ **then**
    $d = r_{k,max}^2$, $D = D_r$, find $\hat{\mathcal{X}}'_{r,k|k} = X$ using (18).
  **end if**
  **if** $\hat{v}_{k|k} \leq v_{min}$ **then**
    $d = v_{min}^2$, $D = D_v$, find $\hat{\mathcal{X}}'_{v,k|k} = X$ using (18).
  **else if** $\hat{v}_{k|k} \geq v_{max}$ **then**
    $d = v_{max}^2$, $D = D_v$, find $\hat{\mathcal{X}}'_{v,k|k} = X$ using (18).
  **end if**
  Redefine $\hat{\mathcal{X}}_{k|k} = \hat{\mathcal{X}}'_{k|k}$ as in Eq. (11).
**end for**

where $p(\mathcal{Y}_0|\mathcal{X}_0)$ represents the likelihood of measurement. Taking negative natural logarithm of Eq. (23) on following [19] we can write,

$$\mathcal{I}_0 = \mathcal{I}_{0,0} + \mathcal{I}_{0,1}, \tag{24}$$

where $\mathcal{I}_{0,0} = P_{0|0}^{-1}$ and

$$\mathcal{I}_{0,1} = \Big[\nabla_{\mathcal{X}_0} tan^{-1}\Big(\frac{\mathcal{X}_0(1,1)}{\mathcal{X}_0(2,1)}\Big)\Big] R_k^{-1} \Big[\nabla_{\mathcal{X}_0} tan^{-1}\Big(\frac{\mathcal{X}_0(1,1)}{\mathcal{X}_0(2,1)}\Big)\Big]^T. \tag{25}$$

Further, equality constrained FIM, $\mathcal{I}_0^{EC}$ can be evaluated as [27]

$$(\mathcal{I}_0^{EC})^{-1} = \mathcal{I}_0^{-1} - \mathcal{I}_0^{-1}G_{r,v}(G_{r,v}^T \mathcal{I}_0^{-1} G_{r,v})^{-1} G_{r,v}^T \mathcal{I}_0^{-1}, \tag{26}$$

where $G_{r,v} \in \mathbb{R}^{4\times 2}$ denotes the gradient of the constraints $x^2+y^2-r^2 = 0$ and $\dot{x}^2+\dot{y}^2-v^2 = 0$, where $r$ and $v$ represents the necessary values of range and velocity, respectively *i.e.,*

$$G_{r,v} = \begin{bmatrix} 2x & 2y & 0 & 0 \\ 0 & 0 & 2\dot{x} & 2\dot{y} \end{bmatrix}^T. \tag{27}$$

From Eq. (26) we can evaluate $P_0^{CCR}$ as follows:

$$P_0^{CCR} = (\mathcal{I}_0^{EC})^{-1}. \tag{28}$$

Under inequality constraint, [28] the FIM becomes $I_0^{IEC} = P_{0|0}^{-1}$. Once initialized as discussed above, the constrained Cramer-Rao bound of the estimation problem can be calculated using traditional Cramer-Rao bound as in Section III, D of [2], [17], [27].

## IV. Simulation Results

### A. Scenarios

In this paper, two scenarios, one moderately nonlinear and another highly nonlinear are taken into consideration. The moderately nonlinear scenario as shown in Fig. 1a is denoted as Scenario I and the highly nonlinear scenario shown in Fig.

1b is denoted as Scenario II [2], [6]. The total time taken for simulation in both the scenarios is 30 min. The sampling time in both scenarios is $\Delta = 1$ min. The process noise intensity, $q$ is considered to be $1.944 \times 10^{-6}$ $\text{km}^2/\text{min}^3$. All the parameters of both scenarios are as in [7]. For Scenario I, the upper and lower velocity limits are considered to be 5 knot and 3 knot, respectively, and for Scenario II they are 17 knot and 13 knot, respectively. The upper and lower limits of the range in both scenarios are considered to be $\pm 1$ km of the respective true values. In Fig. 1, 'Start' represents the beginning of the trajectories and '*' denotes the point from where ownship started maneuvering. In Fig. 1(a) '+' denotes the ownship ended maneuvering.

### B. Performance comparison

The Algorithm 1 is implemented for constrained estimation using the EKF, CKF, and UKF. The single run plots for Scenario I and Scenario II are shown in Fig. 1a and 1b, respectively, where the estimated trajectory is obtained using constrained CKF (CCKF). It can be seen from the figure that the estimated trajectory merges with the truth at the end of the simulation in both the scenarios. The single run plots of range and velocity, using the UKF, for Scenario I are shown in Fig. 2a and 2b, respectively, and for Scenario II are shown in Fig. 3a and 3b, respectively. Using constrained filtering method, both range and velocity values get limited to their maximum and minimum values in both the scenarios.

Fig. 2c and 2d show the RMSE plots of position and velocity for 500 Monte Carlo runs along with the CRLB and constrained CRLB (CCRLB) for Scenario I excluding lost tracks. In our case of inequality constraints the CRLB represents the minimum bound for error. The tracks whose terminal error which is the estimation error at the 30th min is beyond 1 km are considered to be lost. From the figures, we can see that the RMSE plots of the constrained UKF (CUKF) and CCKF after implementing both the range and velocity constraints are higher than the unconstrained filters for Scenario I. This is due to the fact that the track loss % is much lowered on implementing the range and velocity constraints. The RMSE in position and velocity on implementing the range and velocity constraints are the lowest in Scenario II as shown in Fig. 3c and 3d.

The track loss % and relative execution time are shown in Table I. The track loss % is evaluated considering a track bound of 1 km *i.e.,* the maximum allowable error in estimation at the last time step beyond which the track is considered to be lost, evaluated for $10,000$ Monte Carlo runs. In both scenarios, the track loss % obtained using constrained estimators is lower compared to the unconstrained filters. The execution time in the table is relative to the execution time of the unconstrained EKF. It can be observed that the execution time is slightly increased for the constrained filters than for the unconstrained filters.

TABLE I: % of track loss and relative execution time

| Filter | Scenario I | Scenario II | Rel. Exe. Time |
|---|---|---|---|
| EKF | 5.71 | 81.21 | 1 |
| CEKF | 2.18 | 65.88 | 1.55 |
| CKF | 2.2 | 63.89 | 1.41 |
| CCKF | 0.18 | 40.98 | 1.89 |
| UKF | 1.23 | 63.58 | 1.75 |
| CUKF | 0.09 | 38.66 | 2.04 |

## V. Discussion and Conclusion

In this paper, accuracy of the TMA has been enhanced by incorporating the limit of the range and radial velocity of the target as received from an experienced sonar operator. Consequently, a constrained optimization problem has been solved along with a nonlinear estimation technique. The proposed method is applied to two BOT scenarios. The results obtained from the proposed technique are compared to that of the traditional filters in terms of RMSE, track loss %, and relative execution time. It has been observed that the proposed method works better but it requires slightly more execution time.

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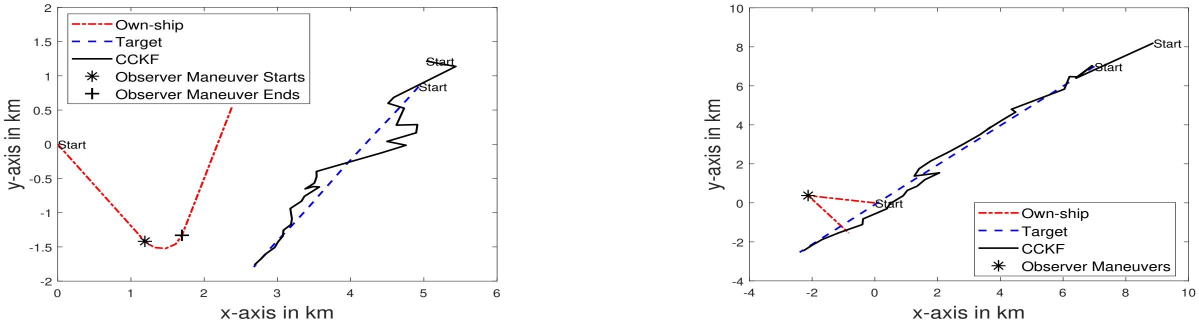
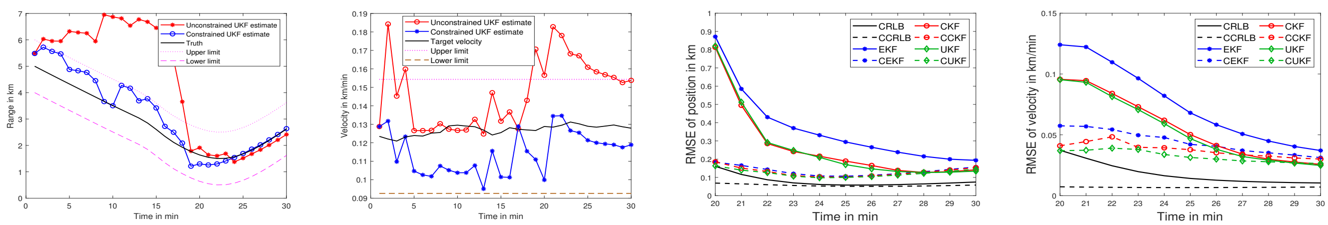


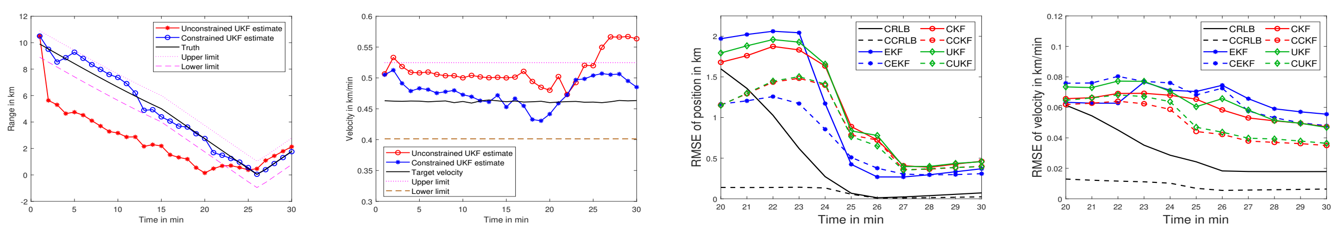

(a) (b)

Fig. 1: (a) Engagement scenario I and (b) Engagement scenario II.




(a) (b) (c) (d)

Fig. 2: (a) Truth and estimated Range, (b) Truth and estimated velocity, (c) RMSE plot of position and (d) RMSE plot of velocity for Scenario I.




(a) (b) (c) (d)

Fig. 3: (a) Truth and estimated Range, (b) Truth and estimated velocity, (c) RMSE plot of position and (d) RMSE plot of velocity for Scenario II.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