# Novel Tensor-based Singular Spectrum Decomposition

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***Abstract*—Tensor-based signal decomposition methods offer a promising avenue for signal decomposition of short, non-stationary and non-linear input signals. A novel Tensor-based Singular Spectrum Decomposition (TSSD) framework is presented that extends Singular Spectrum Decomposition (SSD) to tensors for univariate signals using two tensor decomposition techniques, namely, Multilinear Singular Value Decomposition (MLSVD) and Canonical Polyadic Decomposition (CPD). Results indicate improved performance under the influence of noise and in the presence of sizeable trends. Experiments on real-life data on EEG signals from epileptic seizures further show the strong practical relevance of TSSD as a tool for exploratory signal analysis that helps unveil underlying system(s) in signals.**

***Index Terms*—Signal Decomposition, Singular Spectrum Decomposition, Multi-linear Singular Value Decomposition, Tensor Decomposition**

## I. Introduction

Signal decomposition methods are fundamental and versatile tools for time-series analysis. Applications include feature extraction, denoising, compression and anomaly detection [5] [8] [7]. Given an input signal, the goal is to decompose it into simpler, distinct and interpretable components. Popular methods for signal decomposition include Singular Spectrum Analysis (SSA), Empirical Mode Decomposition (EMD), and Variational Mode Decomposition (VMD) [4].

Another such method is Singular Spectrum Decomposition (SSD) used for signal decomposition of short, non-stationary, non-linear signals [1]. Unlike it's predecessor SSA, SSD is iterative data-driven approach that decomposes a signal in four steps: (1) embedding into a trajectory matrix, (2) matrix decomposition, (3) grouping, and (4) reconstruction. It extends SSA by automating certain parametric choices, substantially improving the decomposition quality. In particular, the embedding dimension is automated using the spectral information of the signal. In doing so, SSD is able to overcome two persistent shortcomings of signal decomposition of short, non-stationary, non-linear input signals: Mode mixing and the extraction of non-harmonic oscillatory shapes. Mode mixing refers to when a single component contains information from different scales or when information of similar scales can be found in several components [4]. This prevents from making the components distinct in nature and thus affect the interpretability of the components. The automation of the parameter choice helps in overcoming these shortcomings because the embedding dimension is tuned to target one scale at a time. This allows for a more targeted separation of frequency components which ensures no mode mixing.

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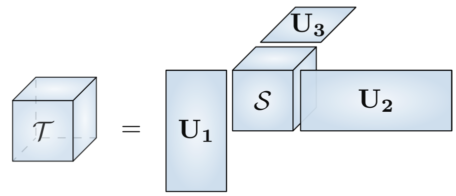



Fig. 1: The MLSVD is a higher-order generalization of the well-known SVD, enabling common task such as dimensionality reduction and subspace analysis for higher-order tensors.

While SSD has several advantages, there are a few limitations as well. In cases where a sizeable trend is present, the decomposition and grouping steps may select one too many components for the reconstruction of the trend which show signs of mode mixing or even generation of spurious components. Currently, to overcome this partially, only the first component in the first iteration is chosen for the reconstruction of trend. A second limitation is the influence of noise in cases where the energy of the signal falls below the noise level, the convergence of the technique is slowed down and calls for improvement to the choice of the stopping threshold.

In this paper, we extend the SSD framework to tensors by embedding the univariate input signal into a higher-order tensor instead of a (second-order) matrix. A tensor is a multi-way array of numerical values, while a vector or a matrix are one-way and two-way arrays, respectively. We show that this can further boost the underlying structure in the signal, improving on the limitations of SSD such as the presence of a sizeable trend and noise, leading to enhanced signal decomposition quality. The tensor is obtained by embedding the input signal into a third-order tensor using a Hankelization approach. Next, in the decomposition step of SSD, we have multiple choices for the tensor decomposition. In this paper, we limit ourselves to Multilinear Singular Value Decomposition (MLSVD) and Canonical Polyadic Decomposition (CPD), and compare both alternatives. Finally, we apply the proposed method on EEG signals from patients suffering from Epilepsy to show the practical relevance of the method in the detection of epileptic seizures. SSA has been been successfully extended to tensors in the past [6]. However, by choosing to extend SSD we aim to focus on further improving this extension.

In the remainder of this section, we discuss the tensor decompositions used in our approach. The paper is then

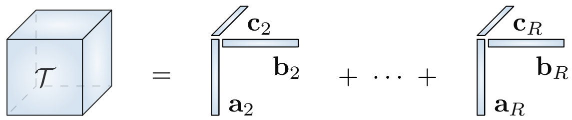



Fig. 2: Illustration of the CPD showcasing the decomposition of a third-order tensor into a sum of $R$ rank-1 tensors. CPD is commonly used in source separation tasks to help reveal underlying sources.

organised as follows: In Section II, we introduce and describe the proposed method. In Section III, results from numerical experiments to test the quality of decomposition in special cases are presented and discussed. Lastly, in Section IV we conclude the paper and present some future research directions.

*Tensor Decomposition Techniques*

MLSVD generalizes the Singular Value Decomposition (SVD) for matrices to tensors [9]. It decomposes a tensor $\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times \ldots \times I_N}$ as:

$$\mathcal{X} = \mathcal{S} \cdot_1 \mathbf{U}^{(1)} \cdot_2 \mathbf{U}^{(2)} \ldots \cdot_n \mathbf{U}^{(n)}$$

where $\mathbf{U}^{(k)} \in \mathbb{R}^{I_k \times I_k}$ for all $i = 1, \cdots, N$ are unitary factor matrices and $\mathcal{S} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$ is an all orthogonal core tensor [9]. This is illustrated in Figure 1.

A polyadic decomposition (PD) decomposes a tensor into $R$ rank-1 tensors [10]. Consider a tensor $\mathcal{T}$ of size $I \times J \times K$. This can be written as a sum of rank-1 tensors as:

$$\mathcal{T} = \sum_{r=1}^{R} a_r \otimes b_r \otimes c_r$$

where $R$ is the rank of the tensor $\mathcal{T}$ and $\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_R] \in \mathbb{R}^{I \times R}$, $\mathbf{B} = [\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_R] \in \mathbb{R}^{J \times R}$ and $\mathbf{C} = [\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_R] \in \mathbb{R}^{K \times R}$ are the factor matrices obtained from the decomposition. When PD is performed using the smallest $R$, it is called the Canonical Polyadic Decomposition (CPD). This is illustrated in Figure 2. Its key feature is that it is unique under mild conditions for tensor orders higher than two [10].

## II. Tensor-based Singular Spectrum Decomposition

In this Section, we introduce the proposed Tensor-based Singular Spectrum Decomposition (TSSD) method using MLSVD or CPD. Similar to SSD, it consists of four steps, which we describe below, and illustrate in Figure 3.

*Embedding into trajectory tensor*

To embed the univariate signal $\mathbf{v} \in \mathbb{R}^N$ into a third-order tensor $\mathcal{H}$, we *Hankelize* the signal into a Hankel tensor. In this paper, we limit ourselves to third-order tensors but an embedding into higher orders is possible and can possibly further boost the underlying structure of the signal. For the Hankel tensor, a window of length $I$ is slided over $\mathbf{v}$ to get the columns of a Hankel matrix $\mathbf{H} \in \mathbb{R}^{I \times J}$ where $I$ is referred to as the embedding dimension. The resulting matrix is called a Hankel matrix because it consists of constant anti-diagonals which is called the Hankel structure and is formally written as:

$$h_{i,j} = v_{i+j-2+1}.$$

This is then further used to obtain the Hankel tensor $\mathcal{H}$ by repeating the same process on the columns (or rows) of the Hankel matrix $\mathbf{H}$ [3]. Formally, Hankel tensor $\mathcal{H}$ of third-order is defined element-wise as:

$$h_{i_1,i_2,i_3} = v_{i_1+i_2+i_3-3+1}.$$

This is also illustrated in Figure 4. For computing the embedding dimension $I$, we follow the same data-driven approach as SSD to construct $\mathcal{H}$ of size $I \times (N-I) \times (K-1)$ where $N$ refers to the length of the signal and $K$ refers to the order of the tensor. Here, mode 1 represents the embedding mode, mode 2 represent the time mode and mode 3 represents the windowed hankel matrices in time, see Figure 4. The choice of mode 1 and 2 especially helps in the interpretation of the decomposed components.

*Selecting number of components*

For TSSD-CPD, we need to determine $\hat{R}$, the number of rank-1 components in the decomposition of $\mathcal{H}$. For TSSD-MLSVD, we need to find a value for the truncation parameter $\hat{R}$, which coincides with the number of mode 1 singular values. In order to find a good estimate for $\hat{R}$ in both cases, we suggest a heuristic. We examine the differenced mode 1 singular values and choose $\hat{R}$ where the largest difference is observed. In this way, we find the elbow of the curve of singular values which would indicate the optimal point between the number of components and the overall information captured. Our heuristic is based on the mode 1 singular values because this mode relates to the embedding information while the mode 2 is interpreted as time.

*Decomposition*

Using the parameter $\hat{R}$ from the previous step, we compute the decomposition. In this step we tested two tensor decomposition techniques: CPD and MLSVD which result in two versions of the proposed method: TSSD-MLSVD and TSSD-CPD. For TSSD-CPD, the decomposition is computed as:

$$\mathcal{H} = \sum_{\hat{r}=1}^{\hat{R}} a_{\hat{r}} \otimes b_{\hat{r}} \otimes c_{\hat{r}}.$$

For TSSD-MLSVD the decomposition using MATLAB notation is computed as:

$$\mathcal{H} = \mathcal{S}_{(1:\hat{R},:,:)} \cdot_1 \mathbf{U}^{(1)}_{(:,1:\hat{R})} \cdot_2 \mathbf{U}^{(2)}_{(:,:)} \cdot_3 \mathbf{U}^{(3)}_{(:,:)}.$$

*Grouping and Reconstruction*

Lastly, for the reconstruction of the $k^{th}$ component in the $k^{th}$ iteration, we use a similar procedure as in SSD. In particular, a subset $M_k$ is created which contains components

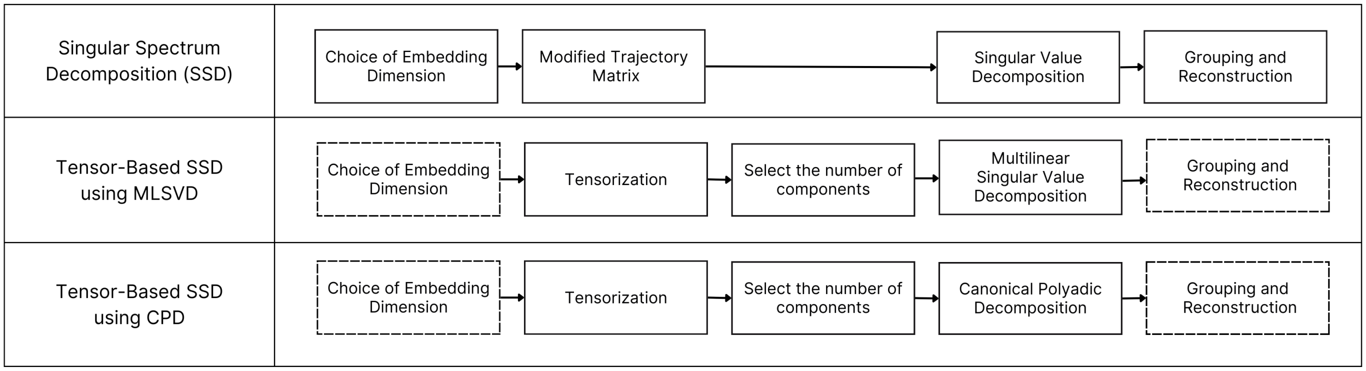



Fig. 3: In this paper, we extend the matrix-based SSD method to tensor-based SSD by replacing the (matrix) embedding by a tensor embedding, adding an estimation of the truncation parameter, and using CPD or MLSVD instead of SVD. The other steps of the SSD method essentially remain the same.

$$\mathbf{H} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{pmatrix}$$

$$\mathcal{H}(:,:,1) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix} \mathcal{H}(:,:,2) = \begin{pmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$$

Fig. 4: Hankelization is used for embedding the input signal into a trajectory tensor $\mathcal{H}$. For $\mathbf{v} = [1,2,3,4,5]$ and an embedding dimension of 2, we construct $\mathbf{H}$. When hankelized further, we get $\mathcal{H}$ with dimensions $2 \times 3 \times 2$ where mode 1 represents the embedding information, mode 2 represents time and mode 3 shows the windowed hankel matrices.

obtained from the decomposition whose dominant frequency falls in the range defined in [1]. For TSSD-CPD, we have:

$$M_k = \{(a_{r_k} \otimes b_{r_k} \otimes c_{r_k})\}_{r_k \in J}$$

For TSSD-MLSVD this is written as:

$$M_k = \{(\mathcal{S}_{(r_k,:,:)} \cdot_1 \mathbf{U}^{(1)}_{(:,r_k)} \cdot_2 \mathbf{U}^{(2)}_{(:,:)} \cdot_3 \mathbf{U}^{(3)}_{(:,:)})\}_{r_k \in J}$$

where $J = \{1, 2, \ldots \hat{R}\}$. The components in set $M_k$ are first dehankelized by taking an average along the anti-diagonal planes of the tensor. Lastly, they are all summed to then obtain the $K^{th}$ component with respect to the input signal.

## III. Results and Discussion

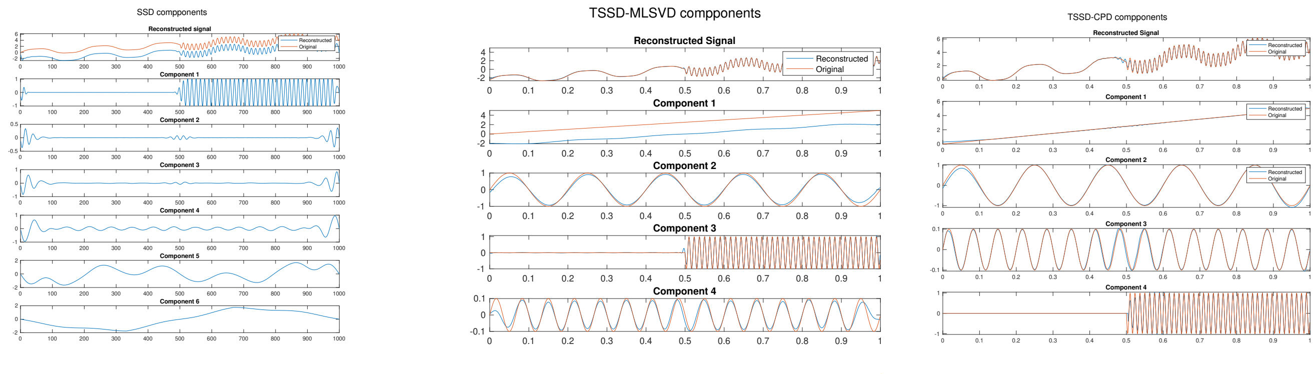
In this Section, we evaluate the proposed method with four experiments: (1) under the influence of trend, (2) under the influence of noise, (3) when the components are very close in frequency content, and lastly (4) we apply it to a real-life dataset. In all experiments we also use results from SSD for comparison of the components. For all tensor computations, we used Tensorlab [12] with default settings of the methods unless specified otherwise. For the evaluation, reconstruction error is computed as $\text{RE}(y, \hat{y}) = \frac{\sum_i^N (y_i - \hat{y}_i)^2}{N}$ where $y$ refers to the input signal, $\hat{y}$ refers to the reconstructed signal using the components derived from the proposed method and $N$ refers to the length of the signal.

### A. *Influence of Trend*

We evaluated the influence of a sizeable linear trend on the input signal. In particular, we wanted to investigate if TSSD-MLSVD or TSSD-CPD successfully extract the trend or generate spurious components. For this, we used a simulated signal of 10 s. It is composed of a sinusoid of unit amplitude and frequency of 5 Hz combined with another sinusoid of 0.1 amplitude and frequency 15 Hz and lastly, a sinusoid of unit amplitude and frequency 75 Hz which is mixed in from 5 s. The sampling frequency is 1000 Hz. The results from SSD, TSSD-MLSVD and TSSD-MLSVD are shown in Figure 5. We observe that TSSD-CPD is able to decompose the components where the trend was extracted as a separate component. For TSSD-MLSVD, we observe that it extracts the sinusoids but is not able to extract the trend fully. On the other hand, SSD results in the generation of spurious components.

### B. *Influence of Noise*

In this experiment we tested the ability of the proposed method to extract components under the influence of noise. For this, a 10 s signal is simulated with triangular wave of unit amplitude and frequency of 5 Hz combined with another triangular wave of unit amplitude and frequency 20 Hz which is added only in the middle while the first and last quarter of the signal is set to 0 to create discontinuity in the time-frequency domain. The sampling frequency is 256 Hz. White Gaussian noise components with signal-to-noise ratio of 10 dB, 20 dB and 30 dB were added to create three input signals with varying levels of noise. Figure 6 shows the mean reconstruction error in comparison with SSD with respect to noise. We can observe that in all cases, TSSD-MLSVD outperforms SSD and TSSD-CPD in extracting the original components back. We interpret this to be because of embedding the signal into a higher-order tensor. We speculate that the error could be further decreased by increasing the order of the trajectory tensor.




(a) Components retrieved by SSD (b) TSSD-MLSVD (c) TSSD-CPD

Fig. 5: In the presence of a sizeable trend, SSD generates spurious components while only retrieving the component with highest frequency. Whereas TSSD-CPD retrieves all components including the trend and TSSD-MLSVD retrieves all sinusoids.

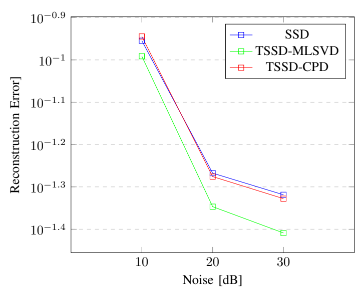



Fig. 6: Mean reconstruction error across 100 experiments for the input signal with an SNR of 10 dB, 20 dB and 30 dB. We see that TSSD-MLSVD shows improved decomposition with a standard deviation of 0.005, 0.004, 0.0003 whereas SSD has a standard deviation of 0.005, 0.002, 0.0004 and TSSD-CPD with 0.005, 0.003, 0.0004 respectively.

### C. *Components with close frequency content*

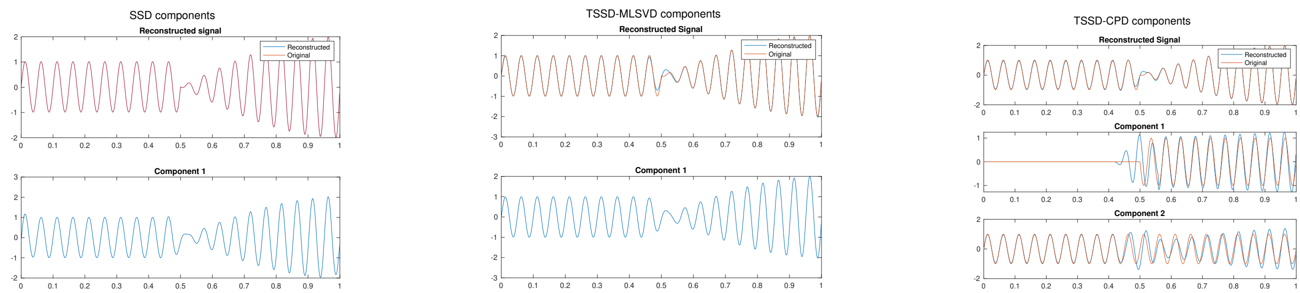
We evaluated the extent to which components with very close frequencies could be extracted with limited mode-mixing. For this, a 10 s signal with a sampling frequency 1000 Hz was simulated. It consisted of two sinusoids of unit amplitude, one with frequency of 20 Hz combined with another of frequency 21 Hz which is only mixed in at 5 s. The results from the proposed methods and SSD are presented in Figure 7. We can observe that TSSD-CPD is able to completely separate the two components with no mode mixing.

### D. *Real-life application*

Lastly, we evaluated the proposed method on a real-life case of diagnosing an epileptic seizure. For this, data from Neurology and Sleep Centre in New Delhi, India is used [11]. The data consists of single channel electroencephalogram (EEG) recordings from ten patient suffering from epileptic seizures. Epilepsy is a neurological disorder which causes abnormal firing of neurons in the patient's brain which causes recurrent seizures and abnormal neuronal brain activity. Research shows that depending upon the presence and the progression of certain frequency bands in the EEG signals during a seizure, one can diagnose the type of epileptic seizures [13]. Furthermore, one can also predict the occurrence of a seizure based on previous phases [2]. However, since the signals are non-stationary and noisy, this is a challenging task. The signals have been divided into three segments each indicating a different phase of an epileptic seizure. The sampling frequency is 200 Hz where each segment is 5.12 s long. In this paper, we show the decomposition of the ictal phase which contains the entire seizure to show the practical relevance of TSSD as an exploratory or pre-processing tool that helps in the diagnosis of seizure. When applied to other phases, it can be used to predict the seizure as in [2]. The results from SSD and TSSD-MLSVD are shown in Figure 8. Since TSSD-CPD performed similarly to SSD under noisy examples, we only compare SSD with TSSD-MLSVD. Here we see that both methods are able to decompose the signal into five components. Each component falls within the theoretically known bands of an EEG signal namely, the delta band (0.5 - 4 Hz), theta (4 - 7 Hz), alpha (8 - 13 Hz), beta (13 - 30 Hz) and gamma (> 30 Hz) [13].

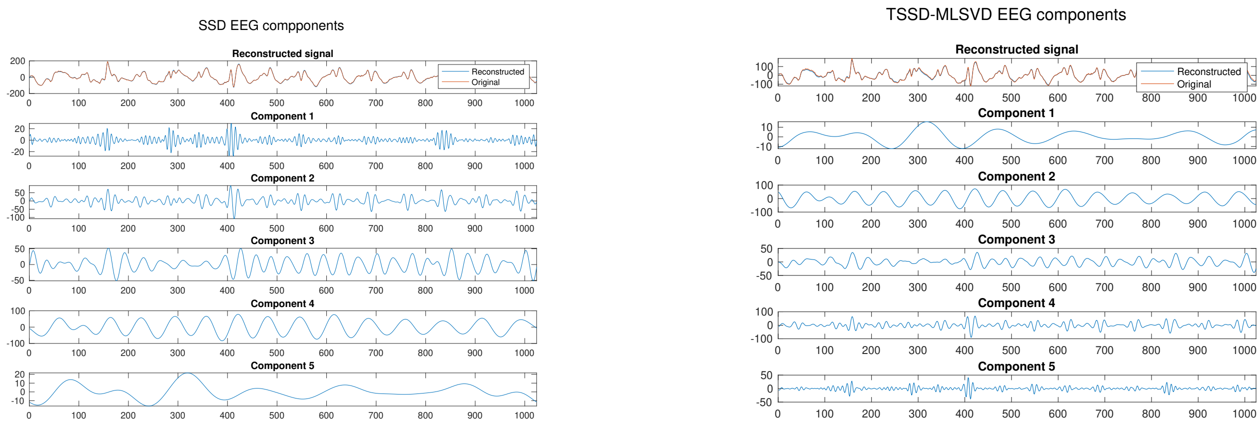
## IV. Conclusion

We introduced TSSD, a tensor-based version of SSD, by using a tensor embedding instead of a matrix embedding. For the decomposition step, we utilized CPD and MLSVD. We validated both alternatives in several experiments, indicating that they can successfully decompose short, non-stationary, non-linear signals into narrow-banded components. The experiments indicated improved performance with TSSD-MLSVD in the noisy case, further demonstrated in the real-life EEG data. TSSD-CPD showed improved performance under the influence of a sizeable trend and for close frequency content. While we made certain parameter choices for TSSD, based on heuristics, further experiments are required to determine better heuristics for the approximation of $\hat{R}$.




(a) Components retrieved by SSD (b) TSSD-MLSVD (c) TSSD-CPD

Fig. 7: Two sinusoids with frequencies 20 Hz and 21 Hz are shown to be retrieved by TSSD-CPD whereas for TSSD-MLSVD and SSD we see one narrow-band component.




(a) Components retrieved by SSD (b) TSSD-MLSVD

Fig. 8: TSSD-MLSVD and SSD successfully decompose the EEG signal from the ictal phase of an epileptic seizure into 5 bands which conform to the 5 theoretical bands called the delta, theta, alpha, beta and gamma.

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