Best Least Squares Paraunitary Approximation for an Extended Range of Polynomial Matrices

Stephan Weiss[1], Sebastian J. Schlecht[2], and Marc Moonen[3]
[1]Department of Electronic & Electrical Engineering, University of Strathclyde, Glasgow, Scotland
[2]Chair of Multimedia Communications & Signal Processing, Friedrich-Alexander Universität Erlangen-Nürnberg, Germany
[3]STADIUS Center for Dynamical Systems, Signal Processing & Data Analytics, ESAT, KU Leuven, Belgium
stephan.weiss@strath.ac.uk, sebastian.schlecht@fau.de, marc.moonen@esat.kuleuven.be

***Abstract*—The analytic Procrustes solution of a polynomial matrix $\boldsymbol{A}(z)$, the best paraunitary approximation in the least squares sense, is based on the analytic singular value decomposition (SVD) of $\boldsymbol{A}(z)$. Except for contrived cases, this analytic SVD exists for $\boldsymbol{A}(z)$ but sometimes may require oversampling to $\boldsymbol{A}(z^2)$. In this paper, we show that in the latter case, an analytic Procrustes solution of $\boldsymbol{A}(z)$ is still possible by compensating a fractional delay in the decomposition. We also show that this result can equivalently be reached by decimating the Procrustes solution of $\boldsymbol{A}(z^2)$. This extends the range of the Procrustes solution to almost all matrices $\boldsymbol{A}(z)$. Our analysis is underpinned by examples and simulations.**

I. Introduction

Assume two multichannel signal vectors $\mathbf{x}[n]$, $\mathbf{y}[n] \in \mathbb{C}^M$ that are connected by a convolutive mixing system $\mathbf{A}[n] \in \mathbb{C}^{M\times M}$, such that $\mathbf{y}[n] = \mathbf{A}[n] * \mathbf{x}[n]$ with $*$ being the convolution operator. In many signal processing scenarios, the desire is for the system $\mathbf{A}[n]$ to be lossless, such that the energy and information in $\mathbf{x}[n]$ remain preserved in $\mathbf{y}[n]$. This then finds numerous applications in compression [1], [2], dimensionality reduction and signal compaction [3]–[7], denoising [8], for general filter bank design [9]–[11], or as reverberation filters and scattering matrices [12]–[14].

In practice, such as in the audio applications in [12]–[14], measured systems $\mathbf{A}[n]$ often deviate from the expected property of being lossless. We are therefore interested in approximating such a matrix $\mathbf{A}[n]$ by a lossless system $\mathbf{Q}[n]$ in the least squares sense. In the z-domain, $\boldsymbol{Q}(z) = \sum_n \mathbf{Q}[n]z^{-n}$, lossless systems are referred to as paraunitary, such that with the parahermitian transposition $\boldsymbol{Q}^{\mathrm{P}}(z) = \{\boldsymbol{Q}(1/z^*)\}^{\mathrm{H}}$ and $\mathbf{I}_M$ the $M \times M$ identifty matrix, $\boldsymbol{Q}(z)$ satisfies

$$\boldsymbol{Q}(z)\boldsymbol{Q}^{\mathrm{P}}(z) = \boldsymbol{Q}^{\mathrm{P}}(z)\boldsymbol{Q}(z) = \mathbf{I}_M \ . \tag{1}$$

The problem is therefore an extension of the Procrustes solution for ordinary matrices [15], [16] to matrices of analytic functions, i.e. matrices whose entries are formed by causal and stable filters [17]–[19]:

$$\boldsymbol{Q}_*(z) = \arg\min_{\boldsymbol{Q}(z)} \oint_{|z|=1} \|\boldsymbol{Q}(z) - \boldsymbol{A}(z)\|_{\mathrm{F}}^2 \frac{\mathrm{d}z}{z} \tag{2}$$
$$\text{s.t.} \quad (1) \quad \text{is} \quad \text{satisfied.}$$

This problem equivalently describes finding the best paraunitary matrix $\boldsymbol{Q}(z)$ that minimises the least squares error between $\boldsymbol{A}_1(z)\boldsymbol{Q}(z)$ and $\boldsymbol{A}_2(z)$, with $N \times M$ matrices $\boldsymbol{A}_i(z)$, $i = 1, 2$, by setting $\boldsymbol{A}(z) = \boldsymbol{A}_1^{\mathrm{P}}(z)\boldsymbol{A}_2(z)$.

The solution in [19] is based on the analytic SVD [20], [21] of the matrix $\boldsymbol{A}(z)$. In a number of scenarios, however, the analytic SVD of $\boldsymbol{A}(z)$ does not exist, and instead only the oversampled system $\boldsymbol{A}(z^2)$ can be decomposed. But what use is the Procrustes solution to $\boldsymbol{A}(z^2)$ if we really want to approximate $\boldsymbol{A}(z)$?

This paper addresses the above question by employing a modification to the analytic SVD proposed in [22], [23]. This modified version, referred to as the complex-valued analytic SVD, allows us to still optimise the problem in (2) without being required to work with an oversampled version of $\boldsymbol{A}(z)$. Based on this, we can also demonstrate that this more general solution can be obtained by decimating the analytic Procrustes solution to $\boldsymbol{A}(z^2)$ according to [19] by a factor of two. Hence we can extend the utility of the analytic Procrustes solution to a wider range of matrices.

In the following, Secs. II and III review the analytic SVD and the analytic Procrustes solution. The modified analytic SVD and its application to the problem in (2) are discussed in Sec. IV. Finally, the implementation of this solution and simulations are addressed in Sec. V. The files to reproduce the results and figures can be found on GitHub[1].

II. Analytic Singular Value Decomposition

For any analytic matrix $\boldsymbol{A}(z)$ and excluding contrived cases[2], there exists an analytic SVD [20], [21],

$$\boldsymbol{A}(z^\kappa) = \boldsymbol{U}(z)\boldsymbol{\Sigma}(z)\boldsymbol{V}^{\mathrm{P}}(z) \ , \tag{3}$$

with $\kappa = 1$ generally but oversampling by $\kappa = 2$ may be required. The diagonal matrix $\boldsymbol{\Sigma}(z) = \mathrm{diag}\{\sigma_1(z), \ldots, \sigma_M(z)\}$ contains the singular values, which are real on the unit circle, i.e. $\sigma_m(\mathrm{e}^{\mathrm{j}\Omega}) \in \mathbb{R}$, $m = 1, \ldots, M$. The corresponding left- and right-singular vectors $\boldsymbol{u}_m(z)$ and $\boldsymbol{v}_m(z)$ form the columns of the paraunitary matrices $\boldsymbol{U}(z)$ and $\boldsymbol{V}(z)$. The analytic singular values are unique up to their ordering. If there are no identical singular values, with an arbitrary allpass $\varphi_m(z)$, $\varphi_m(z)\boldsymbol{u}_m(z)$ and $\varphi_m(z)\boldsymbol{v}_m(z)$ are also valid left- and right-singular vectors.

[1]https://github.com/StephanWeiss5/OversampledProcrustes.git

[2]Multiplexed filters, such as in block processing [10], [24], may require additional oversampling in (3) by the multiplexing factor [21], [23].

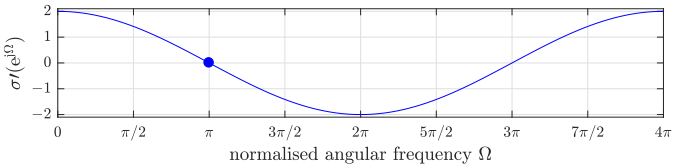

Fig. 1. 4π-periodic singular value of $A(z) = 1 + z^{-1}$ on the unit circle, with the single zero crossing on the interval $[0; 2\pi)$ indicated (•).

Compared to the standard SVD of an ordinary matrix, the analytic SVD in (3) has some unusual properties [21]:

(P.1) In order to be analytic, the singular values $\sigma_m(z)$ cannot be restricted to be nonnegative on the unit circle; this is similar to the real-parameter case in [25], [26].

(P.2) If any singular value has an odd number of zero-crossings on the unit circle, i.e. in the interval $\Omega = [0, 2\pi)$, then the analytic SVD does not exist for $\kappa = 1$ but does exist for $\kappa = 2$.

To highlight these aspects, consider the following example:

Example 1: For the 1×1 matrix $A(z) = (1 + z^{-1})$, we can find

$$A(z) = \underbrace{z^{-\frac{1}{2}}}_{U'(z)} \cdot \big(\underbrace{z^{\frac{1}{2}} + z^{-\frac{1}{2}}}_{\sigma'(z)}\big) \cdot \underbrace{1}_{V'^{\mathrm{P}}(z)} \,. \tag{4}$$

For the singular value to be real-valued on the unit circle, it must be parahermitian i.e. $\sigma'(z) = \sigma'^{\mathrm{P}}(z)$. But the solution in (4) contains fractional powers of z and hence is not analytic[3]. Fig. 1 shows the evaluation of the singular value $\sigma'(z)$ on the unit circle: note that $\sigma'(\mathrm{e}^{\mathrm{j}\Omega}) = 2\cos(\Omega/2)$ changes sign, but does so only once on the interval $[0, 2\pi)$ and is 4π-periodic. Oversampling by two leads to $A(z^2)$ having analytic SVD factors $U(z) = U'(z^2) = z^{-1}$ and $\sigma(z) = \sigma'(z^2) = z + z^{-1}$. △

In practice, property (P.1) may be masked e.g. if $\boldsymbol{A}(z)$ is randomly perturbed, which excludes zero-crossings almost surely [27], [28]. Then property (P.2) is also excluded almost surely. In cases where $\boldsymbol{A}(z)$ is deterministic, or where special steps are taken to find the analytic ground truth eigen- or singular values to a randomly perturbed system [29], [30], (P.1) and (P.2) can be encountered and need to be addressed.

III. Analytic Procrustes Solution

Based on its analytic SVD in (3), the best paraunitary approximation of an analytic matrix $\boldsymbol{A}(z)$ in the least squares sense, i.e. the solution to (2), is given by

$$\boldsymbol{Q}_*(z) = \boldsymbol{U}(z)\boldsymbol{\Pi}(z)\boldsymbol{V}^{\mathrm{P}}(z) \,. \tag{5}$$

The diagonal matrix $\boldsymbol{\Pi}(z) = \mathrm{diag}\{p_1(z), \ldots, p_M(z)\}$ contains allpass filters, such that $\boldsymbol{Q}_*(z)$ is guaranteed to be paraunitary. It is shown in [19] that minimising the cost

[3]Primed variables such as $U'(z)$ indicate functions that are not necessarily analytic in $z \in \mathbb{C}$.

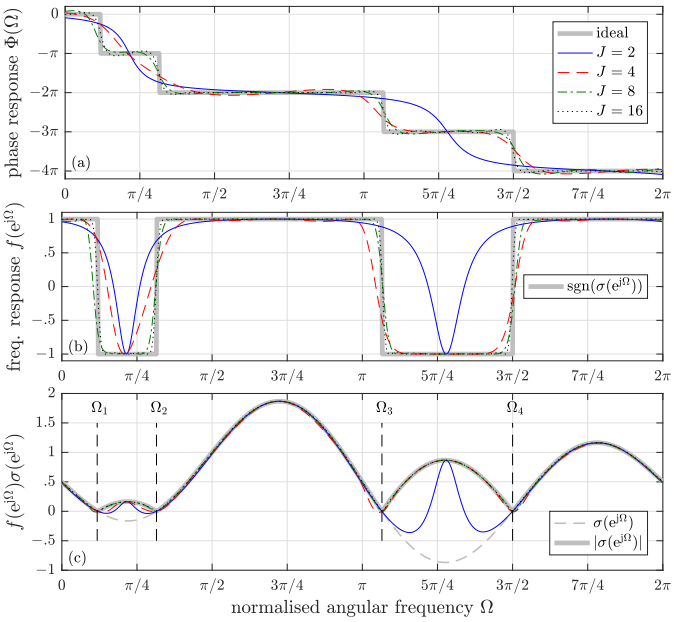

Fig. 2. (a) target phase response of allpass design for different filter orders J, (b) the resulting 'switching' function $f(\mathrm{e}^{\mathrm{j}\Omega})$, and (c) the argument of the maximisation problem in (6).

function in (2) is equivalent to designing the allpass filters $p_m(z)$, $m = 1, \ldots, M$, to maximise

$$\sum_{m=1}^{M} \int_0^{2\pi} \Re\{p_m(\mathrm{e}^{\mathrm{j}\Omega})\sigma_m(\mathrm{e}^{\mathrm{j}\Omega})\}\mathrm{d}\Omega \,. \tag{6}$$

If the mth singular value $\sigma_m(z)$ does not change sign on the unit circle, then $p_m(z) = 1$ and the result in (5) is a direct extension of the Procrustes solution for the ordinary matrix case [17], [18].

If a singular value $\sigma_m(z)$ has sign changes on the circle, then because $\sigma_m(\mathrm{e}^{\mathrm{j}\Omega}) \in \mathbb{R}$, we have $\Re\{p_m(\mathrm{e}^{\mathrm{j}\Omega})\sigma_m(\mathrm{e}^{\mathrm{j}\Omega})\} = \frac{1}{2}(p_m(\mathrm{e}^{\mathrm{j}\Omega}) + p_m^*(\mathrm{e}^{\mathrm{j}\Omega}))\sigma_m(\mathrm{e}^{\mathrm{j}\Omega})$. Reparametrisation with $z = \mathrm{e}^{\mathrm{j}\Omega}$ leads to

$$\frac{1}{2}(p_m(z) + p_m^{\mathrm{P}}(z))\sigma_m(z) = f_m(z)\sigma_m(z) \,. \tag{7}$$

Because $f_m(z) = f_m^{\mathrm{P}}(z)$ is parahermitian and self-inversive [19] and $p_m^{\mathrm{P}}(z)$ has zeros strictly inside the unit circle, based on Cohn's theorem [31] and the findings in [32], the zeros of $f_m(z)$ all lie on the unit circle and are simple. Thus, $f_m(\mathrm{e}^{\mathrm{j}\Omega})$ in (7) is a 'switching' function, whose sign changes, by appropriately designing allpasses $p_m(z)$, can be made to coincide with the sign changes of $\sigma_m(\mathrm{e}^{\mathrm{j}\Omega})$ in order to maximise (6).

Example 2: For a singular value $\sigma(z) = \frac{1}{4\mathrm{j}}(2z^2 - z) + \frac{1}{2} + \frac{1}{4\mathrm{j}}(z^{-1} - 2z^{-2})$ with four zero crossings Ω_k, $k = 1, \ldots, 4$, Fig. 2(a) shows the ideal phase response $\Phi(\Omega) = \angle\{p(\mathrm{e}^{\mathrm{j}\Omega})\}$, as well as the phase responses of allpass designs with four different filter orders $J = 2^\ell$, $\ell = 1, \ldots, 4$. The resulting 'switching' function $f(\mathrm{e}^{\mathrm{j}\Omega})$ is illustrated in Fig. 2(b), and its multiplication with the singular value $\sigma(\mathrm{e}^{\mathrm{j}\Omega})$, i.e. the term that

contributes to the cost function in (6), is depicted in Fig. 2(c). As the allpass filter order J increases, $f(\mathrm{e}^{\mathrm{j}\Omega})$ starts to closely approximate a switching function, and the argument to (6) approaches the ideal $|\sigma(\mathrm{e}^{\mathrm{j}\Omega})|$. △

If the analytic SVD as defined in (3) does not exist for $\boldsymbol{A}(z)$, then the approach in [19] can only find the Procrustes solution for $\boldsymbol{A}(z^2)$, and it has been unclear what use that might be. However, in the following we will demonstrate that an analytic Procrustes solution for $\boldsymbol{A}(z)$ can still be found.

IV. Complex-Valued Analytic SVD and Procrustes Solution

A. Complex-Valued Analytic SVD

In the case that (3) does not exist for $\kappa = 1$ but for $\kappa = 2$, an alternative decomposition of $\boldsymbol{A}(z)$ can always be found, provided that the singular values are no longer constrained to be real-valued on the unit circle [22]. Then we have the complex-valued analytic SVD

$$\boldsymbol{A}(z) = \boldsymbol{U}_{\mathrm{c}}(z)\boldsymbol{S}(z)\boldsymbol{V}_{\mathrm{c}}^{\mathrm{P}}(z), \tag{8}$$

whereby all factors are analytic, but the diagonal matrix $\boldsymbol{S}(z) = \mathrm{diag}\{s_1(z), \ldots, s_M(z)\}$ of singular values no longer has to be parahermitian, i.e. $s_m(z) = s_m^{\mathrm{P}}(z)$ may not hold. Further, it is shown in [23] that a fractional delay can be exchanged between the factors in (3), such that they become analytic. Thus, $s_m[n] \circ\!\!-\!\!\bullet\, s_m(z)$ can still be chosen to be conjugate symmetric, but not necessarily w.r.t. $n = 0$ but to $n = \frac{1}{2}$.

Example 3: For $A(z)$ in Example 1, it is possible to write

$$A(z) = \underbrace{z^{-\frac{1}{2}}}_{U'(z)}\, z^{\frac{1}{2}} z^{-\frac{1}{2}} \underbrace{(z^{\frac{1}{2}} + z^{-\frac{1}{2}})}_{\sigma'(z)} = \underbrace{1}_{U(z)} \underbrace{(1 + z^{-1})}_{s(z)} \underbrace{1}_{V^{\mathrm{P}}(z)} .$$

The factors $U(z)$, $s(z)$, and $V(z) = 1$ are analytic, but $s^{\mathrm{P}}(z) = (1 + z) \neq s(z)$, and hence generally $s(\mathrm{e}^{\mathrm{j}\Omega}) \notin \mathbb{R}$. △

B. Modified Analytic Procrustes Solution

We utilise (8) and assume that w.r.t. (3) (i) for any singular values $\sigma_m(z)$ with an even number of zero crossings on the unit circle, we have $s_m(z) = \sigma_m(z)$, while otherwise (ii) $s_m(z) = z^{-\frac{1}{2}}\sigma'_m(z)$ in the complex-valued analytic SVD. We indicate cases (i) and (ii) by $\kappa_m = 1$ and $\kappa_m = 2$, respectively, for later use. The optimisation of (2) leads to (6) having its integrands replaced by $\Re\{p_m(\mathrm{e}^{\mathrm{j}\Omega})s_m(\mathrm{e}^{\mathrm{j}\Omega})\}$. For case (i), these terms remain unchanged w.r.t. (6). For case (ii), we have

$$\begin{aligned}\Re\{p_m(\mathrm{e}^{\mathrm{j}\Omega})s_m(\mathrm{e}^{\mathrm{j}\Omega})\} &= \Re\{p_m(\mathrm{e}^{\mathrm{j}\Omega})\mathrm{e}^{-\mathrm{j}\Omega/2}\mathrm{e}^{\mathrm{j}\Omega/2}s_m(\mathrm{e}^{\mathrm{j}\Omega})\} \\ &= \Re\{p_m(\mathrm{e}^{\mathrm{j}\Omega})\mathrm{e}^{-\mathrm{j}\Omega/2}\}\sigma'_m(\mathrm{e}^{\mathrm{j}\Omega}) \qquad (9)\\ &= f'_m(\mathrm{e}^{\mathrm{j}\Omega})\sigma'_m(\mathrm{e}^{\mathrm{j}\Omega}) , \qquad (10)\end{aligned}$$

since $\sigma'_m(\mathrm{e}^{\mathrm{j}\Omega}) \in \mathbb{R}$ and with $f'_m(z) = p_m(z)z^{-\frac{1}{2}} + p_m^{\mathrm{P}}(z)z^{\frac{1}{2}}$. Overall, we obtain

$$\boldsymbol{Q}_*(z) = \boldsymbol{U}_{\mathrm{c}}(z)\boldsymbol{\Pi}(z)\boldsymbol{V}_{\mathrm{c}}^{\mathrm{P}}(z) = \boldsymbol{U}'(z)\boldsymbol{\Pi}'(z)\boldsymbol{V}^{\mathrm{P}}(z) , \tag{11}$$

where $\boldsymbol{\Pi}(z) = \mathrm{diag}\{p_1(z), \ldots, p_M(z)\}$ and $\boldsymbol{\Pi}'(z) = \mathrm{diag}\{p'_1(z), \ldots, p'_M(z)\}$, with now generally $p'_m(z) = p_m(z)z^{-\frac{1}{2}(\kappa_m - 1)}$ across both cases (i) and (ii). The notation in (11) means that the Procrustes solution can be obtained either from the factors in (8), or from the factors in (3) for $\boldsymbol{A}(z)$ without oversampling, thus resulting in potential half-sample delays in the singular values $\sigma_m(z)$ and in one of the corresponding left- or right-singular vectors, here assumed to be $\boldsymbol{U}'(z)$. Overall, $\boldsymbol{U}'(z)\boldsymbol{\Pi}'(z)$ is analytic in $z \in \mathbb{C}$.

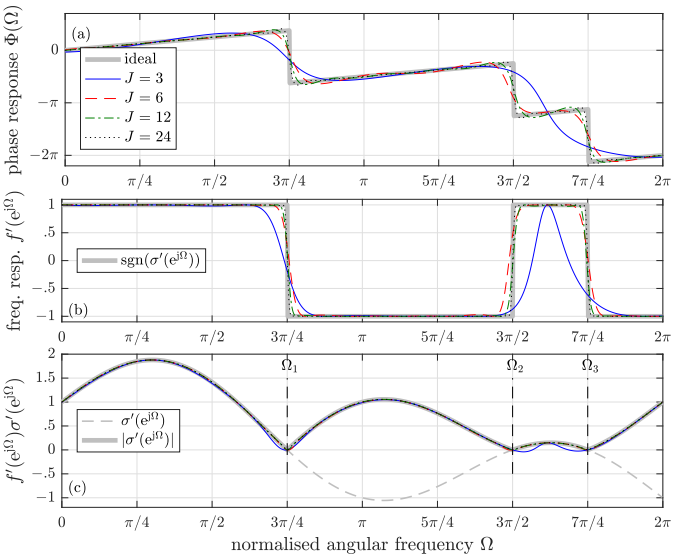

Fig. 3. (a) modified target phase response of an allpass design with an odd number of switching points and designs with different allpass orders J, (b) the resulting 'switching' function $f'(\mathrm{e}^{\mathrm{j}\Omega})$, and (c) the argument of the maximisation problem in (6).

The allpass design and the resulting switching function for $\kappa_m = 1$ remain as outlined in Sec. III. For $\kappa_m = 2$, the allpass $p_m(z)$ needs to provide an odd number of switching frequencies, but also compensate for the phase shift $\mathrm{e}^{-\mathrm{j}\Omega/2}$ in (9). This potential inclusion of a half sample advance in the allpass $p_m(z)$ causes an additional phase increment of π over a 2π interval of Ω. Note that unlike for an even number of zero crossings in Fig. 2(a) of Example 2, an odd number $(2k+1)$, $k \in \mathbb{N}$, of zero-crossings will result in a phase progression of $(2k+1)\pi$. The additional phase term $\mathrm{e}^{\mathrm{j}\Omega/2}$ thus results in a function that can be analytic.

Example 4: The parahermitian but non-analytic singular value $\sigma'(z) = \frac{-\mathrm{j}}{2}(z^{\frac{3}{2}} - z^{-\frac{3}{2}}) + \frac{1}{2}(z^{\frac{1}{2}} + z^{-\frac{1}{2}})$ has an odd number of zero crossings at $\Omega_1 = \frac{3}{4}\pi$, $\Omega_2 = \frac{3}{2}\pi$, and $\Omega_3 = \frac{7}{4}\pi$. Incorporating a half sample advance into the target phase response allows to design an analytic allpass filter in Fig. 3(a). The 'switching' function $f'(\mathrm{e}^{\mathrm{j}\Omega})$ is shown in Fig. 3(b), while the resulting argument of the maximisation problem in (6) is displayed in Fig. 3(c). △

Alternatively, the allpass design specification in [19] can be oversampled by a factor of two, such that switching frequencies occur at $\{\frac{\Omega_i}{2}, \frac{\Omega_i}{2} + \pi\}$, $i = 1, 2, 3$, which is guaranteed to be even. A demonstration of such a design modification is outlined in the following example.

Example 5: By twice oversampling the specification from Example 4, an allpass with 6 switching frequencies can

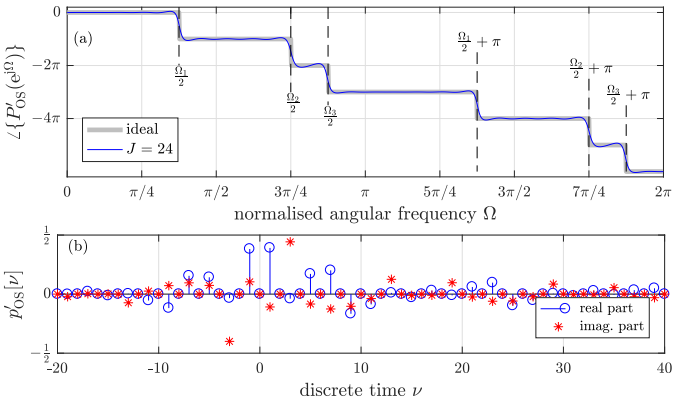

Fig. 4. (a) phase response and (b) impulse response of the allpass design $p'_{\mathrm{OS}}[\nu] \circ\!\!-\!\!\bullet\, P'_{\mathrm{OS}}(\mathrm{e}^{\mathrm{j}\Omega})$ in Example 5, which is the twice oversampled system compared to Fig. 3.

now be constructed. Utilising a design from [19] of order $J = 24$, an oversampled allpass $p'_{\mathrm{OS}}[\nu] \circ\!\!-\!\!\bullet\, P'_{\mathrm{OS}}(\mathrm{e}^{\mathrm{j}\Omega})$, with the subscript OS denoting 'oversampled', is constructed that possesses the phase response shown Fig. 4(a). Since this design is π-periodic, it can be decimated by a factor of two, but the decimation process must take a half-sample delay into account, noting that the impulse response $p'_{\mathrm{OS}}[\nu]$ as shown in Fig. 4(b) has one zero polyphase component, such that $p[n] = p'_{\mathrm{OS}}[2n+1]$ is possible. Note that allpass designs are non-causal as evident in Fig. 4(b) if the allpass order exceeds the number of pairs of switching frequencies [19]. △

C. Relation to Existing Analytic Procrustes Solution

With $\kappa = 2$ in (3), we can calculate the analytic Procrustes solution $\boldsymbol{Q}_{*,\mathrm{OS}}(z)$ for the oversampled system $\boldsymbol{A}(z^2) \bullet\!\!-\!\!\circ\, \mathbf{A}[n/2]$. This now matches $\boldsymbol{Q}_{*,\mathrm{OS}}(z) = \boldsymbol{Q}_*(z^2)$ in (11), such that we can also obtain

$$\boldsymbol{Q}_*(z) = \boldsymbol{Q}_{*,\mathrm{OS}}(z^{\frac{1}{2}}) \,, \tag{12}$$

i.e. by performing a two-fold decimation of the result of [19].

Example 6: An $M = 32$-dimensional square matrix $\boldsymbol{A}(z)$ is generated from the factors in (8). Paraunitary matrices $\boldsymbol{U}_{\mathrm{c}}(z)$ and $\boldsymbol{V}_{\mathrm{c}}(z)$ are formed from random elementary paraunitary matrices [10] as $\mathbf{I}_M - (1 - z^{-1})\mathbf{v}\mathbf{v}^{\mathrm{H}}/\|\mathbf{v}\|_2$, with $\mathbf{v} \sim \mathcal{CN}(\underline{0}, \mathbf{I}_M)$. Half of the singular values are generated from $\varsigma_m = \alpha_{-1}z + \alpha_0 + \alpha_1 z^{-1}$ with $\alpha_i \sim \mathcal{CN}(0,1)$, as $s_m(z) = (\varsigma_m(z) + \varsigma_m^{\mathrm{P}}(z))/C$. These singular values are parahermitian, are real-valued on the unit circle, and either possess none or two zero crossings on the unit circle. The normalisation C is such that $\int_{-\pi}^{\pi} |s_m(\mathrm{e}^{\mathrm{j}\Omega})|^2 \mathrm{d}\Omega = 2\pi$. The remaining singular values are constructed as $s_m(z) = (\mathrm{e}^{-\mathrm{j}\theta_m} + \mathrm{e}^{\mathrm{j}\theta_m} z^{-1})/\sqrt{2}$, with uniformly distributed $\theta_m \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, thus having a single spectral zero at $\Omega = 2\theta_m$. A few entries of the twice oversampled matrix $\mathbf{A}[n/2]$ are characterised in Figs. 5 and 6 in terms of their real and imaginary parts, respectively.

Executing the algorithm in [19] on $\boldsymbol{A}(z^2)$, results in the Procrustes solution $\boldsymbol{Q}_{*,\mathrm{OS}}(z)$. A few entries of this matrix $\mathbf{Q}_{*,\mathrm{OS}}[n] \circ\!\!-\!\!\bullet\, \boldsymbol{Q}_{*,\mathrm{OS}}(z)$ are illustrated in Figs. 5 and 6. The

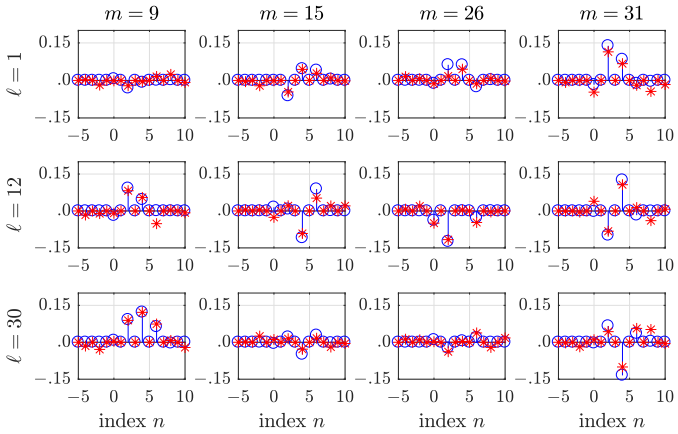

Fig. 5. Real parts $\Re\{A_{\ell,m}[n/2]\}$ (∘) and $\Re\{Q_{*,\mathrm{OS},\ell,m}[n]\}$ (∗), the components in the ℓth row and mth column of $\mathbf{A}[n/2]$ and of $\mathbf{Q}_{*,\mathrm{OS}}[n]$.

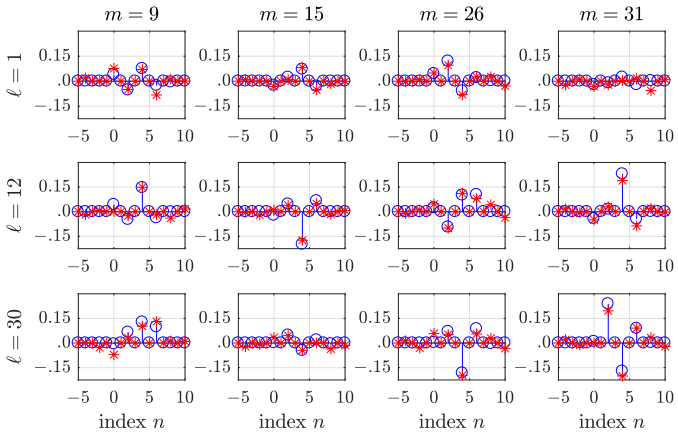

Fig. 6. Imaginary parts $\Im\{A_{\ell,m}[n/2]\}$ (∘) and $\Im\{Q_{*,\mathrm{OS},\ell,m}[n]\}$ (∗).

elements of $\mathbf{Q}_{*,\mathrm{OS}}[n]$ closely match those of $\mathbf{A}[n/2]$, whereby the relative magnitudes match well because of the normalisation of the singular values of $\boldsymbol{A}(z)$. Notice in particular that all odd-indexed coefficients of $\mathbf{Q}_{*,\mathrm{OS}}[n]$ are zero, and hence indeed $\boldsymbol{Q}_{*,\mathrm{OS}}(z)$ can be decimated by two to obtain the Procrustes solution $\boldsymbol{Q}_*(z)$ to $\boldsymbol{A}(z)$. △

V. Simulations and Results

We generate a set of 100 random matrices of the type in Example 6 for different spatial dimensions $M \in \{2, 4, 8, 16, 32\}$, whereby half the singular values have an odd number of zero crossings. We apply the method in [19] to the oversampled matrices $\boldsymbol{A}(z^2)$, and the approach in (11) based on a modified allpass design using a phase correction as shown in Fig. 3(a). We measure the least squares error e_{LS}, i.e. the cost term in (2), the error in paraunitarity $e_{\mathrm{PU}} = \frac{1}{2\pi}\int_{\pi}^{\pi} \|\boldsymbol{Q}_*(\mathrm{e}^{\mathrm{j}\Omega})\boldsymbol{Q}_*^{\mathrm{H}}(\mathrm{e}^{\mathrm{j}\Omega}) - 1\|_{\mathrm{F}}^2 \mathrm{d}\Omega$, and record the execution time T of the implementations.

Both algorithms operate in the discrete Fourier transform (DFT) domain with a maximum DFT size of 2^{10}. Every simulation went to this limit in order to model the allpass filters, yielding similar LS errors and execution times as shown in Fig. 7. However, since for the approach in [19] the result

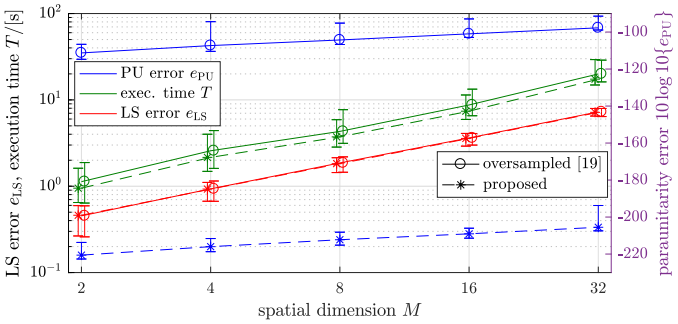

Fig. 7. Median values and 5/95-percentiles of the ensemble results for the least squares error e_{LS}, the paraunitarity error e_{PU}, and the execution time T on an x86 1.1GHz processor under Matlab 2016b.

needs to be decimated, the effective order of the solution is halved compared to the direct implementation of (11). Thus, the paraunitarity error e_{PU} is considerably reduced for the latter.

VI. Discussion and Conclusions

A previous method in [19] to determine the Procrustes solution to a polynomial matrix $\boldsymbol{A}(z)$ was restricted to cases where the analytic SVD of $\boldsymbol{A}(z)$ exists. Otherwise only the Procrustes solution to the oversampled system $\boldsymbol{A}(z^2)$ could be determined. In this paper, we have shown that in that case, a Procrustes solution to $\boldsymbol{A}(z)$ still exists, and can be determined via the existing algorithm and by decimating the solution to $\boldsymbol{A}(z^2)$. However, operating on an oversampled matrix incurs a higher computational complexity than necessary, and we have suggested an algorithmic modification that can dispense with the oversampling / decimation step.

Even though in practice, zero-crossings of singular values may be obscured if $\boldsymbol{A}(z)$ is randomly perturbed [27], the presented results matter even in the case of randomly perturbed systems if algorithms are employed to recover a solution for the unperturbed, ground truth system [29], [30].

References

[1] P. Vaidyanathan, "Theory of optimal orthonormal subband coders," *IEEE Trans. Signal Process.*, vol. 46, no. 6, pp. 1528–1543, Jun. 1998.

[2] P. L. Dragotti, S. D. Servetto, and M. Vetterli, "Optimal Filter Banks for Multiple Description Coding: Analysis and Synthesis," *IEEE Trans. Inf. Theory*, vol. 48, no. 7, pp. 2036–2052, July 2002.

[3] M. Unser, "On the optimality of ideal filters for pyramid and wavelet signal approximation," *IEEE Trans. Signal Process.*, vol. 41, no. 12, pp. 3591–3596, Dec. 1993.

[4] P. Moulin and M. Mihcak, "Theory and design of signal-adapted FIR paraunitary filter banks," *IEEE Trans. Signal Process.*, vol. 46, no. 4, pp. 920–929, Apr. 1998.

[5] I. Tabus, C. Popeea, and J. Astola, "A design procedure for optimal energy compaction IIR filters," *IEEE Trans. Circuits and Systems II*, vol. 48, no. 7, pp. 740–744, jul 2001.

[6] A. Tkacenko and P. Vaidyanathan, "On the spectral factor ambiguity of FIR energy compaction filter banks," *IEEE Trans. Signal Process.*, vol. 54, no. 1, pp. 380–385, Jan. 2006.

[7] V. Neo, C. Evers, S. Weiss, and P. Naylor, "Signal compaction using polynomial EVD for spherical array processing with applications," *IEEE/ACM Trans. Audio, Speech, and Language Process.*, vol. 31, pp. 3537–3549, 2023.

[8] V. W. Neo, C. Evers, and P. A. Naylor, "Enhancement of noisy reverberant speech using polynomial matrix eigenvalue decomposition," *IEEE/ACM Trans. Audio, Speech, and Language Process.*, vol. 29, pp. 3255–3266, 2021.

[9] P. P. Vaidyanathan, "Multirate Digital Filters, Filter Banks, Polyphase Networks, and Applications: A Tutorial," *Proc. IEEE*, vol. 78, no. 1, pp. 56–93, January 1990.

[10] ——, *Multirate Systems and Filter Banks*. Englewood Cliffs: Prentice Hall, 1993.

[11] Z. Cvetkovic and M. Vetterli, "Oversampled filter banks," *IEEE Trans. Signal Process.*, vol. 46, no. 5, pp. 1245–1255, May 1998.

[12] S. J. Schlecht and E. A. P. Habets, "Scattering in feedback delay networks," *IEEE/ACM Trans. Audio, Speech, and Language Process.*, vol. 28, pp. 1915–1924, 2020.

[13] S. J. Schlecht, "Allpass feedback delay networks," *IEEE Trans. Signal Process.*, vol. 69, pp. 1028–1038, 2021.

[14] O. Das, S. J. Schlecht, and E. De Sena, "Grouped feedback delay networks with frequency-dependent coupling," *IEEE/ACM Trans. Audio, Speech, and Language Process.*, vol. 31, pp. 2004–2015, May 2023.

[15] J. R. Hurley and R. B. Cattell, "The Procrustes program: Producing direct rotation to test a hypothesized factor structure," *Behavioral Science*, vol. 7, no. 2, pp. 258–262, 1962.

[16] G. H. Golub and C. F. Van Loan, *Matrix Computations*, 3rd ed. Baltimore, Maryland: John Hopkins University Press, 1996.

[17] S. Weiss, S. Schlecht, O. Das, and E. De Sena, "Polynomial Procrustes problem: Paraunitary approximation of matrices of analytic functions," in *31st European Signal Processing Conference*, Helsinki, Finland, 2023, pp. 1629–1633.

[18] S. Weiss, S. J. Schlecht, and M. Moonen, "Polynomial procrustes solution for randomly perturbed near-paraunitary systems," in *23rd IEEE Statistical Signal Processing Workshop*, Edinburgh, Scotland, Jun. 2025, pp. 231–235.

[19] ——, "Best least squares paraunitary approximation: Analytic procrustes problem," *IEEE Trans. Signal Process.*, vol. 74, pp. 904-916, Feb. 2026.

[20] G. Barbarino and V. Noferini, "On the Rellich eigendecomposition of para-Hermitian matrices and the sign characteristics of $*$-palindromic matrix polynomials," *Linear Algebra Appl.*, vol. 672, pp. 1–27, Sep. 2023.

[21] S. Weiss, I. K. Proudler, G. Barbarino, J. Pestana, and J. G. McWhirter, "On properties and structure of the analytic singular value decomposition," *IEEE Trans. Signal Process.*, vol. 72, pp. 2260–2275, 2024.

[22] G. Barbarino, "On the periodicity of singular vectors and the holomorphic block-circulant SVD on the unit circumference," *Linear Algebra Appl.*, vol. 721, pp. 465–483, Sep. 2025.

[23] S. Weiss and G. Barbarino, "Relations between analytic spectral and singularvalue decompositions," in *33rd European Signal Process. Conf.*, Palermo, Italy, Sep. 2025, pp. 2272–2276.

[24] X.-g. Xia and B. Suter, "Multirate filter banks with block sampling," *IEEE Trans. Signal Process.*, vol. 44, no. 3, pp. 484–496, March 1996.

[25] B. De Moor and S. Boyd, "Analytic properties of singular values and vectors," KU Leuven, Tech. Rep., 1989.

[26] A. Bunse-Gerstner, R. Byers, V. Mehrmann, and N. K. Nicols, "Numerical computation of an analytic singular value decomposition of a matrix valued function," *Numer. Math*, vol. 60, pp. 1–40, 1991.

[27] M. Bakhit, F. A. Khattak, I. K. Proudler, and S. Weiss, "Impact of estimation errors of a matrix of transfer functions onto its analytic singular values and their potential algorithmic extraction," in *IEEE High Performance Extreme Comp. Conf.* Boston, MA, Sep. 2024, pp. 1–7.

[28] M. A. Bakhit, F. A. Khattak, G. W. Rice, I. K. Proudler, and S. Weiss, "Recovering ground truth singular values from randomly perturbed MIMO transfer functions," in *IEEE Statistical Signal Process. Workshop*, Edinburgh, Scotland, Jun. 2025, pp. 246–250.

[29] S. J. Schlecht and S. Weiss, "Reconstructing analytic dinosaurs: Polynomial eigenvalue decomposition for eigenvalues with unmajorised ground truth," in *32nd European Signal Process. Conf.*, Lyon, France, August 2024, pp. 1287–1291.

[30] M. A. Bakhit, F. A. Khattak, S. J. Schlecht, G. W. Rice, and S. Weiss, "Challenges to subcarrier mimo precoding and equalisation with smooth phase responses," in *28th Int. Workshop on Smart Antennas*, Erlangen, Germany, Sep. 2025, pp. 32–38.

[31] A. Cohn, "Über die Anzahl der Wurzeln einer algebraischen Gleichung in einem Kreise," *Math. Zeitschrift*, vol. 14, no. 1, pp. 110–148, 1922.

[32] Y. Choo and Y. Kim, "On the zeros of self-inversive polynomials," *Int. Journal Math. Analysis*, vol. 7, no. 1–4, pp. 187–193, 2013.