Space-Time Coding in Statistical MIMO Radar

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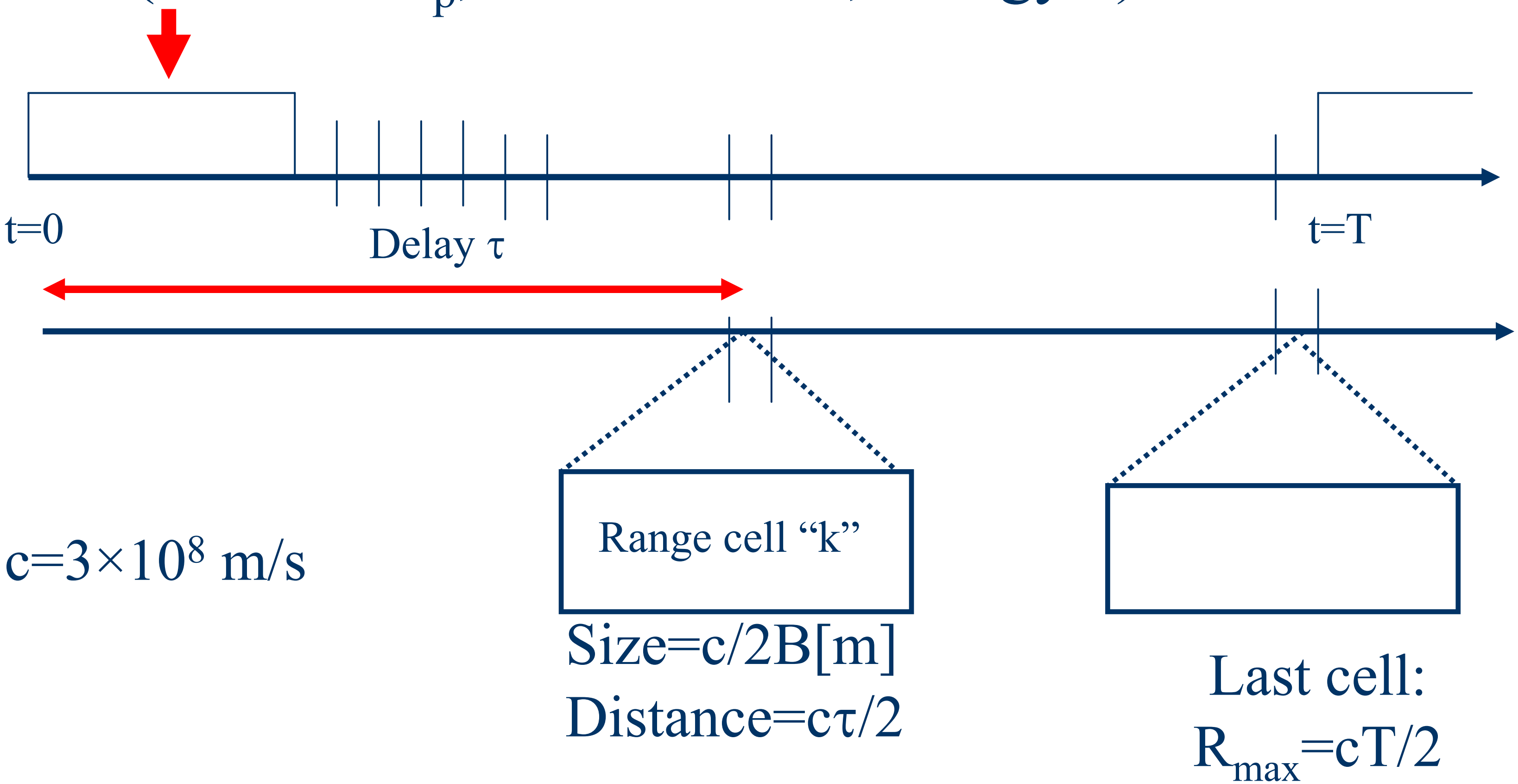
MIMO Detection: context

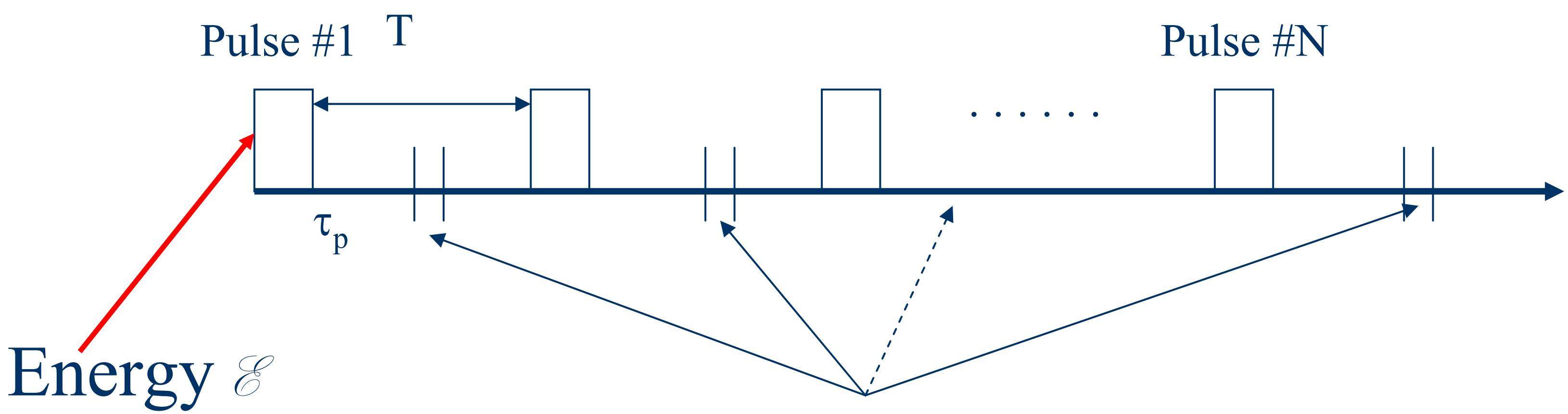
- Multiple Input Multiple Output Radar with widely spaced Antennae.
- Linear Space-Time Coding helps us shed light on the achievable detection performance.
- STC allows shaping the transmit waveform to achieve
  - Diversity (through different aspect angles);
  - Energy Integration (i.e., power multiplexing);
  - A compromise between the two.

RADAR Basic Operation

Pulse (duration τ_p, bandwidth B, energy $\mathscr{E}$)



Pulsed Radars



N echoes from a target in cell k+

N "clutter" echoes, exhibiting time correlation

Basic Receiver

Test on the signal received in the k-th cell:

$$H_1 : \mathbf{r} = \alpha\sqrt{\mathscr{E}}\,\mathbf{1} + \mathbf{w}$$
$$H_0 : \mathbf{r} = \mathbf{w}$$

← N-dimensional vectors

$$\mathrm{E}\left[\mathbf{w}\mathbf{w}^H\right] = \mathbf{M} \qquad \mathbf{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

Matched Filter

$$\left|\mathbf{r}^{H}\mathbf{M}^{-1}\mathbf{1}\right|^{2} \underset{H_0}{\overset{H_1}{\gtrless}} T$$

$P_{fa}=\mathbb{P}\{\text{Declare } H_1|H_0\}$ fixed (T)

$P_d=\mathbb{P}\{\text{Declare } H_1|H_1\}$ maximum

Performance

If α is complex Normal with variance σ_a^2 (Swerling-I):

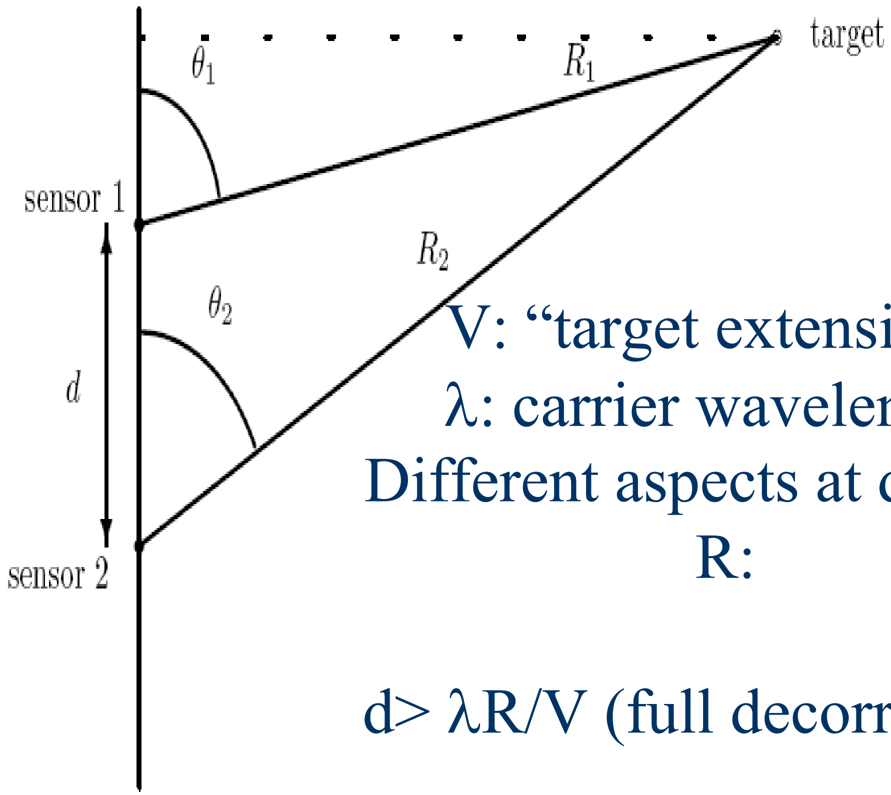
$$SCR_T = \mathscr{E}\left(\mathbf{1}^T \mathbf{M}^{-1}\mathbf{1}\right)\sigma_a^2 = N \cdot SCR$$

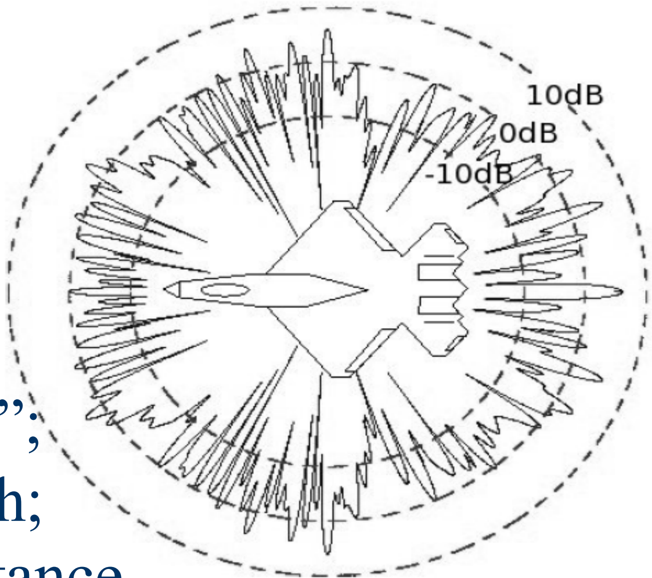
$$P_d = P_{fa}^{\frac{1}{1+N\cdot SCR}} \approx 1 - \frac{\left|\ln P_{fa}\right|}{N \cdot SCR} \qquad SCR \to \infty$$

N: Coherent Integration Gain

SCR: Signal-to-Clutter Ratio per pulse

Angle Diversity

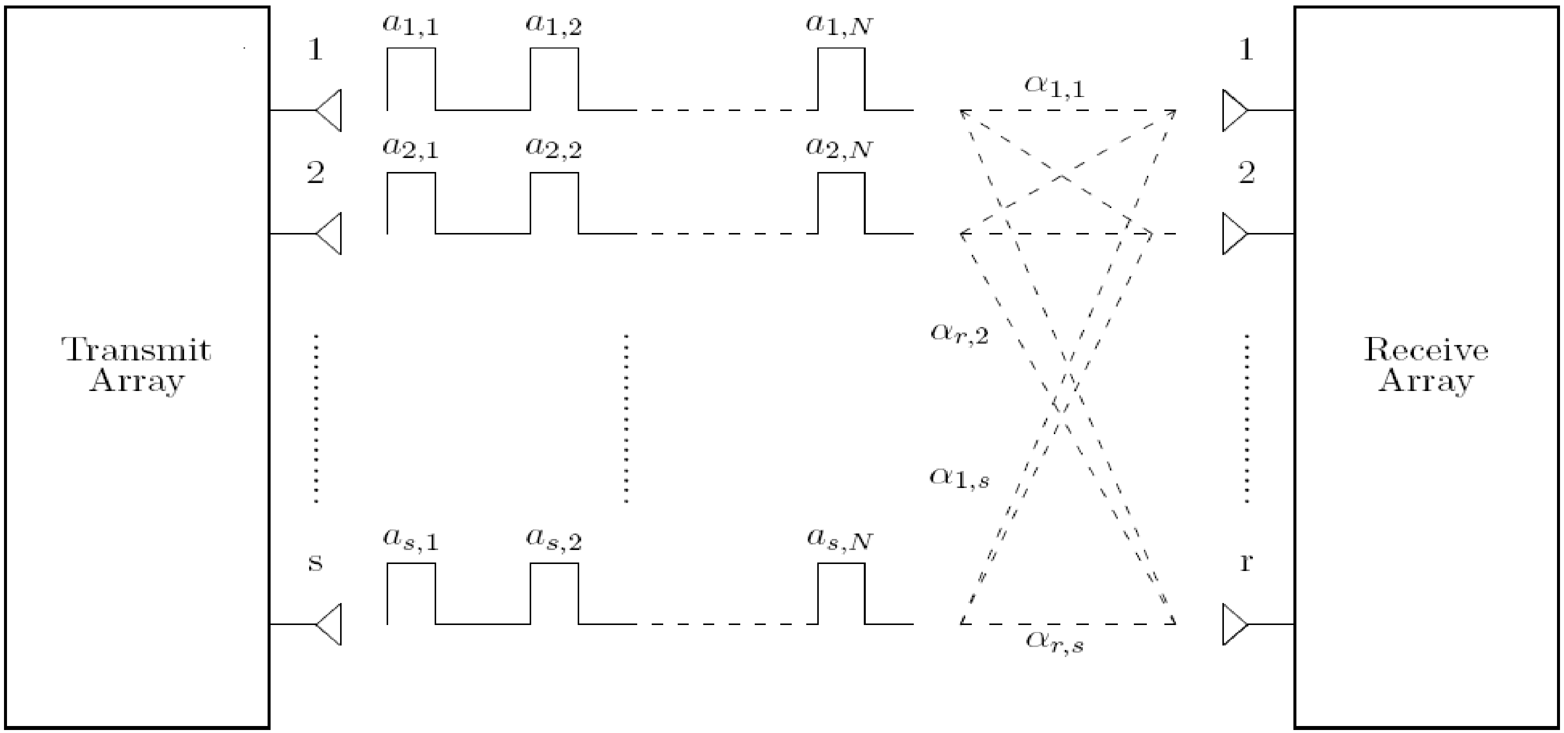




V: “target extension”;
λ: carrier wavelength;
Different aspects at distance R:

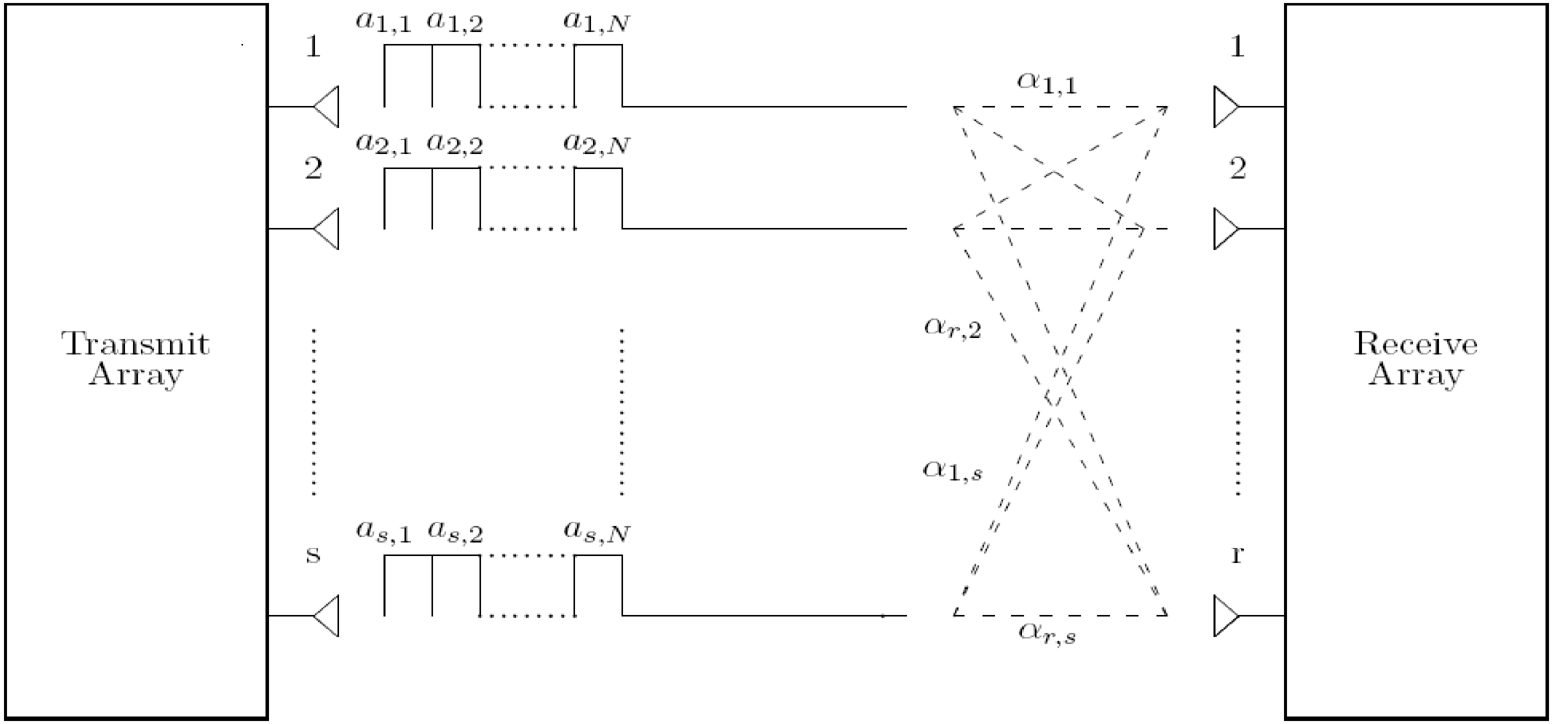
d> λR/V (full decorrelation)

Pulsed MIMO Radar: Scheme



Pulsed Waveforms

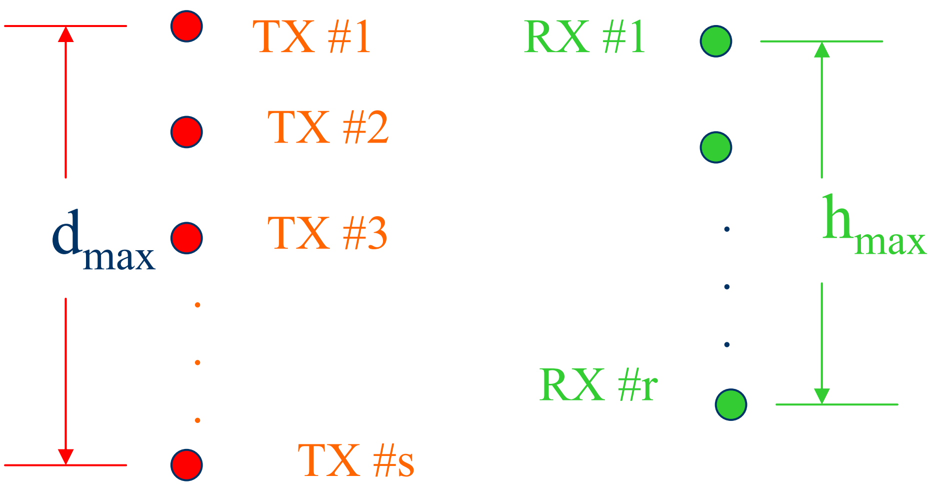
Scheme #2



Transmit Array
1
2
s
$a_{1,1}$ $a_{1,2}$ $a_{1,N}$
$a_{2,1}$ $a_{2,2}$ $a_{2,N}$
$a_{s,1}$ $a_{s,2}$ $a_{s,N}$
$\alpha_{1,1}$
$\alpha_{r,2}$
$\alpha_{1,s}$
$\alpha_{r,s}$
1
2
r
Receive Array

Underlying Assumption

- "Narrow-band" assumption on the transmitted pulses: The target is seen in the same range-cell by all of the antennae.



$d_{max}+h_{max} << c/B$

Ex: B=1MHz we have:

$d_{max}+h_{max} << 300m$

Degrees of Freedom

The number of degrees of freedom is

$$s \cdot \min(r, N) = \begin{cases} sr & \text{(tall code matrix)} \\ sN & \text{(fat code matrix)} \end{cases}$$

Why? Because the transmitted signal space has at most dimension N, whereby a “fat” code matrix means that the codewords are not linearly independent!

MIMO Radar: The Model

$$H_1: \quad \mathbf{r}_i = \mathbf{A}\boldsymbol{\alpha}_i + \mathbf{w}_i \qquad i = 1, \ldots, r$$

$$H_0: \quad \mathbf{r}_i = \mathbf{w}_i \qquad i = 1, \ldots, r$$

received at antenna "i"

Code-Matrix (N×s)

Coefficients from the TX to "i" RX ant.

Clutter with Covariance **M**

$$H_1: \quad \mathbf{R} = \mathbf{A}\mathbf{H} + \mathbf{W}$$

$$H_0: \quad \mathbf{R} = \mathbf{W}$$

Parameters

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & \cdots & a_{s,1} \\ \vdots & \vdots & \vdots \\ a_{1,N} & \cdots & a_{s,N} \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_s \end{bmatrix}$$

$$\boldsymbol{\alpha}_i = \begin{bmatrix} \alpha_{i,1} \\ \vdots \\ \alpha_{i,s} \end{bmatrix}$$

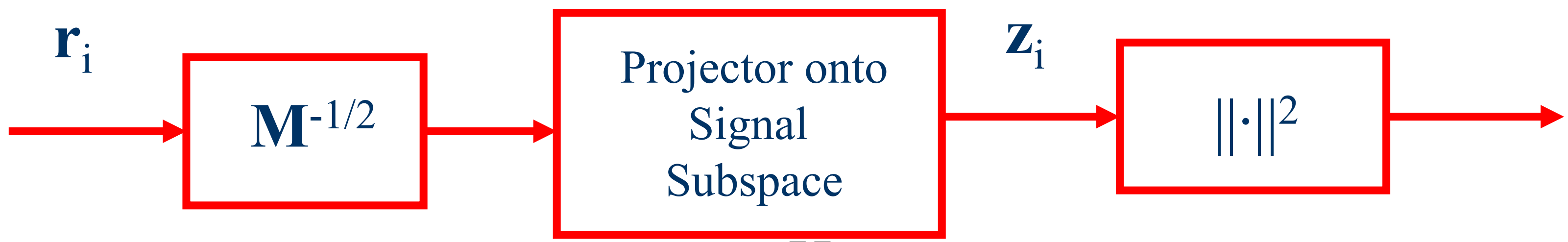
$$\mathrm{E}\left[\mathbf{w}_i \mathbf{w}_i^H\right] = \mathbf{M}; \qquad \mathrm{E}\left[\mathbf{w}_i \mathbf{w}_j^H\right] = \mathbf{M}\delta_{i,j}$$

Designing the test

$$\frac{\max\limits_{\boldsymbol{\alpha}_1,\ldots,\boldsymbol{\alpha}_r} f\left(\mathbf{r}_1,\ldots,\mathbf{r}_r \middle| \boldsymbol{\alpha}_1,\ldots,\boldsymbol{\alpha}_r,\mathbf{M};H_1\right)}{f\left(\mathbf{r}_1,\ldots,\mathbf{r}_r \middle| \mathbf{M};H_0\right)} \underset{H_0}{\overset{H_1}{\gtrless}} T \qquad \text{(GLRT)}$$

$$\mathbf{M} = \mathrm{E}\left[\mathbf{w}_i \mathbf{w}_i^H\right]$$

GLR Test Statistics



$$\sum_{i=1}^{r}\left\|\mathbf{z}_i\right\|^2=\sum_{i=1}^{r}\left\|\mathbf{P}_{\mathbf{M}^{-1/2}\mathbf{A}}\mathbf{M}^{-1/2}\mathbf{r}_i\right\|^2 \underset{H_0}{\overset{H_1}{\gtrless}} T$$

GLRT Test Statistics

$\mathbf{P}_{\mathbf{V}}$=projector onto the range span of V.

Remark

The maximum number of degrees of freedom (maximum diversity order, under full-rank space-time coding) is r min (N,s)= rδ. In fact:

$$\mathbf{P}_{\mathbf{M}^{-1/2}\mathbf{A}} = \begin{cases} \mathbf{M}^{-1/2}\mathbf{A}\left(\mathbf{A}^H\mathbf{M}^{-1}\mathbf{A}\right)^{-1}\mathbf{A}^H\mathbf{M}^{-1/2} & N \geq s \\ \mathbf{I}_N & N \leq s \end{cases}$$

Special cases

White noise, N=s, and orthogonal codes

$$\sum_{i=1}^{r} \|\mathbf{r}_i\|^2 \underset{H_0}{\overset{H_1}{\gtrless}} T$$

(Incoherent Integrator)

White noise, N=s, and un-coded waveforms (i.e. rank-1 coding, $\mathbf{A}=\mathbf{1}\mathbf{1}^T$)

$$\sum_{i=1}^{r} \left|\mathbf{1}^T \mathbf{r}_i\right|^2 \underset{H_0}{\overset{H_1}{\gtrless}} T$$

(Coherent Integrator)

Performances

Under rank-θ coding the performance of the GLRT admits the general expression:

$$P_{fa} = e^{-T} \sum_{k=0}^{\theta r-1} \frac{T^k}{k!}$$

$$P_d = \mathrm{E}\left[Q_{\theta r}\left(\sqrt{2\sum_{i=1}^{r} \boldsymbol{\alpha}_i^H \mathbf{A}^H \mathbf{M}^{-1} \mathbf{A} \boldsymbol{\alpha}_i} ; \sqrt{2T} \right) \right]$$

Code Selection

- No uniformly optimum (i.e., for any signal-to-clutter ratio and for any backscattering pdf) strategy exists;
- First Step: Assume Gaussian Scattering, and study the impact of diverse design strategies aimed at optimizing some figures of merit under some constraint.

Figures of merit (to be maximized)

- Lower Chernoff Bound (LCB) to the detection probability:

$$P_d \geq 1 - \min_{\gamma \geq 0} e^{\gamma T} \prod_{j=1}^{\theta} \left[\frac{1}{1+\gamma\left(\lambda_j \sigma_a^2 + 1\right)} \right]^r$$

- Average received SCR per pulse per receive antenna

$$SCR = \frac{\sigma_a^2}{N} \mathrm{tr}\left(\mathbf{A}^H \mathbf{M}^{-1} \mathbf{A}\right) = \frac{\sigma_a^2}{N} \sum_{i=1}^{\theta} \lambda_i$$

Eigenvalues of $\mathbf{M}^{-1/2}\mathbf{A}\mathbf{A}^{H}\mathbf{M}^{-1/2}$

- Mutual Information (MI):

$$I(\mathbf{r}_1,..,\mathbf{r}_r;\alpha_1,\ldots,\alpha_r)$$

LCB-Optimal codes

- Chose the space-time code A so as to maximize the LCB subject to constraints on:

1) Average Received Signal-to-Clutter Ratio per pulse and per receive antenna, SCR;
2) Rank of the code matrix (θ);

Solution: generate as many independent and identically distributed paths as possible.

MI-Optimal codes

- Chose the space-time code A so as to maximize the MI subject to constraints on:

1) Average Received Signal-to-Clutter Ratio per pulse and per receive antenna, SCR;
2) The Maximum Rank of the code matrix (i.e., rank($\mathbf{A}$)≤θ)

Solution: generate as many (i.e., θr) independent and identically distributed paths as possible. Thus with a definite constraint on the rank MI and LCB are equivalent.

SCR-Optimal codes

- Chose the space-time code **A** so as to maximize the average received SCR subject to a constraint on the transmitted energy

  Solution: Use rank-1 coding (i.e., transmit all of the energy along the least interfered direction of the signal space): note the consistency with the results concerning capacity (Boche et alii, 2005).

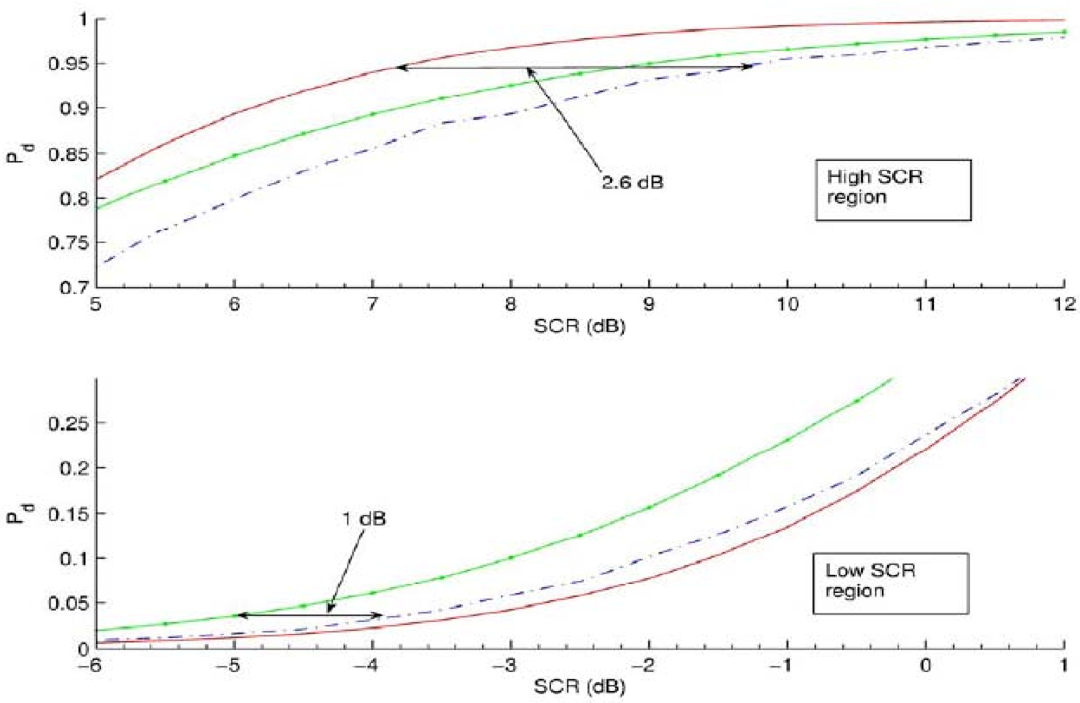
Other possible strategies

- Both LCB and MI could be maximized under a constraint on the transmitted power.
- The corresponding solution (“waterfilling”) leads to a performance improvement on the previous strategies, but is useless (in that it requires knowledge of the average power of a target whose presence we are trying to ascertain).

Semi-definite vs Definite rank constraints

- With a definite rank constraint we are left free only of allocating the power on a pre-determined number of modes;
- With a semi-definite constraint we determine the optimum transmit policy, i.e. we can decide between diversity and beam-forming.
- Diversity maximization is not uniformly optimal for either P_d or LCB, an indirect evidence that the uncertainty as to the target presence plays a fundamental role.
- Conclusion: MI is not a good figure of merit for waveform design in MIMO radar with surveillance functions.

Which Strategy is the best?



N=r=s=2; white noise; LCB-optimal; Un-coded+coherent; Un-coded+incoherent

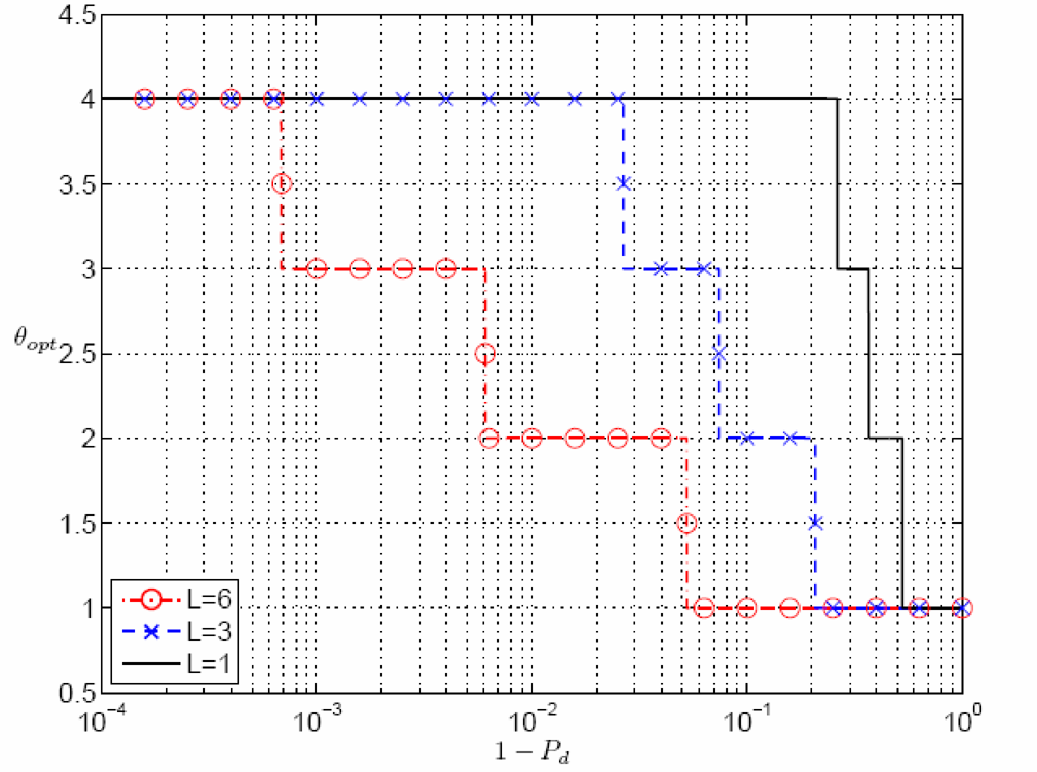
Diversity order and Energy Integration

$$P_d \approx 1 - \frac{1}{(\theta r)\,!}\left(\frac{T}{N \cdot SCR / \theta}\right)^{\theta r}$$

$$\text{rank}(\mathbf{A}) = \theta$$

Here the trade-off is apparent: increasing the number of diversity paths (i.e., θ, the code-matrix rank) results in a larger number of paths with lower average SCR.

Performance



Optimal rank of the code matrix (i.e., requiring minimum SCR to achieve the miss probability on the abscissas).

$P_{fa}=10^{-4}$;

N=s=4;

Some conclusions

- Unlike LCB, MI does not appear very suitable a figure of merit (oblivious as to the uncertainty as to the target presence);
- Mathematically, we need a figure of merit which is neither Schur-Concave nor Schur-convex, in order to study the trade-offs between diversity and integration, and the corresponding interplay with the policy transmission.
- Unfortunately, a semi-definite optimization problem involving the LCB is not very credible, nor feasible (the LCB is not tight at low SNR's).

Alternative: Kullback-Leibler Distance

Define the short-hand notation $f_i(\mathbf{r}_k)=f(\mathbf{r}_k|H_i)$ and recall the Kullback-Leibler distance (or divergence):

$$D\left(f_1(\mathbf{r}) \| f_0(\mathbf{r}) \right) = D_{1,0} = \int f_1(\mathbf{r}) \ln \frac{f_1(\mathbf{r})}{f_0(\mathbf{r})} d\mathbf{r}$$

$$D\left(f_0(\mathbf{r}) \| f_1(\mathbf{r}) \right) = D_{0,1} = \int f_0(\mathbf{r}) \ln \frac{f_0(\mathbf{r})}{f_1(\mathbf{r})} d\mathbf{r}$$

A new figure of merit

$$d\left(\mathbf{A}\right) = \lambda D\left(f_1\left(\mathbf{r}\right) \| f_0\left(\mathbf{r}\right)\right) + \left(1-\lambda\right) D\left(f_0\left(\mathbf{r}\right) \| f_1\left(\mathbf{r}\right)\right)$$

Design strategy: chose λ and design **A** according to

$$d\left(\mathbf{A}\right) = \max$$

subject to a constraint on Average received/Transmitted SCR

General Comments

- For $\lambda=0$ the criterion admits a nice interpretation in the light of Stein's lemma, and represents the asymptotically optimum criterion for fixed sample size multi-frame MIMO radars;
- For non-zero λ the criterion can be nicely interpreted as an optimality criterion for sequential multi-frame MIMO radars.

Other interpretations

- Maximizing D_{01} distances has a nice interpretation in the light of Stein's lemma for fixed-sample size tests;
- Maximizing D_{10} and D_{01} corresponds to some form of optimization (in terms of Average Sample Number) in a sequential MIMO-radar procedure (see Wald's theory).

Sequential MIMO?

- Defer detection until the returns from several frames (i.e., pulse trains) each with different carrier have been received and solve

$$\mathbf{R}(m) = \begin{cases} \mathbf{AH}(m) + \mathbf{W}(m) \\ \mathbf{W}(m) \end{cases}$$

The decision process proceeds until a given pair (Pd, Pfa) is reached, by using a random number of samples and two detection thresholds. The Average Sample Number corresponds to the expected observation number under the two alternatives.

An example of performance

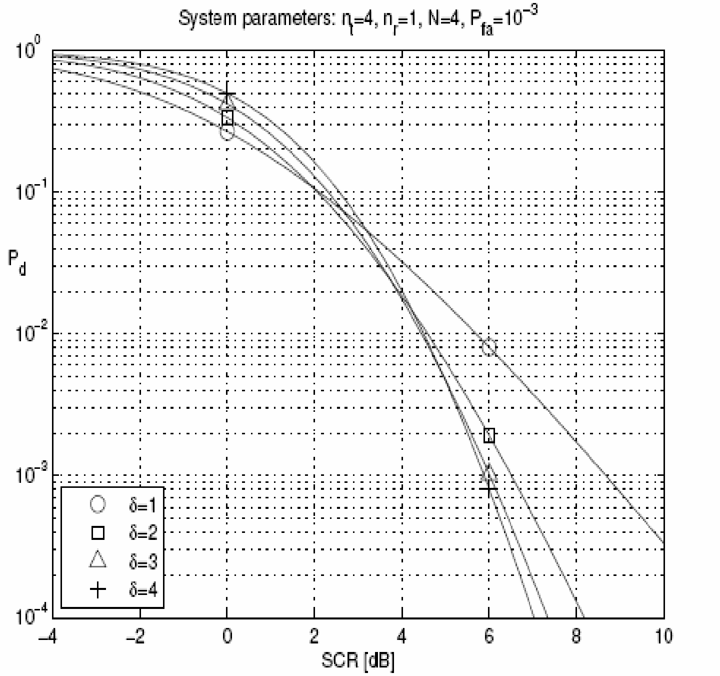


Figure 2. P_{miss} as a function of the available received SCR for various coding ranks.

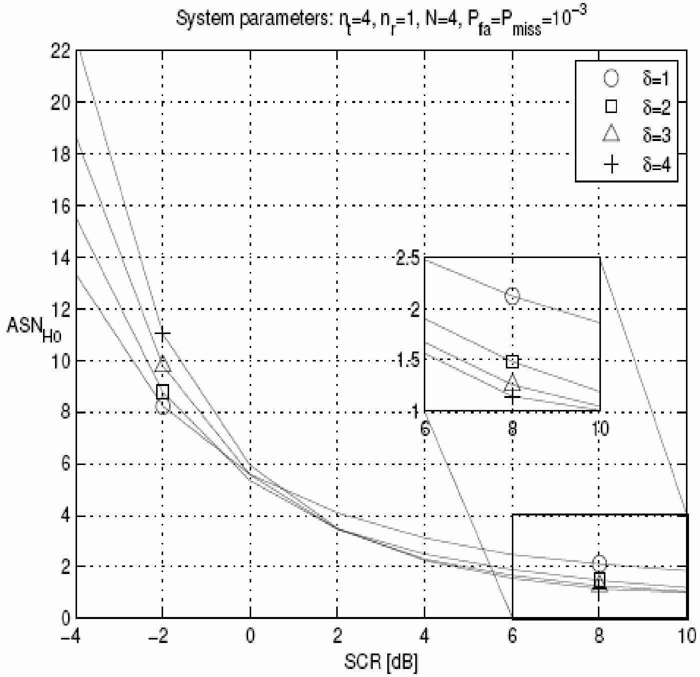


Figure 3. ASN_{H_0} as a function of the available received SCR for various coding ranks.

Problem

- Everything hinges upon uncorrelated Gaussian scattering;
- Notice that target scattering correlation is not under the control of the designer, since it depends on the antennas spacing, the target distance and its extension (the distance is in particular variable);
- The Gaussian assumption looks less dramatic.

Schur-Convexity

Let $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^n$

a) Majorization: $\mathbf{x} \prec \mathbf{y}$ iff

$$\sum_{k=1}^{m} x_k \le \sum_{k=1}^{m} y_k \quad m = 1, \ldots n-1 \text{ with } \sum_{k=1}^{n} x_k = \sum_{k=1}^{n} x_k$$

b) $\varphi(\mathbf{x})$ is Schur-Convex iff

$$\mathbf{x} \prec \mathbf{y} \Rightarrow \quad \varphi(\mathbf{x}) \le \varphi(\mathbf{y})$$

Robust Coding (Unpublished, HWC)

Denote

$$\mathbf{R}_{\boldsymbol{\alpha}} = \mathrm{E}\left[\boldsymbol{\alpha}_i \boldsymbol{\alpha}_i^H\right]$$

Denote

$$f\left(\zeta_i\left(\mathbf{R}_{\boldsymbol{\alpha}}^{1/2}\mathbf{A}^H\mathbf{M}^{-1}\mathbf{A}\mathbf{R}_{\boldsymbol{\alpha}}^{1/2}\right)\right)$$

a figure of cost which is Schur-convex. Here and in the following $\left\{\zeta_i\left(\mathbf{X}\right)\right\}_{i=1}^{n}$ denote the eigenvalues of

$$\mathbf{X} \in C^{n \times n}.$$

Robust Space-Time Coding

Mini-Max Design

Assume $N \geq s,$ so we have s eigenvalues to play with.

$\mathbf{A}$ is chosen as the solution to the constrained problem:

$$\min_{\mathbf{A}} \max_{\mathbf{R}_{\alpha}} f\left(\left\{\zeta_i\left(\mathbf{R}_{\alpha}^{1/2}\mathbf{A}^H\mathbf{M}^{-1}\mathbf{A}\mathbf{R}_{\alpha}^{1/2}\right)\right\}_{i=1}^{s}\right)$$

subject to constraints on:

transmitted energy/received SCR;

Trace of the scattering matrix $\mathbf{R}_{\alpha}$;

maximum rank (s).

Robust codes

- **A** must be full-rank (i.e., *s*);
- The amount of power to be poured in each mode should ensure equal powers of the modes at the receiver, i.e. we have to equalize all the modes, which means that the most interfered modes require more power (dramatically different from water-filling).

What cost functions satisfy the conditions stated above?

- Any decreasing function of the SCR, defined as SCR=Trace($\mathbf{A}\mathbf{R}_{\alpha}\mathbf{A}^{H}\mathbf{M}^{-1}$);
- Any increasing function of the MMSE of a linear estimator of the scattering;
- Under Gaussian scattering, any decreasing function of the MI between received signal and target scattering.

Non-Gaussian Target Scattering

Why shouldn't α_i be circularly symmetric Gauss?

- The target RCS fluctuation might be more constrained than exponential;
- The target scattering might be similar to "composite surface scattering", wherein a "slowly varying" texture is modulated by a rapidly decorrelating speckle.

How to select the Code under NG?

Set $\mathbf{A}^H\mathbf{M}^{-1}\mathbf{A}=\mathbf{CDC}^H$ with $\mathbf{C}$ unitary $s\times s$, $\mathbf{D}$ diagonal with $\theta\leq s$ positive entries, and define

$$\boldsymbol{\beta}_i=\mathbf{C}^H\boldsymbol{\alpha}_i, \quad i=1,\ldots,r$$

The $\boldsymbol{\beta}_i$ are defined exchangeable (a la de Finetti) if

$$f_{\beta 1,\ldots,\beta r}(\mathbf{x}_1,\ldots,\mathbf{x}_r)= f_{\beta 1,\ldots,\beta r}(\pi(\mathbf{x}_1,\ldots,\mathbf{x}_r))$$

If the r.v.'s $\boldsymbol{\alpha}_i$ are independent unitarily invariant (i.e., their pdf is invariant under unitary transformations), then define

$$\boldsymbol{\beta}=\text{vec}(\boldsymbol{\beta}_1,\ldots,\boldsymbol{\beta}_r)$$

LCB-optimal codes

If the α_i are independent unitarily invariant and the β_i are exchangeable, then:

The LCB is strictly concave for given θ=rank(**A**) and is maximized as we generate θr independent paths (i.e., as we maximize diversity, which can be done through equalization under no power limitation).

MI-optimal codes

Under the same invariance condition, the solution to the constrained (semi-definite) problem:

$$\begin{cases} \max I\left(\mathbf{r}_1,\ldots,\mathbf{r}_r;\boldsymbol{\alpha}_1,\ldots,\boldsymbol{\alpha}_r\right) \\ \text{s.t.}\quad \text{Trace}\left(\mathbf{A}^H\mathbf{M}^{-1}\mathbf{A}\right)\le\nu \\ \qquad\qquad \text{rank}\left(\mathbf{A}\right)\le\theta \end{cases}$$

is solved by equalizing θr paths, i.e. by maximizing the diversity order.

On Unitary Invariance

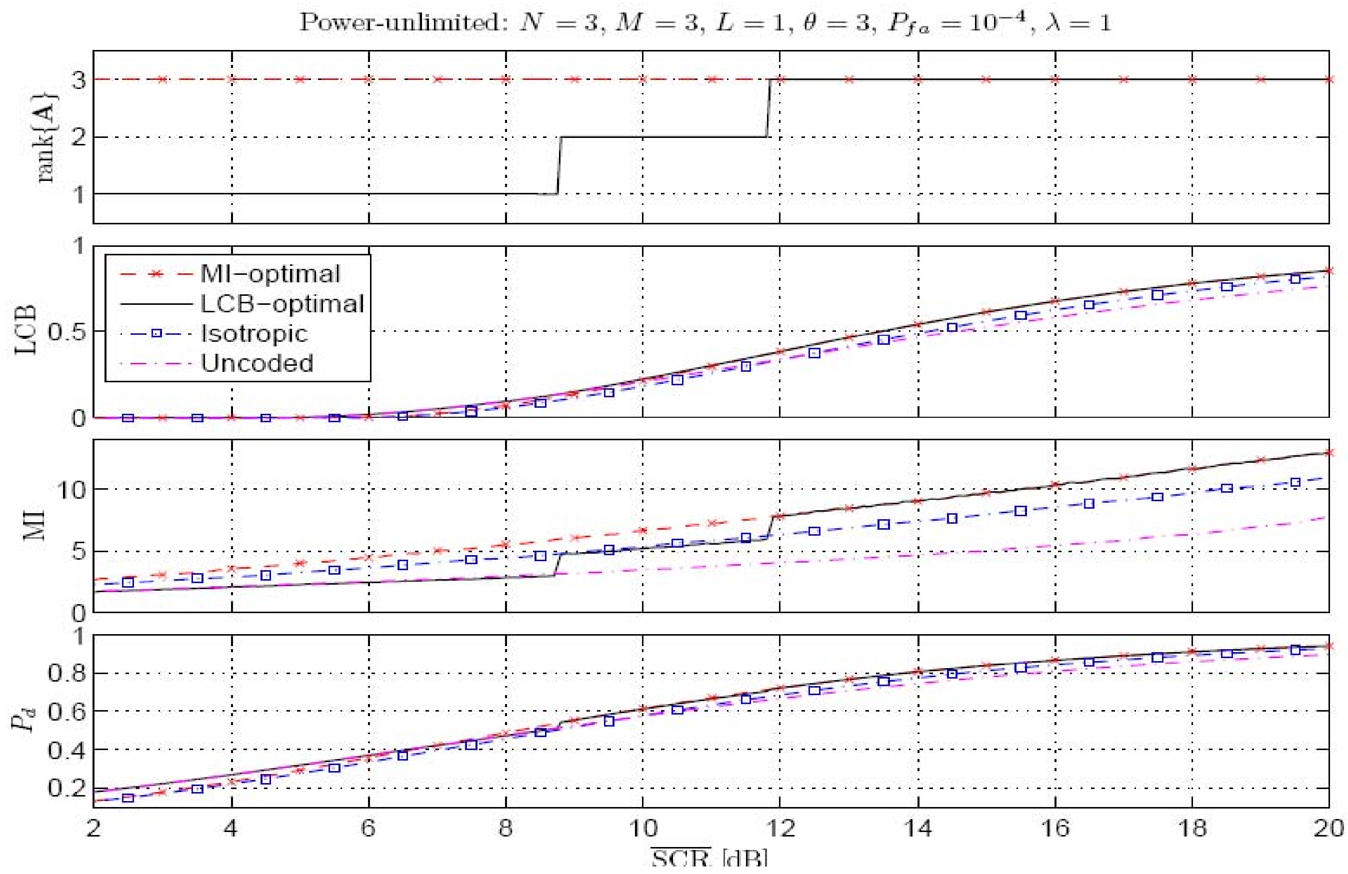
Is this condition restrictive? Yes, but a very important family of non-Gaussian processes, the Spherically Invariant Random Processes (SIRP), used to model composite scattering, are UI. Remark that **x** is an SIRP iff (Yao, 1973):

$$\mathbf{x} = s\mathbf{g}$$

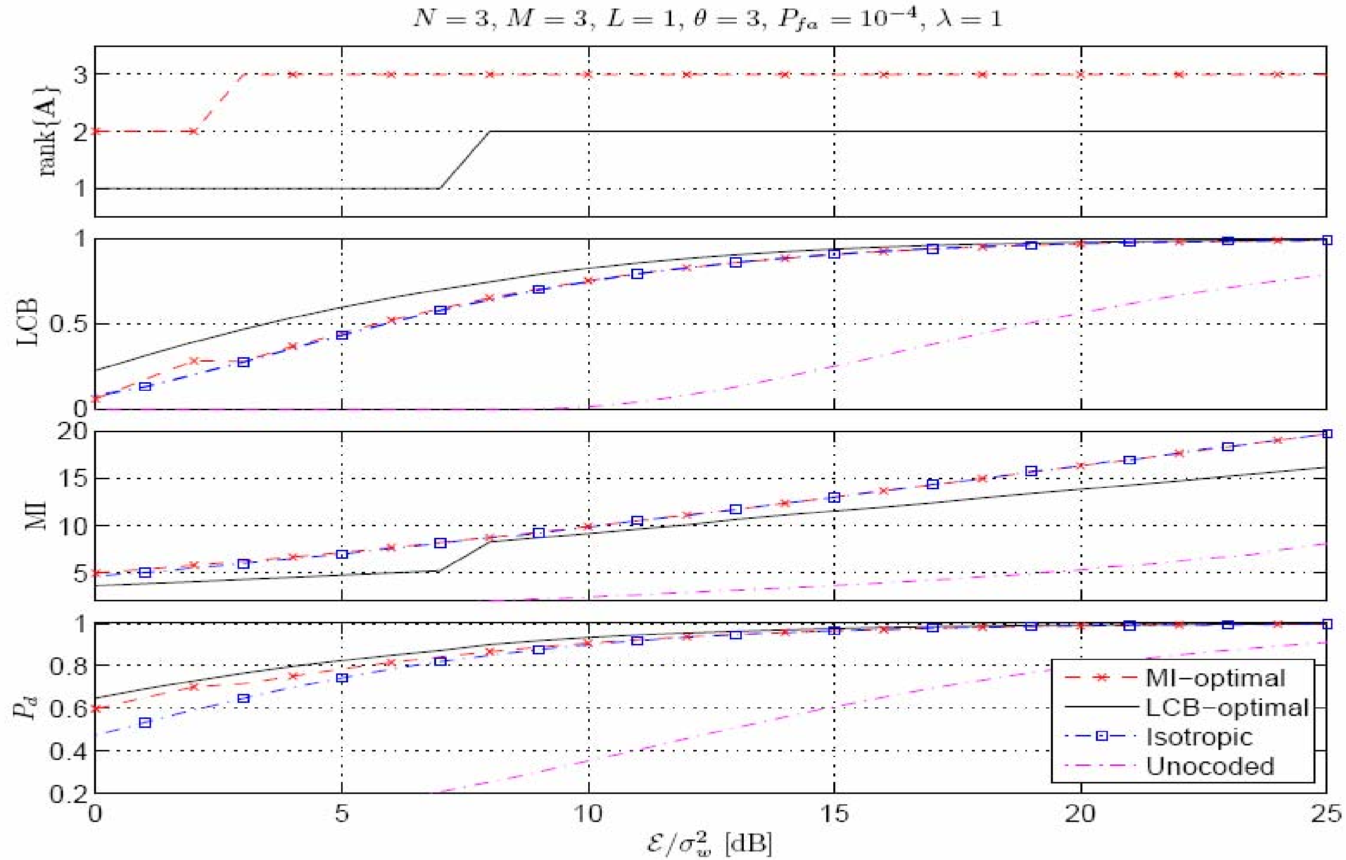
with **g** a complex white circularly symmetric Gaussian vector and s a non-negative random variable, whereby:

$$f_{x_j}(z) = \int \frac{1}{2\pi s^2} \exp\left(-\frac{z^2}{2s^2} \right) dF(s)$$

Performance under K-distributed scattering: "power unlimited"



Performance under K-distributed scattering: "power limited"



Next…

- Optimized STC with constraints on the accuracy (e.g., in the range-doppler plane);
- What happens for non-colocated antennas (i.e., the delays vary for loose synchronization)?
- For highly mobile targets, the doppler frequencies vary across the antennas, leading to a frequency-dispersive MIMO channel. What’s the model and the coding strategy?
- Can non linear codes help in any of the previous situations?
- What happens to these trade-offs if we use a (random) number of frames to make decisions, i.e. if we merge the concepts of Space-Time Coding and Sequential decision making?

Next….

- Conditioned to the previous results, is it possible to interleave several space-time codes (i.e., for detection of far-away and nearby objects)?
- Is MIMO radar useful as a diversity generator or we realistically have to use it just as a “power multiplexer”?